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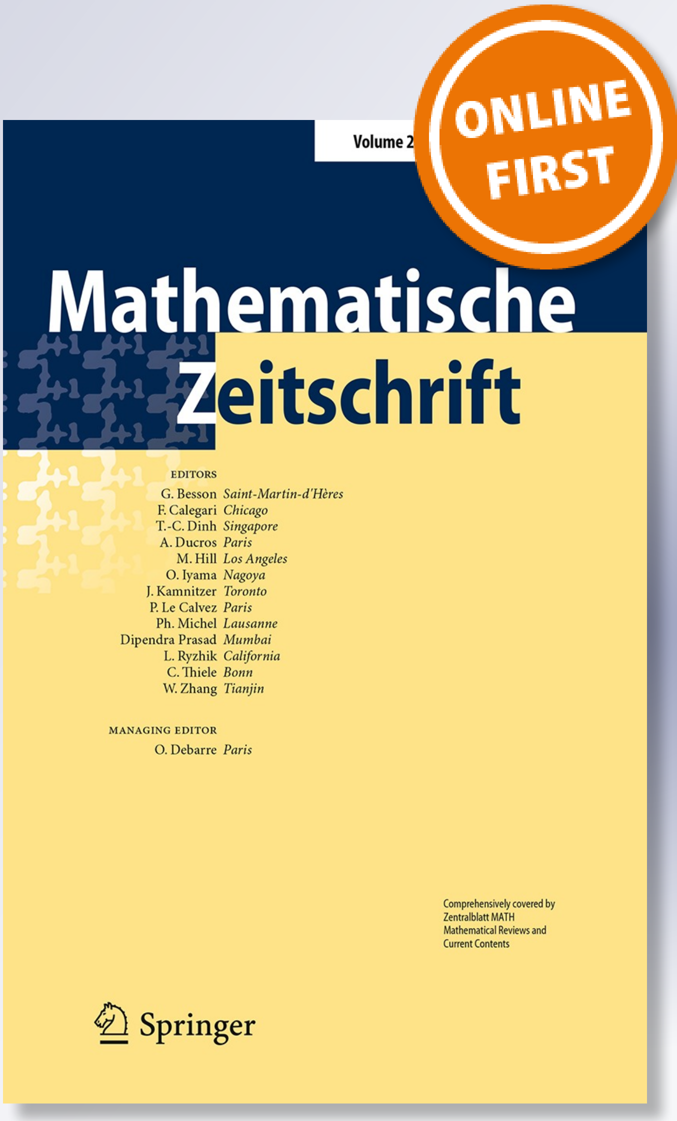

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A Miyawaki type lift for $GSpin(2, 10)$

Henry H. Kim[1,2] · **Takuya Yamauchi**[3]

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Abstract Let $\mathfrak{T}_2$ (resp. $\mathfrak{T}$) be the Hermitian symmetric domain of $Spin(2, 10)$ (resp. $E_{7,3}$). In previous work (Compos. Math 152(2):223–254, 2016), we constructed holomorphic cusp forms on $\mathfrak{T}$ from elliptic cusp forms with respect to $SL_2(\mathbb{Z})$. By using such cusp forms we construct holomorphic cusp forms on $\mathfrak{T}_2$ which are similar to Miyawaki lift constructed by Ikeda (Duke Math J 131:469–497, 2006) in the context of symplectic groups. It is conditional on the conjectural Jacquet–Langlands correspondence from $PGSO(2, 10)$ to $PGSO(6, 6)$.

Keywords Miyawaki type lift · Langlands functoriality · Ikeda lift · Tube domain associated to orthogonal groups

Mathematics Subject Classification Primary 11F55; Secondary 11F70 · 22E55 · 20G41

1 Introduction

For a reductive group G over $\mathbb{Q}$ with Hermitian symmetric domain $\mathfrak{D}$, it is important to construct cuspidal representations of $G(\mathbb{A})$ which give rise to holomorphic cusp forms on $\mathfrak{D}$, where $\mathbb{A} = \mathbb{A}_{\mathbb{Q}}$ is the ring of adeles of $\mathbb{Q}$. In general it would be difficult to construct cusp forms directly. Sometimes, however, techniques predicted or guided by Langlands functorial-

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ity can yield indirect constructions of cusp forms or proofs of their existence. To explain a bit more, given a smaller group H with an L-group homomorphism $r : {}^L H \longrightarrow {}^L G$, Langlands functoriality predicts the existence of a 'functorial lift' from automorphic representations of $H(\mathbb{A})$ to those of $G(\mathbb{A})$. This idea guides constructions of cusp forms using the trace formula or the theta lift. These are very powerful tools, but with the following limitation. The former never gives any explicit construction for classical forms. Further, while the latter construction can be made explicit with a careful choice of test functions, but yields only automorphic representations which are generic, away from holomorphic forms whenever we consider the level one form. On the other hand, Ikeda [19] gave an explicit construction of cusp forms for the symplectic group Sp_{2n} (with $\mathbb{Q}$-rank n) attached to elliptic cusp forms of $GL_2(\mathbb{A})$ with respect to $SL_2(\mathbb{Z})$. Such a cusp form is called Ikeda lift. In [20] Ikeda studied an integral similar to (1.1) below, obtained by substituting the role of Eisenstein series in the usual pull-back formula (cf. [14]) with an Ikeda lift. Then under the assumption of nonvanishing of the integral, he showed that it gives rise to an essentially new cusp form for symplectic groups which is called a Miyawaki lift. The existence of the Ikeda lift (resp. the Miyawaki lift) is compatible with Arthur's multiplicity formula ([1], Theorem 1.5.2).

In this paper we pursue an analogue of Miyawaki lift for $GSpin(2, 10)$ by using our previous work [24]. We now explain the main theorem. We refer to the next section (or Section 2 of [24]) for notations which appear below.

Let G be a form of the exceptional group of type $E_{7,3}$ over $\mathbb{Q}$ that has real rank 3 and splits at all finite primes p, and let $G' = GSpin(2, 10)$ be the spinor group with a similitude defined in Sect. 2.1 which splits at all finite primes p. Let $\mathfrak{T}_2$ (resp. $\mathfrak{T}$) be the Hermitian symmetric domain of $PGSpin(2, 10)(\mathbb{R})^0$ (resp. $E_{7,3}(\mathbb{R})$). Elements of $\mathfrak{T}$ and $\mathfrak{T}_2$ can be described in terms of Cayley numbers $\mathfrak{C}_{\mathbb{C}}$, in such a way that any $g \in \mathfrak{T}$ gets represented in the form $g = \begin{pmatrix} Z & w \\ {}^t\overline{w} & \tau \end{pmatrix}$ with $Z \in \mathfrak{T}_2$, $w \in \mathfrak{C}_{\mathbb{C}}^2$, and $\tau \in \mathbb{H} = \{z \in \mathbb{C} \mid \operatorname{Im} z > 0\}$. Let $S_{2k}(SL_2(\mathbb{Z}))$ be the space of elliptic cusp forms of weight $2k \geq 12$ with respect to $SL_2(\mathbb{Z})$. For each normalized Hecke eigenform $f = \sum_{n=1}^{\infty} c(n)q^n$, $q = \exp(2\pi\tau\sqrt{-1})$, $\tau \in \mathbb{H}$, in $S_{2k}(SL_2(\mathbb{Z}))$, in [24], we attached to f a Hecke eigen cusp form of weight $2k + 8$ with respect to $G(\mathbb{Z})$, that lives on $\mathfrak{T}$. Denote this form by F_f.

For a normalized Hecke eigenform $h \in S_{2k+8}(SL_2(\mathbb{Z}))$, consider the integral

$$\mathcal{F}_{f,h}(Z) = \int_{SL_2(\mathbb{Z})\backslash\mathbb{H}} F_f \begin{pmatrix} Z & 0 \\ 0 & \tau \end{pmatrix} \overline{h(\tau)} (\operatorname{Im}\tau)^{2k+6}\, d\tau. \tag{1.1}$$

We prove in Lemma 7.1 that the integral is well-defined and $\mathcal{F}_{f,h}(Z)$ is a cusp form of weight $2k + 8$ with respect to Γ_2 defined by (2.2).

For each prime p, let $\{\alpha_p, \alpha_p^{-1}\}$ and $\{\beta_p, \beta_p^{-1}\}$ be the Satake parameters of f and h at p, resp. Let π_f and π_h be the cuspidal representations attached to f and h resp., and let $L(s, \pi_f)$ and $L(s, \pi_h)$ be their automorphic L-functions.

For technical reasons, we assume the existence of Jacquet–Langlands correspondence that associates to automorphic representations of $PGSpin(2, 10)(\mathbb{A})$ those of its split inner form $PGSpin(6, 6)(\mathbb{A})$ (Conjecture 6.1). Since $PGSpin(6, 6) = PGSO(6, 6)$, we can relate automorphic representations of $PGSO(6, 6)(\mathbb{A})$ to those of $SO(6, 6)(\mathbb{A})$ with trivial central character, and then we can use Arthur's work [1] to transfer automorphic representations of $SO(6, 6)(\mathbb{A})$ to $GL_{12}(\mathbb{A})$. Therefore, under our assumption, we have the Langlands functorial transfer of everywhere unramified automorphic representations from $PGSpin(2, 10)(\mathbb{A})$ to $GL_{12}(\mathbb{A})$: Namely, given a cuspidal automorphic representation of $PGSpin(2, 10)(\mathbb{A})$ which is unramified at every prime p, there exists an automorphic representation of $GL_{12}(\mathbb{A})$ which is unramified at every prime p such that their Satake parameters correspond under the L-group homomorphism ${}^L GSpin(2, 10) = GSO(12, \mathbb{C}) \hookrightarrow GL_{12}(\mathbb{C})$. We prove

Theorem 1.1 *Assume that $\mathcal{F}_{f,h}$ is not identically zero. Assume* Conjecture 6.1. *Then*

(1) *The cusp form $\mathcal{F}_{f,h}$ is a Hecke eigenform, and hence gives rise to a cuspidal representation $\Pi_{f,h}$ of $G'(\mathbb{A})$ with trivial central character, which is unramified at every prime p.*
(2) *Let $\Pi_{f,h} = \Pi_\infty \otimes \otimes'_p \Pi_p$. The transfer of $\Pi_{f,h}$ to $GL_{12}(\mathbb{A})$ is $(\pi_f \boxtimes \pi_h) \boxplus 1_{GL_7} \boxplus 1$, or $(\pi_f \boxtimes \pi_h) \boxplus 1_{GL_5} \boxplus 1_{GL_3}$, where 1_{GL_n} is the trivial representation of $GL_n(\mathbb{A})$.*

We first show (Proposition 5.1) that the multiset of Satake parameters of Π_p is of the form

$$\{\varepsilon_p(\beta_p\alpha_p)^{\pm 1}, \quad \varepsilon_p(\beta_p\alpha_p^{-1})^{\pm 1}, \quad b_p^{\pm 1}, \quad (b_p p)^{\pm 1}, \quad (b_p p^2)^{\pm 1}, \quad (b_p p^3)^{\pm 1}\},$$

where $\varepsilon_p \in \{\pm 1\}$ and $b_p \in \mathbb{C}^\times$. Using the functorial transfer, in Sect. 7, we remove the sign ambiguity and show $b_p = 1$ for all p, or $b_p = p^{-1}$ for all p. It is expected that $b_p = 1$ for all p, and the transfer to $GL_{12}(\mathbb{A})$ is $(\pi_f \boxtimes \pi_h) \boxplus 1_{GL_7} \boxplus 1$. This would agree with Conjecture 6.2 and the classification in [8, Section 7.2.2] which would imply that $\Pi_{f,h}$ would come from the endoscopic transfer from $\pi_f \boxtimes \pi_h$ on $PGSO(2, 2)$ and the trivial representation on the anisotropic $SO(8)$. However we are not able to prove it.

Remark 1.2 If we take $h = E_{2k+8}$, the Eisenstein series of weight $2k + 8$, the integral (1.1) still makes sense (cf. Lemma 7.1) and defines a cusp form $\mathcal{F}_{f,E_{2k+8}}$ of weight $2k + 8$ with respect to Γ_2. If $\mathcal{F}_{f,E_{2k+8}}$ is not zero, then it gives rise to a cuspidal representation $\Pi_{f,E_{2k+8}}$ of $GSpin(2, 10)$.

Remark 1.3 Here Π_∞ is a holomorphic discrete series of lowest weight $2k+8$. Since f and h have different weights, they can never be equal. Therefore, $L(s, \pi_f \times \pi_h)$ is entire.

Remark 1.4 Note that ${}^L Spin(2, 10) = PGSO(12, \mathbb{C})$, and $PGSO(12, \mathbb{C})$ does not have a 12-dimensional representation. By Weyl's dimension formula, the minimum possible dimension for a nontrivial algebraic irreducible representations of $PGSO(12, \mathbb{C})$ is 66. The 66-dimensional algebraic irreducible representation of $PGSO(12, \mathbb{C})$ is given by

$$\mathrm{Ad} : PGSO(12, \mathbb{C}) \longrightarrow GL(\mathrm{Lie}(PGSO(12, \mathbb{C})) \cong GL_{66}(\mathbb{C}).$$

Therefore, given a cuspidal representation π of $Spin(2, 10)$, we cannot define the degree 12 standard L-function of π. However, ${}^L GSpin(2, 10) = GSO(12, \mathbb{C})$, and $GSO(12, \mathbb{C})$ has a 12-dimensional representation. Since $PGSpin(2, 10) = PGSO(2, 10)$, our form $\Pi_{f,h}$ can be considered as a cuspidal representation of $GSpin(2, 10)$ with the trivial central character.

This situation is similar to that of Siegel cusp forms. Given a Siegel cusp form F on a degree 2 Siegel upper half plane, we need to consider a cuspidal representation π_F of GSp_4, rather than Sp_4, in order to define the degree 4 spin L-function.

Remark 1.5 We give a conjectural Arthur parameter for $\Pi_{f,h}$: Let $\phi_f, \phi_h : \mathcal{L} \longrightarrow SL_2(\mathbb{C})$ be the hypothetical Langlands parameter attached to f, h, resp. We have the tensor product map $SL_2(\mathbb{C}) \times SL_2(\mathbb{C}) \longrightarrow SO_4(\mathbb{C})$. [[31], page 88. Use the identification $SL_2(\mathbb{C}) = Sp_1(\mathbb{C})$, and we have a representation of $SL_2(\mathbb{C}) \times SL_2(\mathbb{C})$ on $\mathbb{C}^2 \otimes \mathbb{C}^2 \simeq \mathbb{C}^4$. It defines a symmetric, non-degenerate bilinear form on $\mathbb{C}^4$.] Then we have $\phi_f \otimes \phi_h : \mathcal{L} \longrightarrow SO_4(\mathbb{C})$. The distinguished unipotent orbit $(7, 1)$, or $(5, 3)$ of $SO_8(\mathbb{C})$ gives rise to a map $SL_2(\mathbb{C}) \longrightarrow SO_8(\mathbb{C})$. (We expect that $(7, 1)$ is the correct one.) Hence it defines a map $\phi_u : \mathcal{L} \times SL_2(\mathbb{C}) \longrightarrow SO(8, \mathbb{C})$. Then consider

$$\phi = (\phi_h \otimes \phi_f) \oplus \phi_u : \mathcal{L} \times SL_2(\mathbb{C}) \longrightarrow SO_4(\mathbb{C}) \times SO_8(\mathbb{C}) \subset GSO_{12}(\mathbb{C}).$$

We expect that ϕ parametrizes $\Pi_{f,h}$.

This paper is organized as follows. In Sect. 2, we recall several facts about the Hermitian symmetric domain of $Spin(2, 10)$ or $PGSpin(2, 10) = PGSO(2, 10)$, and holomorphic modular forms on it. In Sect. 3, we recall our previous work [24]. In Sects. 5 and 7, following Ikeda [20], we study the integral expression (1.1) for $\mathcal{F}_{f,h}$, which gives rise to a cusp form on $\mathfrak{T}_2$. We follow essentially the same strategy as Ikeda, but have to rely on roots to describe a certain double coset space that features crucially in this strategy. Section 4 will be devoted to the calculation of these double cosets. In Sect. 6, we describe a conjectural Jacquet–Langlands correspondence that associates to automorphic representations of $PGSpin(2, 10)(\mathbb{A})$ to those of its split inner form $PGSpin(12, \mathbb{A})$. In Sect. 8, we compute $\mathcal{F}_{f,h}$ explicitly using the Fourier–Jacobi expansion of F_f, so as to furnish evidence that it is nonvanishing for all f, h.

2 Preliminaries

2.1 Cayley numbers and the exceptional domain

In this section we refer to Section 2 of [24] (also [4,22]). For any field K whose characteristic is different from 2 and 3, the Cayley numbers $\mathfrak{C}_K$ over K are an eight-dimensional vector space over K, equipped with a basis $\{e_0 = 1, e_1, e_2, e_3, e_4, e_5, e_6, e_7\}$, and a non-associative non-commutative multiplication satisfying the following rules:

(1) $xe_0 = e_0 x = x$ for all $x \in \mathfrak{C}_K$,
(2) $e_i^2 = -e_0$ for $i = 1, \ldots, 7$,
(3) $e_i(e_{i+1}e_{i+3}) = (e_i e_{i+1})e_{i+3} = -e_0$ for any i (mod 7).

For each $x = \sum_{i=0}^{7} x_i e_i \in \mathfrak{C}_K$, the map $x \mapsto \bar{x} = x_0 e_0 - \sum_{i=1}^{7} x_i e_i$ defines an anti-involution on $\mathfrak{C}_K$. The trace and the norm on $\mathfrak{C}_K$ are defined by

$$\mathrm{Tr}(x) := x + \bar{x} = 2x_0, \quad N(x) := x\bar{x} = \sum_{i=0}^{7} x_i^2.$$

We denote by $\mathfrak{o}$ a $\mathbb{Z}$-submodule of $\mathfrak{C}_K$ given by the basis

$$\alpha_0 = e_0, \quad \alpha_1 = e_1, \quad \alpha_2 = e_2, \quad \alpha_3 = -e_4, \quad \alpha_4 = \frac{1}{2}(e_1 + e_2 + e_3 - e_4),$$
$$\alpha_5 = \frac{1}{2}(-e_0 - e_1 - e_4 + e_5),$$
$$\alpha_6 = \frac{1}{2}(-e_0 + e_1 - e_2 + e_6), \quad \alpha_7 = \frac{1}{2}(-e_0 + e_2 + e_4 + e_7).$$

The elements of $\mathfrak{o}$ are called the integral Cayley numbers. It is known that $\mathfrak{o}$ is stable under the operations of the anti-involution, multiplication, and addition. Further if K has characteristic zero, we have $\mathrm{Tr}(x), N(x) \in \mathbb{Z}$ if $x \in \mathfrak{o}$. By using this integral structure, for any $\mathbb{Z}$-algebra R, one can consider $\mathfrak{C}_R = \mathfrak{o} \otimes_{\mathbb{Z}} R$ so that $\mathfrak{C}_R$ inherits anti-involution and multiplication.

Let $\mathfrak{J}_K$ be the exceptional Jordan algebra consisting of elements of the form:

$$X = (x_{ij})_{1 \le i,j \le 3} = \begin{pmatrix} a & x & y \\ \bar{x} & b & z \\ \bar{y} & \bar{z} & c \end{pmatrix}, \tag{2.1}$$

where $a, b, c \in Ke_0 = K$ and $x, y, z \in \mathfrak{C}_K$.

By using integral Cayley numbers, we define a lattice

$$\mathfrak{J}(\mathbb{Z}) := \{X = (x_{ij}) \in \mathfrak{J}_{\mathbb{Q}} \mid x_{ii} \in \mathbb{Z}, \quad \text{and } x_{ij} \in \mathfrak{o} \text{ for } i \neq j\},$$

and put $\mathfrak{J}(R) = \mathfrak{J}(\mathbb{Z}) \otimes_{\mathbb{Z}} R$ for any $\mathbb{Z}$-algebra R. Note that $\mathfrak{J}(R)$ is a non-associative algebra.

We define

$$R_3(K) = \{X \in \mathfrak{J}_K \mid \det(X) \neq 0\}$$

and define $R_3^+(K)$ to be set of squares of elements in $R_3(K)$. It is known that $R_3^+(\mathbb{R})$ is an open, convex cone in $\mathfrak{J}_{\mathbb{R}}$. We denote by $\overline{R_3^+(\mathbb{R})}$ the closure of $R_3^+(\mathbb{R})$ in $\mathfrak{J}_{\mathbb{R}} \simeq \mathbb{R}^{27}$ with respect to Euclidean topology. For any subring A of $\mathbb{R}$, set

$$\mathfrak{J}(A)_+ := \mathfrak{J}(A) \cap R_3^+(\mathbb{R}), \quad \mathfrak{J}(A)_{\geq 0} := \mathfrak{J}(A) \cap \overline{R_3^+(\mathbb{R})}.$$

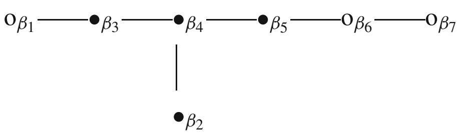
We define the exceptional domain as follows:

$$\mathfrak{T} := \{Z = X + Y\sqrt{-1} \in \mathfrak{J}_{\mathbb{C}} \mid X, Y \in \mathfrak{J}_{\mathbb{R}}, \quad Y \in R_3^+(\mathbb{R})\}$$

which is a complex analytic subspace of $\mathbb{C}^{27}$.

Let G be the exceptional Lie group of type $E_{7,3}$ over $\mathbb{Q}$ which acts on $\mathfrak{T}$. Then $G(\mathbb{R})$ is of real rank 3 (cf. [4]). In loc.cit. Baily constructed an integral model $\mathcal{G}_{\mathbb{Z}}$ of G over Spec $\mathbb{Z}$, and $G(\mathbb{Q}_p)$ is a split group of type E_7 for any prime p since $\mathfrak{o}$ splits for any prime p.

The Satake diagram of $E_{7,3}$ is



The $\mathbb{Q}$-root system is of type C_3, and the extended Dynkin diagram of C_3 is

$$\circ_{\lambda_0} \Longrightarrow \circ_{\lambda_1} \text{——} \circ_{\lambda_2} \Longleftarrow \circ_{\lambda_3},$$

where λ_1 corresponds to β_1, λ_2 to β_6, λ_3 to β_7, and $-\lambda_0$ is the maximal root in C_3. Here λ_1, λ_2 have multiplicity 8, and λ_3 has multiplicity 1.

Let $G_1 = SL_2$, $G_2 = Spin(2, 10)$.[1] Then (G_1, G_2) is a dual pair inside $G = E_{7,3}$ (cf. [10]). They are given as follows: If we remove the root λ_1 in the extended Dynkin diagram, the remaining diagram is an almost direct product G_1G_2. More precisely, let $\theta = h_{\lambda_0}(-1)$. Then θ is an involution whose centralizer $H = C_{E_7}(\theta)$ as an algebraic group is the almost direct product G_1G_2. Then $G_1 \cap G_2 = Z = \langle h_{\lambda_0}(-1)\rangle \simeq \{\pm 1\}$. Since G_1 and G_2 are simply connected algebraic groups, one has the following exact sequence

$$\begin{aligned} 1 &\longrightarrow \mu_2(k) \longrightarrow G_1(k) \times G_2(k) \longrightarrow H(k) \longrightarrow H^1(\mathrm{Gal}(k^{\mathrm{sep}}/k), \\ &\mu_2(\overline{k})) = k^\times/(k^\times)^2 \longrightarrow 1 \end{aligned}$$

for any field k. This means that $H(k)$ is strictly bigger than $G_1(k)G_2(k) \subset E_7(k)$. Furthermore the 2 to 1 isogeny $G_1 \times G_2 \longrightarrow H$ induces a natural inclusion $X^*(T_H) \hookrightarrow X^*(T_{G_1}) \times X^*(T_{G_2})$ of index 2 where for $A = G_1, G_2$ or H, T_A is an appropriate split maximal torus of A, and $X^*(T)$ stands for the character group of a torus T.

[1] Since we are not dealing with the exceptional group of type G_2, we hope that our notation will not cause any confusion.

We remark that $G_2(k)$ is a split group for any p-adic field k. The $\mathbb{Q}$-root system of G_2 is of type C_2. It is the group $\mathfrak{L}_2$ in [4] described from the bottom in page 527 to page 528, and it acts on the boundary component $\mathfrak{T}_2$ defined in Sect. 2.2. The algebraic group G_2 is obtained by an almost direct factor of the centralizer of the torus S^2 (in the notation of Baily) in $E_{7,3}$. This construction in G_2 also works over $\mathbb{Z}$ by using integral model of $E_{7,3}$ and we denote it by $\mathcal{G}_2$. The centralizer of $S^2/\mathbb{F}_p$ is always reductive since so is $E_{7,3}/\mathbb{F}_p$. It follows from Theorem 3.1 and Corollary 4.4 in [9] that the integral model $\mathcal{G}_2$ over $\mathbb{Z}$ of G_2 we chose as above is a smooth integral model of G_2.

Henceforth we fix this integral model of G_2 obtained as above and regard

$$\Gamma_2 := \mathcal{G}_2(\mathbb{Z}) \tag{2.2}$$

with an arithmetic subgroup of "a level one" in $G_2(\mathbb{Q})$. By construction one can check that the operations (2.3) in Sect. 2.3 are contained in Γ_2. It is interesting to specify Γ_2 explicitly toward studying the classical forms but we do not pursue it in this paper.

Recall that $G' = GSpin(2, 10) = \{h_0(t)\} \ltimes G_2$ where $\{h_0(t)\} \simeq GL_1$ is the torus for which it gives GE_7 in p. 104 of [15]. It inherits the integral structure coming from $\mathcal{G}_2$ and $GL_1/\mathbb{Z}$, and we denote by $G'(\mathbb{Z})$ an arithmetic subgroup corresponding to the integral model.

2.2 Hermitian symmetric domain for *Spin*(2, 10)

For any $\mathbb{Z}$-algebra R, define $\mathfrak{J}_2(R)$ as the set of all matrices of the form

$$X = \begin{pmatrix} a & x \\ \bar{x} & b \end{pmatrix}, \ a, b \in R, \ x \in \mathfrak{C}_R.$$

We define the inner product on $\mathfrak{J}_2(R) \times \mathfrak{J}_2(R)$ by $(X, Y) := \frac{1}{2}\mathrm{Tr}(XY + YX)$. For any such X, we define $\det(X) := ab - N(x)$. For X as above, $r \in R$, and $\xi = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}$, $\xi_i \in \mathfrak{C}_R$ $(i = 1, 2)$, we have $\begin{pmatrix} X & \xi \\ {}^t\bar{\xi} & r \end{pmatrix} \in \mathfrak{J}(R)$. For any subring A of $\mathbb{R}$, define

$$\mathfrak{J}_2(A)_+ = \left\{ \begin{pmatrix} a & x \\ \bar{x} & b \end{pmatrix} \in \mathfrak{J}_2(A) \,\middle|\, a, b \in A \cap \mathbb{R}_{>0}, \quad ab - N(x) > 0 \right\},$$

and

$$\mathfrak{J}_2(A)_{\geq 0} = \left\{ \begin{pmatrix} a & x \\ \bar{x} & b \end{pmatrix} \in \mathfrak{J}_2(A) \,\middle|\, a, b \in A \cap \mathbb{R}_{\geq 0}, \quad ab - N(x) \geq 0 \right\}.$$

We also define

$$\mathfrak{T}_2 := \{X + Y\sqrt{-1} \in \mathfrak{J}_2(\mathbb{C}) \mid X, Y \in \mathfrak{J}_2(\mathbb{R}), \quad Y \in \mathfrak{J}_2(\mathbb{R})_+\}.$$

It is well-known that $\mathfrak{T}_2$ is the Hermitian symmetric domain for $G_2(\mathbb{R})$ which is a tube domain of type (IV). Since $Spin(2, 10)(\mathbb{R})/\{\pm 1\} \simeq SO(2, 10)(\mathbb{R})$, where $\{\pm 1\}$ is a subgroup in the center of $Spin(2, 10)(\mathbb{R})$, $\mathfrak{T}_2$ is also the symmetric domain for $SO(2, 10)(\mathbb{R})$ (See Section 6 of Appendix in [34]). For us, it is more convenient to consider $\tilde{G} = PGSO(2, 10) = PGSpin(2, 10)$. In this case, $\mathfrak{T}_2$ is also the symmetric domain for $PGSO(2, 10)(\mathbb{R})^0$. Then modular forms on $\mathfrak{T}_2$ can be considered as automorphic forms on $GSpin(2, 10)(\mathbb{A})$ with trivial central character.

2.3 Modular forms on $\mathfrak{T}_2$

Recall the integral model $\mathcal{G}$ of $G_2 = Spin(2,10)$ over $\mathbb{Z}$ from Sect. 2.1. Then one can define the arithmetic group $\Gamma_2 = \mathcal{G}(\mathbb{Z})$ of "level one". Recall that $G' = GSpin(2,10)$. For any $g \in G'(\mathbb{R})$ and $Z \in \mathfrak{T}_2$, one can define a holomorphic automorphy factor $j(g,Z) \in \mathbb{C}$ which satisfies the cocycle condition. (See [5], page 456.) More explicitly, $j(p_B, Z) = j(t_U, Z) = 1$ and $j(\iota, Z) = \det(Z)$.

Let F be a holomorphic function on $\mathfrak{T}_2$ which for some integer $k > 0$ satisfies

$$F(\gamma Z) = F(Z) j(\gamma, Z)^k, \quad Z \in \mathfrak{T}_2,\ \gamma \in \Gamma_2.$$

Then F is called a modular form on $\mathfrak{T}_2$ of weight k with respect to Γ_2. For example, F satisfies

$$F(Z+B) = F(Z), \quad F({}^t\bar{U}ZU) = F(Z), \quad F(-Z^{-1}) = \det(Z)^k F(Z), \tag{2.3}$$

where $B \in \mathfrak{J}_2(\mathbb{Z})$, and where $U = \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}$ for some $u \in \mathfrak{o}$.

We denote by $\mathcal{M}_k(\Gamma_2)$ the space of such forms. By Koecher principle, the holomorphy at the cusps is automatic. For a holomorphic function $F : \mathfrak{T}_2 \longrightarrow \mathbb{C}$, consider, for $\tau \in \mathbb{H}$,

$$\Phi F(\tau) = \lim_{y \to \infty} F\begin{pmatrix} \tau & 0 \\ 0 & iy \end{pmatrix}.$$

If $\Phi F = 0$, F is called a cusp form. Let $\mathcal{S}_k(\Gamma_2)$ be the space of cusp forms of weight k with respect to Γ_2.

By [32], Theorem 7.12, the strong approximation theorem holds with respect to $S = \{\infty\}$, namely, $G'(\mathbb{A}) = Z_{G'}(\mathbb{R})^+ G'(\mathbb{Q}) G_2(\mathbb{R}) \mathcal{G}_2(\widehat{\mathbb{Z}})$, and $\Gamma_2 = \mathcal{G}_2(\mathbb{Z}) = G'(\mathbb{Q}) \cap G_2(\mathbb{R}) \mathcal{G}_2(\widehat{\mathbb{Z}})$. Notice that $G'(\widehat{\mathbb{Z}}) \subset Z_{G'}(\mathbb{R})^+ G'(\mathbb{Q}) \mathcal{G}_2(\widehat{\mathbb{Z}})$. Hence one can associate a Hecke eigen cusp form in $S_k(\Gamma_2)$ with an automorphic form on $G'(\mathbb{A})$ which is fixed by $G'(\widehat{\mathbb{Z}})$, and then we obtain a cuspidal automorphic representation of $G'(\mathbb{A})$ with trivial central character.

Remark 2.1 In [12], Eie and Krieg considered an arithmetic subgroup $\Gamma' \subset \Gamma_2$, generated by the following elements. For $Z \in \mathfrak{T}_2$, let $Z = \begin{pmatrix} z_1 & w \\ \bar{w} & z_2 \end{pmatrix}$, where $z_1, z_2 \in \mathbb{H}$, and $w = x + y\sqrt{-1}$ with $x, y \in \mathfrak{C}_{\mathbb{R}}$, and $\bar{w} = \bar{x} + \bar{y}\sqrt{-1}$. Let $det(Z) = z_1 z_2 - w\bar{w}$:

(1) $p_B : Z \mapsto Z + B$, $B \in \mathfrak{J}_2(\mathbb{Z})$;
(2) $t_U : Z \mapsto {}^t\bar{U}ZU$, $U = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ or $U = \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}$ for $u \in \mathfrak{o}$;
(3) $\iota : Z \mapsto -Z^{-1}$, where $Z^{-1} = \frac{1}{\det(Z)} \begin{pmatrix} z_2 & -w \\ -\bar{w} & z_1 \end{pmatrix}$.

If we consider Γ' as a subgroup of $G(\mathbb{Z})$, p_B is the element $p_{B'}$ in [24] with $B' = \begin{pmatrix} 0 & 0 \\ 0 & B \end{pmatrix}$ and $B \in \mathfrak{J}_2(\mathbb{Z})$; and $\iota = \iota_{e_2}\iota_{e_3}$ in [24]. Also for $U = \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}$, t_U equals $m_{ue_{23}} \in M(\mathbb{Z})$ in the notation of [24]. If $U = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, t_U equals $m_{e_{23}} m_{-e_{32}} m_{e_{23}}$.

It is likely that $\Gamma' = \Gamma_2$, but we have not shown it yet.

3 Ikeda type lift for $E_{7,3}$

In this section we recall the Ikeda type construction for $G = E_{7,3}$ in [24]. Let $P = MN$ be a Siegel parabolic subgroup of G so that the derived group $M^D = [M, M]$ of the Levi subgroup M is of type E_6. Let $\nu : M \longrightarrow GL_1$ be the similitude character (see Section 2 of [24]). It can be naturally extended to P. In [4], Baily used the set $\mathfrak{o}$ of the integral Cayley numbers to define an arithmetic subgroup of $G(\mathbb{R})$. Denote this subgroup by $\Gamma = G(\mathbb{Z})$. Let $f = \sum_{n\geq 1} c(n)q^n \in S_{2k}(SL_2(\mathbb{Z}))$ be a nonzero Hecke eigen cusp form, where $k \geq 6$. To f, we associated in [24] a nonzero Hecke eigen cusp form $F_f(Z)$ in $S_{2k+8}(\Gamma)$. Let us briefly recall the construction of F_f: For a positive integer $k \geq 6$, let E_{2k+8} be the Siegel Eisenstein series on $\mathfrak{T}$ of weight $2k+8$ with respect to Γ. Then it has the Fourier expansion of form

$$E_{2k+8}(Z) = \sum_{T\in\mathfrak{J}(\mathbb{Z})_+} a_{2k+8}(T)\exp(2\pi\sqrt{-1}(T, Z)), \quad Z \in \mathfrak{T},$$

$$a_{2k+8}(T) = C_{2k+8}\det(T)^{\frac{2k-1}{2}} \prod_{p|\det(T)} \widetilde{f}_T^p\left(p^{\frac{2k-1}{2}}\right),$$

where $C_{2k+8} = 2^{15}\prod_{n=0}^{2} \frac{2k+8-4n}{B_{2k+8-4n}}$, and $\widetilde{f}_T^p(X)$ is a Laurent polynomial over $\mathbb{Q}$ in X that depends only on T and p.

Let $S_{2k}(SL_2(\mathbb{Z}))$ be the space of elliptic cusp forms of weight $2k \geq 12$ with respect to $SL_2(\mathbb{Z})$. For each normalized Hecke eigenform $f = \sum_{n=1}^{\infty} c(n)q^n$, $q = \exp(2\pi\sqrt{-1}\tau)$, $\tau \in \mathbb{H}$ in $S_{2k}(SL_2(\mathbb{Z}))$ and each rational prime p, we define the Satake p-parameter $\{\alpha_p, \alpha_p^{-1}\}$ by $c(p) = p^{\frac{2k-1}{2}}(\alpha_p + \alpha_p^{-1})$. For such f, consider the following formal series on $\mathfrak{T}$:

$$F_f(Z) = \sum_{T\in\mathfrak{J}(\mathbb{Z})_+} A(T)\exp(2\pi\sqrt{-1}(T, Z)),\ Z \in \mathfrak{T}, \quad A(T) = \det(T)^{\frac{2k-1}{2}} \prod_{p|\det(T)} \widetilde{f}_T^p(\alpha_p).$$

Here note that $\widetilde{f}_T^p(X) = \widetilde{f}_T^p(X^{-1})$ by Corollary 6.2 of [24] and therefore $\widetilde{f}_T^p(\alpha_p)$ does not depend on the choice of α_p. Then we showed

Theorem 3.1 [24] *The function $F_f(Z)$ is a non-zero Hecke eigen cusp form on $\mathfrak{T}$ of weight $2k + 8$ with respect to Γ.*

We call F_f the Ikeda type lift of f. Then $F = F_f$ gives rise to a cuspidal automorphic representation $\pi_F = \pi_\infty \otimes \otimes'_p \pi_p$ of $\mathbf{G}(\mathbb{A})$. Then π_∞ is a holomorphic discrete series of the lowest weight $2k + 8$ associated to $-(2k + 8)\varpi_7$ in the notation of [7] (cf. [25], page 158). For each prime p, π_p is unramified. In fact, π_p turns out to be a degenerate principal series

$$\pi_p \simeq \mathrm{Ind}_{\mathbf{P}(\mathbb{Q}_p)}^{\mathbf{G}(\mathbb{Q}_p)} |\nu(g)|^{2s_p}, \tag{3.1}$$

where $p^{s_p} = \alpha_p$ (see Section 11 of [24]). Let $L(s, \pi_f) = \prod_p (1 - \alpha_p p^{-s})(1 - \alpha_p^{-1} p^{-s})$ be the automorphic L-function of the cuspidal representation π_f of $GL_2(\mathbb{A})$ attached to f. Then

Theorem 3.2 [24] *The degree* 56 *standard L-function* $L(s, \pi_F, St)$ *of* π_F *is given by*

$$L(s, \pi_F, St) = L(s, \mathrm{Sym}^3\pi_f)L(s, \pi_f)^2 \prod_{i=1}^{4} L(s \pm i, \pi_f)^2 \prod_{i=5}^{8} L(s \pm i, \pi_f),$$

where $L(s, \mathrm{Sym}^3\pi_f)$ *is the symmetric cube L-function.*

4 Double coset decomposition

In order to prove Theorem 1.1, following [20], we need to compute a suitable set of representatives of the double coset space over a p-adic field related to the unwinding method.

This section is mainly due to R. Lawther. We thank him for very detailed notes [28]. He gave an explicit double coset space related to what we need, but he worked over an algebraically closed field because he relied on the results in [27]. In what follows we modify his argument so that it would work over any p-adic field. Let p be any rational prime, and G a split simply-connected algebraic group of type E_7 over a p-adic field k. For simplicity, let $G = G(k)$, $G_1 = G_1(k)$, and $G_2 = G_2(k)$. Let T be a fixed maximal torus of G. Let B be the standard Borel subgroup containing T. Let $\{\beta_1, \ldots, \beta_7\}$ be the set of simple roots of T in B, numbered as in Bourbaki [7]. Write roots of E_7 as strings of coefficients of simple roots, so that for example, the highest root is $\tilde{\beta} = 2234321$. Let Φ (resp. Φ^+) be the set of all roots (resp. all positive roots). The extended Dynkin diagram is

$$\begin{array}{ccccccccccccc}
\circ_{-\tilde{\beta}} & — & \circ_{\beta_1} & — & \circ_{\beta_3} & — & \circ_{\beta_4} & — & \circ_{\beta_5} & — & \circ_{\beta_6} & — & \circ_{\beta_7} \\
 & & & & & & | & & & & & & \\
 & & & & & & \circ_{\beta_2} & & & & & &
\end{array}$$

Let $\theta = h_{\beta_7}(-1)$. Then θ is an involution whose centralizer $H = C_{E_7}(\theta)$ is of the form $A_1 D_6$. Explicitly, the roots whose root subgroups lie in H are those whose β_6-coefficient is even. In Lawther's notes [28], Lawther decided to take the simple roots of the D_6 consisting of positive roots of E_7, namely, $\gamma_1 = 0112221$, and $\gamma_2 = \beta_1, \gamma_3 = \beta_3, \gamma_4 = \beta_4, \gamma_5 = \beta_2, \gamma_6 = \beta_5$, and that of the A_1 is β_7. (We could take $-\tilde{\beta}$ instead of γ_1. It will not affect our result since we would be dealing with a conjugate subgroup.) Then $Z := G_1(k) \cap G_2(k) = \langle h_{\beta_7}(-1)) \rangle \simeq \{\pm 1\}$. Note that $h_{\gamma_1}(-1)h_{\gamma_3}(-1)h_{\gamma_6}(-1) = h_{\beta_7}(-1)$ and $H(k)$ contains $G_1 G_2 \simeq G_1(k) \times G_2(k)/Z$.

For the readers' convenience we summarize positive roots for $G_2 = Spin(12)$ in E_7. All positive roots (total 30 roots) are

γ_1	0112221
$\gamma_1+\gamma_2$	1112221
$\gamma_1+\gamma_2+\gamma_3$	1122221
$\gamma_1+\gamma_2+\gamma_3+\gamma_4$	1123221
$\gamma_1+\gamma_2+\gamma_3+\gamma_4+\gamma_5$	1223221
γ_2	1000000
$\gamma_2+\gamma_3$	1010000
$\gamma_2+\gamma_3+\gamma_4$	1011000
$\gamma_2+\gamma_3+\gamma_4+\gamma_5$	1111000
γ_3	0010000
$\gamma_3+\gamma_4$	0011000
$\gamma_3+\gamma_4+\gamma_5$	0111000
γ_4	0001000
$\gamma_4+\gamma_5$	0101000
γ_5	0100000
$\gamma_1+\gamma_2+\gamma_3+\gamma_4+\gamma_6$	1123321
$\gamma_2+\gamma_3+\gamma_4+\gamma_6$	1011100
$\gamma_3+\gamma_4+\gamma_6$	0011100
$\gamma_4+\gamma_6$	0001100
γ_6	0000100
$\gamma_1+2(\gamma_2+\gamma_3+\gamma_4)+\gamma_5+\gamma_6$	2234321
$\gamma_1+\gamma_2+2(\gamma_3+\gamma_4)+\gamma_5+\gamma_6$	1234321
$\gamma_1+\gamma_2+\gamma_3+2\gamma_4+\gamma_5+\gamma_6$	1224321
$\gamma_1+\gamma_2+\gamma_3+\gamma_4+\gamma_5+\gamma_6$	1223321
$\gamma_2+2(\gamma_3+\gamma_4)+\gamma_5+\gamma_6$	1122100
$\gamma_2+\gamma_3+2\gamma_4+\gamma_5+\gamma_6$	1112100
$\gamma_2+\gamma_3+\gamma_4+\gamma_5+\gamma_6$	1111100
$\gamma_3+2\gamma_4+\gamma_5+\gamma_6$	0112100
$\gamma_3+\gamma_4+\gamma_5+\gamma_6$	0111100
$\gamma_4+\gamma_5+\gamma_6$	0101100

We set $\gamma_7=\beta_7$.

4.1 Double coset space

We choose a Chevalley system for G, i.e., for each root α, choose a collection of isomorphisms $x_\alpha:\mathbb{G}_a\longmapsto$ root subgroup for α satisfying Chevalley commutator relations. Put

$$n_\alpha=x_\alpha(1)x_{-\alpha}(-1)x_\alpha(1),\quad y_\alpha=x_\alpha(1)n_\alpha x_\alpha\left(\tfrac{1}{2}\right),\ \text{and}$$
$$h_\alpha(c)=x_\alpha(c)x_{-\alpha}(-c^{-1})x_\alpha(c)n_\alpha^{-1},\quad c\in k^\times.$$

Put $h_i=h_{\beta_i}$ for simplicity.

Let P be the Siegel parabolic subgroup of G corresponding to $\{\beta_1,\ldots,\beta_6\}$ which is of type $E_6T_1U_{27}$ over an algebraic closed field where T_i denotes an i-dimensional torus, and U_j is a unipotent group of dimension j. For each element $g\in G(k)$, put $Q_g=g^{-1}P(k)g\cap H(k)$ and we denote by T_g the maximal split torus included in the standard Borel subgroup of H. Then we have the following lemma.

Lemma 4.1 *The double coset space $P(k)\backslash G(k)/H(k)$ is a finite set. For any $g\in G(k)$, there exists g' so that $P(k)gH(k)=P(k)g'H(k)$, and $g'Q_{g'}g'^{-1}$ coincides with yQ_yy^{-1}, and $T_{g'}=T_y$ for some $y\in\{1,n,y_{\beta_7}n,y_{\gamma_1}y_{\beta_7}n\}$, where $n=n_{\beta_6+\beta_7}$.*

The proof of this lemma involves several steps that feature lengthy computations. It is not clear to us if the natural map $P(k)\backslash G(k)/H(k) \longrightarrow P(\bar{k})\backslash G(\bar{k})/H(\bar{k})$ is injective. If it is, then the finiteness of the double coset space would follow from known results (cf. [18]). Let C be the complete system of the representatives of the Weyl group $W = N(k)/T(k)$ which are given in terms of words of reflections.

Lemma 4.2 *A complete system of representatives of the double coset space $B(k)\backslash G(k)/H(k)$ is a finite set and it consists of the elements of form*

$$y_{\alpha_r} \cdots y_{\alpha_1} n', \ \alpha_1, \dots, \alpha_r \in \Phi^+, \quad 0 \le r \le 7$$

where $\alpha_1, \dots, \alpha_r$ are mutually orthogonal and $n' \in N = N_{G(k)}(T(k))$ runs over the set

$$\Big(\{1\} \cup \{n' \in C \mid {}^{n'}\theta := n'\theta n'^{-1} \neq \theta\}\Big) \cap \{n' \in C \mid (\alpha_1, \dots, \alpha_r) : \text{admissible for } {}^{n'}\theta\}.$$

Here the admissibility condition of $(\alpha_1, \dots, \alpha_r)$ for ${}^{n'}\theta$ is given at the middle part of p. 118 in [27].

Proof The proof is almost same as in Section 3 of [27] but we have to take into consideration that the base field is not algebraically closed; the results in [27] are stated for which the base field is algebraically closed.

Put $S = \{g\theta(g)^{-1} \mid \theta(g) := \theta g\theta, \ g \in G(k)\}$. Define the action of $G(k)$ on S by

$$g * s = gs\theta(g)^{-1} \ s \in S, \quad g \in G(k).$$

Let us define a bijective map

$$B(k)\backslash G(k)/H(k) \longrightarrow \{O_{B(k)}(s) \mid s \in S\}, \quad BxH \mapsto O_{B(k)}(x\theta(x)^{-1})$$

where $O_{B(k)}(s)$ stands for the orbit of s for $B(k)$ with respect to the action $*$ as above. By Proposition 6.6 of [18], $O_{B(k)}(s) \cap N \neq \emptyset$, hence there exists $b \in B(k)$ such that $b * s \in N$. Since $\theta \in N$, if $s = x\theta(x)^{-1}$ we also have $(b * s)\theta = bx\theta(bx)^{-1} =: {}^{bx}\theta \in N$ which is an involution (hence $({}^{bx}\theta)^2 = 1$). Take another $b' \in B(k)$ so that $b' * s \in N$ if exists. Put $g_1 = bb'^{-1} \in B(k)$. Then ${}^{bx}\theta$ is conjugate by g_1 to ${}^{b'x}\theta$. By using Bruhat decomposition, ${}^{bx}\theta$ is conjugate by an element of $T(k)$ to ${}^{b'x}\theta$. Hence ${}^{bx}\theta$ is unique up to conjugation by $T(k)$. This way, for any given x, we choose a $b \in B(k)$ such that ${}^{bx}\theta \in N$, and in fact denote such a b by b_x to indicate its dependence on x. Summing up we have an injective map

$$B(k)\backslash G(k)/H(k) \hookrightarrow \{n \in N \mid n^2 = 1\}/ \overset{T}{\sim}, \quad BxH \mapsto {}^{b_x x}\theta$$

where $\overset{T}{\sim}$ stands for the equivalence relation of the conjugation by elements in T. We now describe the image of this map. Let $g = {}^{b_x x}\theta \in N$ be an involution for some $x \in G(k)$ and $b_x \in B(k)$. Then by the proof of Lemma 2 of [27] (noting that $n_{-\alpha} = n_\alpha^{-1} = n_\alpha t$ for some $t \in T(k)$), there exists $\theta' \in T(k)$ and $t \in T(\bar{k})$ ($t = t_2 t_1$ for t_1 at line 3, p. 119 of [27] and t_2 at line 11,p. loc.cit.) such that

$${}^{t}g = \theta' n_{\alpha_1} \cdots n_{\alpha_r}, \quad 0 \le r \le 7$$

such that $\alpha_1, \dots, \alpha_r \in \Phi^+$ are mutually orthogonal and $(\alpha_1, \dots, \alpha_r)$ is admissible for θ'.

We now descend t to an element in $T(k)$. Put $n = n_{\alpha_1} \cdots n_{\alpha_r}$. Let $Z_{T(\bar{k})}(g)$ be the centralizer of the involution g in $T(\bar{k})$ as an algebraic group over $\bar{k}$. Since g is of finite order, it is diagonalizable and it has eigenvalues ± 1. It follows from this that $Z_{T(\bar{k})}(g)$ is a split torus. We define a one-cocycle on $\mathrm{Gal}(\bar{k}/k)$ takes the values in $Z_{T(\bar{k})}(g)$ by

$$\sigma \mapsto t(t^{-1})^{\sigma}.$$

Since $H^1(\mathrm{Gal}(\bar{k}/k), Z_{T(\bar{k})}(g)) = 1$ by Hilbert Theorem 90, there exists $s \in Z_{T(\bar{k})}(g)$ such that $t(t^{-1})^{\sigma} = s(s^{-1})^{\sigma}$ for any $\sigma \in \mathrm{Gal}(\bar{k}/k)$. This means that $s^{-1}t \in T(k)$ and we have

$${}^{s^{-1}t}g = (s^{-1}t)g(t^{-1}s) = t(s^{-1}gs)t^{-1} = tgt^{-1} = {}^{t}g = \theta' n.$$

On the other hand θ' is conjugate to θ since $\theta' n = {}^{y_{\alpha_r}\cdots y_{\alpha_1}}\theta'$ by the admissibility condition. It follows that they have to be conjugate by some $n' \in N$, hence $\theta' = {}^{n'}\theta$. This gives us the claim. The finiteness is then clear from the above description. □

Remark 4.3 The proof of Lemma 4.2 shows that Corollary 3 of [27] holds for any field k.

We are ready to prove Lemma 4.1.

Proof The finiteness follows from the natural surjection $B(k)\backslash G(k)/H(k) \longrightarrow P(k)\backslash G(k)/H(k)$ and Lemma 4.2.

Henceforth we will make use of the mathematica code implemented by [30]. By direct computation n' runs over the set $R = \{1\} \cup \{n_\alpha \mid \alpha \in X\}$ where

$$\begin{aligned} X = \{&0000010, 0000110, 0000011, 0001110, 0000111, 0101110, 0011110, 0001111,\\ &1011110, 0111110, 0101111, 0011111, 1111110, 1011111, 0112110, 0111111,\\ &1112110, 1111111, 0112210, 0112111, 1122110, 1112210, 1112111, 0112211,\\ &1122210, 1122111, 1112211, 1123210, 1122211, 1223210, 1123211, 1223211\}. \end{aligned}$$

In fact, the condition that ${}^{n_\alpha}\theta = n_\alpha h_7(-1)n_\alpha^{-1} = h_7(-1)h_\alpha((-1)^{\langle\beta_7,\alpha\rangle}) \neq h_7(-1) = \theta$ is equivalent to the condition that $\langle\beta_7, \alpha\rangle$ is odd. Here we used $\alpha(h_\beta(t)) = t^{\langle\alpha,h_\beta\rangle}$ where $\langle\alpha, h_\beta\rangle = \frac{2\langle\alpha,\beta\rangle}{\langle\beta,\beta\rangle}$.

At first we discard all elements among the $y_{\alpha_1}\ldots y_{\alpha_r}n', n' \in R$ which never satisfy both of orthonormality and admissibility. Then next we seek g' in the statement.

Recall that $E_6(k) \subset P(k)$ (resp. H) consists of roots generated by $\beta_1, \ldots, \beta_6$ (resp. $\gamma_1, \gamma_2 = \beta_1,\ \gamma_3 = \beta_3, \gamma_4 = \beta_4, \gamma_5 = \beta_5, \gamma_6 = \beta_2, \beta_7$).

Assume $r = 0$. We further assume that $P(k)n_\alpha H(k) \neq P(k)H(k)$ for $\alpha \in R$. Recall from the statement of the lemma that $n = n_{\beta_6+\beta_7}$. Using the aforementioned mathematica code, we compute $n_\beta n n_\beta^{-1}$ as β ranges over Φ^+, and find that for one such β we have $n_\alpha = n_\beta n n_\beta^{-1}$ and $P(k)n_\alpha H(k) = P(k)nH(k)$.

Assume $r = 1$. For $\alpha = \sum_{i=1}^{7} a_i\beta_i \in \Phi^+$, clearly $y_\alpha \in P(k)$ if $a_7 = 0$. Therefore we may assume that $a_7 > 0$. For each $n' \in R$ we compute the set $R_1(n')$ consisting of α so that $a_7 > 0$ and $\alpha({}^{n'}\theta) = -1$ (see p. 118 of [27]). For example,

$$\begin{aligned} R_1(1) = \{&0000011, 0000111, 0001111, 0101111, 0011111, 1011111, 0111111, 1111111,\\ &0112111, 1112111, 0112211, 1122111, 1112211, 1122211, 1123211, 1223211\}. \end{aligned}$$

By direct calculation for any $n' \in R$ and $\alpha \in R_1(n')$ one checks that there exists $g' \in G(k)$ such that

$$P(k)g'H(k) = P(k)y_\alpha n'H(k) \quad \text{and } Q_{g'} = Q_{y_7 n}.$$

Let us give an example. For $n' = 1$ and $\alpha = 0000011$ we see that

$$g := y_\alpha = n_6 y_7 n_6^{-1} \equiv y_7 n_6 \equiv y_7 n_6 n_7 \equiv y_7 n n_6 \bmod (P(k), H(k)),$$

where $n_i = n_{\beta_i}$. (Here $a \equiv b \bmod (P(k), H(k))$ means $P(k)aH(k) = P(k)bH(k)$.) We used the relation $n = n_\alpha = n_6 n_7 n_6^{-1}$. Put $g' = y_7 n n_6$. Then one can check that $g' Q_{g'} g'^{-1} = y Q_y y^{-1}$ and $T_{g'} = T_y$ for $y = y_7 n$. This is done by using explicit realization in GL_{56} given by positive roots given before.

The remaining cases for $2 \leq r \leq 7$ are done similarly. These are routine and lengthy computations, and are hence omitted. □

4.2 An explicit structure of Q_g

By Lemma 4.1 we may focus on the following four elements to consider $Q_g = g^{-1}P(k)g \cap H(k)$, $g \in G(k)$. The following table is made by Lawther. Here we put $H(\bar{k}) = C_{G(\bar{k})}(\theta)$ and T_i, $i = 1, 2$ in the table means a split torus of dimension i.

Let $g = y_{\alpha_r} \cdots y_{\alpha_1} n$ be an element in Table 1. Let us consider

$$gQ_g g^{-1} = P(k) \cap gH(k)g^{-1} = C_{P(k)}({}^g\theta) = C_{P(k)}({}^n\theta n_{\alpha_1} \cdots n_{\alpha_r}).$$

Let T_B (resp. U_P) be the maximal split torus contained the standard Borel subgroup B (resp. the unipotent radical of $P(k)$) of $G(k)$ and U_g be the unipotent radical of $gQ_g g^{-1}$. Here B is defined by the positive roots fixed in [30]. Let $C_{T_B}({}^n\theta n_{\alpha_1} \cdots n_{\alpha_r})$ (resp. $C_{U_P}({}^n\theta n_{\alpha_1} \cdots n_{\alpha_r})$) be the centralizer of ${}^n\theta n_{\alpha_1} \cdots n_{\alpha_r}$ in T_B (resp. U_P). By using mathematica code we can compute these spaces easily and hence one can check each type which is given in Table 1.

For $i = 0, 1, 2, 3$, put $Q_i = g_i^{-1} P(k) g_i \cap H(k)$. Let $Q_i = M_i N_i$ be the Levi decomposition so that N_i is included in the unipotent subgroup defined by the positive roots of H given at the beginning of Sect. 4.1 and T_i the maximal split torus in M_i (no confusion should happen with a diagonal torus of dimension i just mentioned before). By the computation explained as above we would know T_i, N_i by conjugate, and the values of the modulus character δ_{Q_i} (resp. the modulus character $\delta_{P(k)}$) on T_i (resp. on $g_i T_i g_i^{-1} \subset P(k)$).

In what follows we denote by $|\cdot|$ the normalized valuation of k so that $|\varpi| = q^{-1}$ for a uniformizer ϖ of k where q stands for the cardinality of the residue field of k.

4.2.1 Case $Q_g = Q_0$

In this case the maximal split torus is given by means of simple roots and therefore we have

$$T_0 = \{h_{\gamma_1}(t_1)h_{\gamma_2}(t_2)h_{\gamma_3}(t_3)h_{\gamma_4}(t_4)h_{\gamma_5}(t_5)h_{\gamma_6}(t_6)h_{\gamma_7}(t_7) \mid t_1, \ldots, t_7 \in k^\times\}$$

Table 1 Double coset representatives

$g \in G(k)$	$g^{-1}P(\bar{k})g \cap H(\bar{k})$
$g_0 = 1$	$D_5 T_2 U_{11}$
$g_1 = n$	$A_5 A_1 T_1 U_{15}$
$g_2 = y_{\beta_7} n$	$A_4 T_2 U_{21}$
$g_3 = y_{\gamma_1} y_{\beta_7} n$	$B_3 A_1 T_1 U_{17}$

and $N_0 = \{\prod_{\alpha\in\Phi_0} x_\alpha(c_\alpha) \mid c_\alpha \in \mathbb{G}_a\}$, where

$$\Phi_0 = \{0000001, 0112221, 1112221, 1122221, 1123221, 1123321, 1223221, 1223321, 1224321, 1234321, 2234321\}.$$

Then for $t = h_{\gamma_1}(t_1)h_{\gamma_2}(t_2)h_{\gamma_3}(t_3)h_{\gamma_4}(t_4)h_{\gamma_5}(t_5)h_{\gamma_6}(t_6)h_{\gamma_7}(t_7)$ one has

$$\delta_{Q_0}(t) = |t_1|^{10}|t_2|^2 \text{ and } \delta_{P(k)}(t) = |t_1t_7|^{18}.$$

Since $\delta_{P(k)}(t) = |\nu(t)|^{18}$ (see Section 6 of [24]), one concludes $\nu(t) = t_1t_2u$ for some unit u in $\mathcal{O}_k$. In particular $\omega\circ\nu(t) = \omega(t_1t_2)$ for any unramified character ω of $k^\times$ where $\nu : P \longrightarrow GL_1$ is the similitude character.

It is easy to see that $G_1\cap T_0 = \{h_{\gamma_7}(t_7)\, t_7 \in k^\times\}$ and $G_2\cap T_0 = \{h_{\gamma_1}(t_1)\cdots h_{\gamma_6}(t_6) \mid t_1,\dots, t_6 \in k^\times\}$.

4.2.2 Case $Q_g = Q_1$

In this case we have

$$T_1 = \{h_{\gamma_1}(t_1)h_{\gamma_2}(t_2)h_{\gamma_3}(t_3)h_{\gamma_4}(t_4)h_{\gamma_5}(t_5)h_{\gamma_6}(t_6)h_{\gamma_7}(t_7) \mid t_1,\dots,t_7 \in k^\times\}$$

and $N_1 = \{\prod_{\alpha\in\Phi_1} x_\alpha(c_\alpha) \mid c_\alpha \in \mathbb{G}_a\}$, where

$$\Phi_1 = \{0000100, 0001100, 0101100, 0011100, 1011100, 0111100, 1111100, 0112100, 1112100, 1122100, 1123321, 1223321, 1224321, 1234321, 2234321\}.$$

Then for $t = h_{\gamma_1}(t_1)h_{\gamma_2}(t_2)h_{\gamma_3}(t_3)h_{\gamma_4}(t_4)h_{\gamma_5}(t_5)h_{\gamma_6}(t_6)h_{\gamma_7}(t_7)$ one has

$$\delta_{Q_1}(t) = |t_5|^{10} \text{ and } \delta_{P(k)}(g_1tg_1^{-1}) = |t_5|^{18}.$$

As seen before $\omega\circ\nu(g_1tg_1^{-1}) = \omega(t_5)$ for any unramified character ω of $k^\times$. We also have $G_1\cap T_1 = \{h_{\gamma_7}(t_7)\, t_7 \in k^\times\}$ and $G_2\cap T_1 = \{h_{\gamma_1}(t_1)\cdots h_{\gamma_6}(t_6) \mid t_1,\dots,t_6 \in k^\times\}$.

4.2.3 Case $Q_g = Q_2$

In this case we have

$$T_2 = \{h_{\gamma_1}(t_5t_7)h_{\gamma_2}(t_2)h_{\gamma_3}(t_3)h_{\gamma_4}(t_4)h_{\gamma_5}(t_5)h_{\gamma_6}(t_6)h_{\gamma_7}(t_7) \mid t_2,\dots,t_7 \in k^\times\} \tag{4.1}$$

and $N_2 = \{\prod_{\alpha\in\Phi_2} x_\alpha(c_\alpha) \mid c_\alpha \in \mathbb{G}_a\}$, where

$$\Phi_2 = \{-0000001, 0000100, 0001100, 0101100, 0011100, 1011100, 0111100, 1111100, 0112100, 1112100, 1122100, 0112221, 1112221, 1122221, 1123221, 1223221, 1123321, 1223321, 1224321, 1234321, 2234321\}.$$

Then for $t = h_{\gamma_1}(t_5t_7)h_{\gamma_2}(t_2)h_{\gamma_3}(t_3)h_{\gamma_4}(t_4)h_{\gamma_5}(t_5)h_{\gamma_6}(t_6)h_{\gamma_7}(t_7)$ one has

$$\delta_{Q_2}(t) = |t_5|^{14}|t_7|^4 \text{ and } \delta_{P(k)}(g_2tg_2^{-1}) = |t_5|^{18}.$$

As seen before, $\omega\circ\nu(g_2tg_2^{-1}) = \omega(t_5)$ for any unramified character ω of $k^\times$. We also have $G_1\cap T_2 = 1$ and $G_2\cap T_2 = 1$.

The Levi subgroup of Q_2 is of type A_4T_2 [here T_2 means a 2-dimensional torus but it is regarded with (4.1)] and the simple roots for Q_2 that contains T_2, with respect to T_2, are restriction to T_2 of $\beta_1,\ \gamma_3 = \beta_3,\ \gamma_4 = \beta_4,\ \gamma_6 = \beta_2$. One can check that the centralizer $Z_{T_2}(A_4) = \{t \in T_2 \mid tg = gt \text{ for any } g \in A_4\}$ is given by

$$T := \{h_T(a) := h_{\gamma_1}(a^2)h_{\gamma_2}(a^2)h_{\gamma_3}(a^2)h_{\gamma_4}(a^2)h_{\gamma_5}(a)h_{\gamma_6}(a)\\ h_{\gamma_7}(a) \mid a \in k^\times\} \subset G_1G_2 = H.$$

We see that GL_1 is diagonally embedded in H via $\Delta : GL_1 \longrightarrow T,\ a \mapsto h_T(a)$. Put $T' := \{h_{\gamma_1}(t_7)h_{\gamma_2}(t_2)h_{\gamma_3}(t_3)h_{\gamma_4}(t_4)h_{\gamma_6}(t_6)h_{\gamma_7}(t_7)\}$ and $T'' := \{h_{\gamma_1}(t)h_{\gamma_7}(t) \mid t \in k^\times\}$. Then $T'A_4 = T'' \ltimes A_4$ makes up GL_5 and it is given explicitly by using mathematica code. Then the natural projection $T'A_4 \longrightarrow T'' \simeq GL_1$ corresponds to the determinant.

4.2.4 Case $Q_g = Q_3$

The situation is a little bit more complicated than the other cases. Let us first observe that

$$P(k) \cap g_3H(k)g_3^{-1} = P(k) \cap g_3C_{G(k)}(\theta)g_3^{-1} = C_{P(k)}(g_3\theta g_3^{-1}) = C_{P(k)}({}^n\theta n_{\gamma_1}n_{\beta_7}).$$

By direct computation explained before we get $\dim N_3 = 17$. On the other hand let U_{17} denote the unipotent subgroup of $P \cap gHg^{-1}$ defined as the product of the following 17 copies of $\mathbb{G}_a$. For 16 of the 17 root groups in U_{17}, there is then a 1-dimensional unipotent group diagonally embedded in the product of the two root groups, of the form

$$\{x_\alpha(t)g'x_\alpha(t)g'^{-1} : t \in k\} = \{x_\alpha(t)x_{g'(\alpha)}(\pm t) : t \in k\},$$

where the sign in the second term is determined by the structure constants. The 17th root subgroup is simply the root subgroup corresponding to the highest root 2234321. The 16 pairs of positive roots α, $g'(\alpha)$ interchanged by g' are as follows:

α	$g'(\alpha)$	α	$g'(\alpha)$	α	$g'(\alpha)$	α	$g'(\alpha)$
1000000	1112221	1011110	1011111	1010000	1122221	1111110	1111111
1011000	1123221	1112110	1112111	1011100	1123321	1122110	1122111
1111000	1223221	1112210	1112211	1111100	1223321	1122210	1122211
1112100	1224321	1123210	1123211	1122100	1234321	1223210	1223211

By matching dimensions we have $g_3^{-1}U_{17}g_3 = N_3$ since one can check $g_3^{-1}U_{17}g_3 \subset N_3$ by mathematica code. On the other hand we have

$$T_3 = \{h_{\gamma_1}(t_5t_7)h_{\gamma_2}(t_5^2)h_{\gamma_3}(t_3)h_{\gamma_4}(t_4)h_{\gamma_5}(t_5)h_{\gamma_6}(t_6)h_{\gamma_7}(t_7) \mid t_3, \ldots, t_7 \in k^\times\}.$$

Then for $t = h_{\gamma_1}(t_5t_7)h_{\gamma_2}(t_5^2)h_{\gamma_3}(t_3)h_{\gamma_4}(t_4)h_{\gamma_5}(t_5)h_{\gamma_6}(t_6)h_{\gamma_7}(t_7)$ one has

$$\delta_{Q_3}(t) = |t_5|^{18} \text{ and } \delta_{P(k)}(g_3tg_3^{-1}) = |t_5|^{18}.$$

As seen before, $\omega \circ \nu(g_3tg_3^{-1}) = \omega(t_5)$ for any unramified character ω of $k^\times$. We also have $G_1 \cap T_3 = \{h_{\gamma_7}(t_7) \mid t_7 \in k^\times\}$. We also have $G_1 \cap T_2 = 1$ and $G_2 \cap T_2 = 1$. Finally we remark that $G_1 = SL_2$ is a common subgroup of G_1 and G_2, hence there exists a 2 to 1 homomorphism

$$\Delta : SL_2 \longrightarrow G_1 \times G_2 \longrightarrow H \tag{4.2}$$

onto the image given by on the diagonal torus, Δ sends $\mathrm{diag}(a, a^{-1})$ to $(h_{\gamma_7}(a), h_{\gamma_1}(a)h_{\gamma_7}(a))$. Let $\iota : SL_2 \longrightarrow SL_2$ be the isomorphism defined by $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \mapsto \begin{pmatrix} \alpha & -\beta \\ -\gamma & \delta \end{pmatrix}$. The image of Δ is naturally isomorphic to

$$\{(\gamma, \iota(\gamma)) \mid \gamma \in SL_2\}/\{\pm(I_2, I_2)\}. \tag{4.3}$$

The situation is similar to the case of the symplectic groups in [20].

5 Computation of Satake parameters

In this section, we prove Proposition 5.1 below, which is key to the proof of Theorem 1.1. It is an analogue of Proposition 3.1 of [20]. Recall that $G_1 = SL_2(\mathbb{Q}_p)$ and $G_2 = Spin(12)(\mathbb{Q}_p)$. Let π_2' be a spherical representation of $G' = GSpin(12)(\mathbb{Q}_p)$ with trivial central character. Since the group G_2 appears as a subgroup of E_7, we need to consider the restriction $\pi_2 = \pi_2'|_{Spin(12)}$. Since ${}^L GSpin(12) = GSO(12, \mathbb{C})$, the Satake parameters of π_2' is of the form

$$(b_1, b_2, \ldots, b_6, b_6^{-1}b_0, \ldots, b_2^{-1}b_0, b_1^{-1}b_0) \in GSO(12, \mathbb{C})$$

for some $b_1, \ldots, b_6 \in \mathbb{C}^\times$, and where $b_0 = \omega_{\pi_2'}(p)$. Since the central character is trivial, $b_0 = 1$.

Let π_i, $i = 1, 2$, be a spherical representation of G_i. Then π_i, $i = 1, 2$, is a subquotient of an unramified principal series representation of G_i, i.e., $\mathrm{Ind}_{B_i}^{G_i} \chi_i$, where B_1, B_2 are the standard Borel subgroups of G_1, G_2, resp. and $\chi_i : B_i \longrightarrow \mathbb{C}^\times$ is an unramified quasi-character. Here “Ind” stands for normalized induction and we will denote by “c-Ind” compact normalized induction. The modulus character of each B_i is given by

$$\delta_{B_1}(h_{\gamma_7}(t_7)) = |t_7|^2, \quad \delta_{B_2}(h_{\gamma_1}(t_1) \cdots h_{\gamma_6}(t_6)) = \prod_{i=1}^{6} |t_i|^2.$$

We may write the Satake parameters of π_1 as $\{\beta^{\pm 1}\}$, where

$$\chi_1(h_{\gamma_7}(p^{-1})) = \beta^2. \tag{5.1}$$

Also we have

$$\chi_2(h_{\gamma_i}(p^{-1})) = \frac{b_i}{b_{i+1}}, \quad 1 \le i \le 4, \quad \chi_2(h_{\gamma_6}(p^{-1})) = \frac{b_5}{b_6}, \quad \chi_2(h_{\gamma_5}(p^{-1})) = b_5 b_6. \tag{5.2}$$

Recall that $H = C_G(\theta) \simeq (G_1 \times G_2)/Z$ where $Z \simeq \{\pm 1\}$ is diagonally embedded in both centers. Let $\phi : G_1 \times G_2 \longrightarrow H$ be the isogeny. As seen before $\phi(G_1(\mathbb{Q}_p) \times G_2(\mathbb{Q}_p))$ is a finite index subgroup of $H(\mathbb{Q}_p)$. Let B_H be a Borel subgroup of H. Let χ be a character of B_H and let $\pi(\chi)$ be the spherical subquotient of $\mathrm{Ind}_{B_H}^H \chi$. Let $\widetilde{\chi} = \chi \circ \phi$, so that $\widetilde{\chi}$ is a character of $B_1 \times B_2$ and let $\pi(\widetilde{\chi})$ be the spherical subquotient of $\mathrm{Ind}_{B_1 \times B_2}^{G_1 \times G_2} \widetilde{\chi}$. Thus we have a surjective map between unramified L-packets:

$$\Pi(H(\mathbb{Q}_p)) \longrightarrow \Pi(G_1(\mathbb{Q}_p) \times G_2(\mathbb{Q}_p)), \quad \pi(\chi) \mapsto \pi(\widetilde{\chi}).$$

Given χ_1, χ_2, unramified characters of B_1, B_2, resp., there exist finitely many characters χ of B_H such that $\chi_1 \otimes \chi_2^{-1} = \widetilde{\chi}$. Let $\pi_H = \pi(\chi)$ for any such χ. Then π_H is a subquotient of $\mathrm{Ind}_{\phi(G_1(\mathbb{Q}_p) \times G_2(\mathbb{Q}_p))}^{H(\mathbb{Q}_p)} \pi_1 \otimes \widetilde{\pi}_2$, where π_i is the spherical subquotient of $\mathrm{Ind}_{B_i}^{G_i} \chi_i$, $i = 1, 2$, and $\widetilde{\pi}_2$ stands for the contragredient of π_2; if $\pi_1 \otimes \widetilde{\pi}_2$ is unitary, then π_H is unitary (Lemma

2.3 of [29]). We call π_H a lift of $\pi_1 \otimes \widetilde{\pi_2}$ by abuse of notation. Let $\omega : \mathbb{Q}_p^\times \to \mathbb{C}^\times$ be an unramified unitary character and let $\alpha = \omega(p^{-1})$.

Proposition 5.1 *Assume that $\pi_1 \otimes \widetilde{\pi_2}$ is unitary. If $\mathrm{Hom}_H(\mathrm{Ind}_P^G(\omega^{-2} \circ \nu)|_H, \pi_H) \neq 0$ for some lift π_H of $\pi_1 \otimes \widetilde{\pi_2}$, then as a multiset, $\{b_1^{\pm 1}, \ldots, b_6^{\pm 1}\}$ is equal to the following:*

$$\{\varepsilon(\beta\alpha)^{\pm 1},\ \varepsilon(\beta\alpha^{-1})^{\pm 1}, b^{\pm 1},\ (bp)^{\pm 1},\ (bp^2)^{\pm 1},\ (bp^3)^{\pm 1}\},$$

where $\varepsilon \in \{\pm 1\}$, and $b \in \mathbb{C}^\times$.

Proof As in Sect. 4.2, we take coset representatives $\{g_i'\}_{i=1}^r$ of $P(k)\backslash G(k)/H(k)$ so that $Q_i = g_i'^{-1} P g_i' \cap H$. For $g' \in Q_i$, let $\omega_i(g) = \delta_{P(k)}^{\frac{1}{2}}(g_i' g g_i'^{-1})\omega^{-2} \circ \nu(g_i' g g_i'^{-1})$. Then in the Grothendieck group of the category of admissible representations of H we have

$$\mathrm{Ind}_P^G(\omega^{-1} \circ \nu)|_H = \sum_{i=0}^r m_i \left(\text{c-Ind}_{Q_i}^H \omega_i \delta_{Q_i}^{-\frac{1}{2}}\right), \quad m_i \in \mathbb{Z}_{\geq 0}.$$

For $i = 0, 1, 2, 3$ put $g_i' = g_i$ where g_i is the element in Table 1. Then by assumption we may assume that there exists i, $i = 0, 1, 2, 3$,

$$\begin{aligned} 0 &\neq \mathrm{Hom}_H\left(\text{c-Ind}_{Q_i}^H \omega_i \delta_{Q_i}^{-\frac{1}{2}}, \pi_H\right) \\ &= \mathrm{Hom}_H\left(\widetilde{\pi_H}, \mathrm{Ind}_{Q_i}^H \omega_i^{-1} \delta_{Q_i}^{\frac{1}{2}}\right) \\ &= \mathrm{Hom}_{Q_i}(\widetilde{\pi_H}|_{Q_i}, \omega_i^{-1}) \text{ (by Frobenius reciprocity)} \end{aligned}$$

In the case of Q_0, we observe the action of $h_{\gamma_7}(p^{-1}) \in Q_0$ on both spaces. Then one has $\beta^2 = p^{-9}$ which contradicts to the unitarity of π_1. Similarly we observe the action of $h_{\gamma_7}(p^{-1})$ for Q_1. Then it gives a contradiction that $p\beta^2 = 1$.

In the case of Q_2, applying (5.1) and (5.2) to the following elements

$$h_{\gamma_2}(p^{-1}),\ h_{\gamma_3}(p^{-1}),\ h_{\gamma_4}(p^{-1}),\ h_{\gamma_6}(p^{-1}),\ h_{\gamma_1}(p^{-1})h_{\gamma_5}(p^{-1}),\ h_{\gamma_1}(p^{-1})h_{\gamma_7}(p^{-1}) \in T_2$$

respectively, we have

$$\begin{gathered} p\frac{b_2}{b_3} = 1, \quad p\frac{b_3}{b_4} = 1, \quad p\frac{b_4}{b_5} = 1, \quad p\frac{b_5}{b_6} = 1, \\ \left(p\frac{b_1}{b_2}\right)(pb_5b_6) = p^9\alpha_p^2, \quad (p^{-1}\beta^{-2})\left(p\frac{b_1}{b_2}\right) = 1. \end{gathered} \tag{5.3}$$

From this, we obtain the Satake parameters

$$\{\varepsilon(\beta\alpha)^{\pm 1},\ \ \varepsilon(\beta\alpha^{-1})^{\pm 1},\ \ \varepsilon(\beta\alpha^{-1}p)^{\pm 1},\ \ \varepsilon(\beta\alpha^{-1}p^2)^{\pm 1},\ \ \varepsilon(\beta\alpha^{-1}p^3)^{\pm 1},\ \ \varepsilon(\beta\alpha^{-1}p^4)^{\pm 1}\} \tag{5.4}$$

for some $\varepsilon \in \{\pm 1\}$.

Finally we consider the case of Q_3. For $t = h_{\gamma_1}(t_5t_7)h_{\gamma_2}(t_5^2)h_{\gamma_3}(t_3)h_{\gamma_4}(t_4)h_{\gamma_5}(t_5)h_{\gamma_6}(t_6)$ $h_{\gamma_7}(t_7)$, we see $\omega_3^{-1}(t) = \omega^2(t_5)|t_5|^9\delta_{B_2}^{\frac{1}{2}}(t) = \prod_{i=1}^6 |t_i|$. In this case, applying (5.1) and (5.2) to the following elements

$$\begin{gathered} h_{\gamma_3}(p^{-1}),\ h_{\gamma_4}(p^{-1}),\ h_{\gamma_6}(p^{-1}),\ h_{\gamma_1}(p^{-1})h_{\gamma_2}(p^{-2}) \\ h_{\gamma_5}(p^{-1})h_{\gamma_6}(p),\ h_{\gamma_1}(p^{-1})h_{\gamma_7}(p^{-1}) \in T_3 \end{gathered}$$

respectively, we have

$$p\frac{b_3}{b_4}=1,\quad p\frac{b_4}{b_5}=1,\quad p\frac{b_5}{b_6}=1,\quad p^3\frac{b_1b_2b_6^2}{b_3^2}=p^9\alpha^2,\quad (p^{-1}\beta^{-2})\left(p\frac{b_1}{b_2}\right)=1. \tag{5.5}$$

From the first four equalities, we have $b_1b_2=\alpha^2$. From the last equation, $\frac{b_1}{b_2}=\beta^2$. Hence $b_1^2=(\alpha\beta)^2$. Hence $b_1=\varepsilon\alpha\beta$ and $b_2=\varepsilon\frac{\alpha}{\beta}$, where $\varepsilon=\pm1$.

It follows from (5.5) that

$$b_4=pb_3,\quad b_5=p^2b_3,\quad b_6=p^3b_3, \tag{5.6}$$

where $b_3\in\mathbb{C}^\times$. Hence the Satake parameters of π_2' are

$$\{\varepsilon(\alpha\beta)^{\pm1},\quad \varepsilon(\alpha\beta^{-1})^{\pm1},\quad (bp^3)^{\pm1},\quad (bp^2)^{\pm1},\quad (bp)^{\pm1},\quad b^{\pm1}\},$$

where $\varepsilon\in\{\pm1\}$ and $b\in\mathbb{C}^\times$. Now (5.4) is a special case of when $b=\varepsilon\beta\alpha^{-1}p$. $\square$

6 Jacquet–Langlands correspondence

In this section we give a conjectural description of Jacquet–Langlands correspondence which is expected to hold in conjunction with Arthur's classification for automorphic representations. For any connected reductive group G over $\mathbb{Q}$, we denote by $\prod(G)$ (resp. $\prod^\circ(G)$) the set of isomorphism classes of automorphic (resp. cuspidal) representations of $G(\mathbb{A})$. By [1] we have a map $\iota:\prod(PGSO(6,6))^\circ\longrightarrow\prod(GL_{12})$ so that unramified local components go to unramified components.

We expect the Jacquet–Langlands correspondence which is a map from $\prod(PGSO(2,10))$ to $\prod(PGSO(6,6))$ so that unramified local components go to unramified components. Let us formulate it as the following conjecture:

Conjecture 6.1 *There exists a map $JL:\prod^\circ(PGSO(2,10))\longrightarrow\prod^\circ(PGSO(6,6))$ so that for any $\pi=\otimes'_p\pi_p\in\prod(PGSO(2,10))^\circ$, $JL(\pi)=\otimes'_p\Pi_p$ satisfies that Π_p is unramified if π_p is unramified.*

Then we have the map $\iota\circ JL:\prod^\circ(PGSO(2,10))\longrightarrow\prod(GL_{12})$.

Furthermore, according to the calculation of Satake parameters, it seems natural to expect that the cuspidal representation $\Pi_{f,h}$ on $PGSpin(2,10)$ we constructed in the next section, comes from $((\pi_f\boxtimes\pi_h),1_{PGSO(8)})$ on $PGSO(2,2)\times PGSO(8)$ by an endoscopic transfer, where $1_{PGSO(8)}$ is the trivial representation of $PGSO(8)$. This agrees with [8, Section 7.2.2]. (Note from Remark 1.5 the tensor product map from $PGL_2\times PGL_2$ to $PGSO(2,2)$, and $PGSO(2,2)$ is isogeneous to $PGL_2\times PGL_2$.)

We formulate a conjecture on the endoscopic transfer map:

Conjecture 6.2 *There exists a transfer $E:\prod^\circ(PGSO(2,2)\times PGSO(8))\longrightarrow\prod(PGSO(2,10))$ such that if π_1,π_2 are cuspidal representations of PGL_2, and τ is a cuspidal representation of $PGSO(8)$, then $\iota\circ JL\circ E((\pi_1,\pi_2),\tau)=(\pi_1\boxtimes\pi_2)\boxplus\Pi$, where Π is the image of τ under the map $\prod^\circ(PGSO(8))\longrightarrow\prod(GL_8)$, which is the composition of $\prod^\circ(PGSO(8))\longrightarrow\prod^\circ(PGSO(4,4))$ and $\prod^\circ(PGSO(4,4))\longrightarrow\prod(GL_8)$.*

7 Proof of Theorem 1.1

For $f \in S_{2k}(SL_2(\mathbb{Z}))$ and $h \in S_{2k+8}(SL_2(\mathbb{Z}))$, let $\mathcal{F}_{f,h}$ be defined by the integral (1.1).

Lemma 7.1 *The integral is well-defined and $\mathcal{F}_{f,h}(Z)$ is a cusp form of weight $2k+8$ with respect to Γ_2 defined by* (2.2).

Proof By Theorem 3.1, F_f is a cusp form of weight $2k+8$ with respect to Γ. Hence for a fixed $Z \in \mathfrak{T}_2$, $F_f \begin{pmatrix} Z & 0 \\ 0 & \tau \end{pmatrix}$ is a cusp form of weight $2k+8$ with respect to $SL_2(\mathbb{Z})$. Hence the integral is well-defined. Also if we replace h by the Eisenstein series E_{2k+8}, the integral is still well-defined. The modularity and cuspidality are clear. □

Let $\mathcal{H}(G_i(\mathbb{A}_f))$ $(i = 1, 2)$ be the Hecke algebra for the finite adele group $G_i(\mathbb{A}_f)$. Then $\mathcal{H}(G_1(\mathbb{A}_f)) \cdot h$ and $\mathcal{H}(G_2(\mathbb{A}_f)) \cdot \mathcal{F}_{f,h}$ are the finite part of the cuspidal automorphic representations of $G_1(\mathbb{A})$ and $G_2(\mathbb{A})$ generated by h and $\mathcal{F}_{f,h}$, resp. Here $\mathcal{H}(G_1(\mathbb{A}_f)) \cdot h$ is an irreducible representation of $G_1(\mathbb{A}_f)$. Let π_1 be the p-component of $\mathcal{H}(G_1(\mathbb{A}_f)) \cdot h$. Then π_1 is an unramified principal series representation with the Satake parameters $\{\beta_p^{\pm 1}\}$. On the other hand, since $\mathcal{F}_{f,h}(Z)$ is a cusp form, the representation $\mathcal{H}(G'(\mathbb{A}_f)) \cdot \mathcal{F}_{f,h}$ of $G'(\mathbb{A}_f)$ is unitary and of finite length (cf. [6, Proposition 4.5]), where we recall that $G' = GSpin(2, 10)$. We consider the restriction to $G_2(\mathbb{A}_f)$, and let π_2 be the p-component of some irreducible direct summand of that restriction. Then π_2 is also an unramified principal series representation.

Note that $\det(\mathrm{Im} Z)^{-10} dZ$ is the invariant measure on $G'(\mathbb{Z})\backslash\mathfrak{T}_2$ (cf. [37, p. 250]). Then if $\mathcal{F}_{f,h} \neq 0$,

$$\int_{G'(\mathbb{Z})\backslash\mathfrak{T}_2} \int_{SL_2(\mathbb{Z})\backslash\mathbb{H}} \overline{F\begin{pmatrix} Z & 0 \\ 0 & \tau \end{pmatrix}} h(\tau)\mathcal{F}_{f,h}(Z)(\mathrm{Im}\,\tau)^{2k+6} \det(\mathrm{Im}\,Z)^{2k-2}\, dZ d\tau$$
$$= \langle \mathcal{F}_{f,h}, \mathcal{F}_{f,h} \rangle \neq 0.$$

It follows from this that for each prime p,

$$0 \neq \mathrm{Hom}_{H(\mathbb{Q}_p)}\left(\mathrm{Ind}_{P(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)}(\omega_p^{-2} \circ \det)|_{\phi(G_1(\mathbb{Q}_p)\times G_2(\mathbb{Q}_p))}, \pi_1 \otimes \tilde{\pi}_2\right)$$
$$= \mathrm{Hom}_{H(\mathbb{Q}_p)}\left(\mathrm{Ind}_{P(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)}(\omega_p^{-2} \circ \det)|_{H(\mathbb{Q}_p)}, \mathrm{Ind}_{\phi(G_1(\mathbb{Q}_p)\times G_2(\mathbb{Q}_p))}^{H(\mathbb{Q}_p)} \pi_1 \otimes \tilde{\pi}_2\right)$$

and this implies

$$\mathrm{Hom}_{H(\mathbb{Q}_p)}\left(\mathrm{Ind}_{P(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)}(\omega_p^{-2} \circ \det)|_{H(\mathbb{Q}_p)}, \pi_H\right) \neq 0$$

for some lift π_H to $H(\mathbb{Q}_p)$ of $\pi_1 \otimes \tilde{\pi}_2$ defined as in Proposition 5.1, where $\omega_p : \mathbb{Q}_p^\times \longrightarrow \mathbb{C}^\times$ is the unramified character determined by $\omega_p(p^{-1}) = \alpha_p$. By Proposition 5.1, any irreducible component of $\mathcal{H}(G'(\mathbb{A}_f)) \cdot \mathcal{F}_{f,h}$ has the Satake p-parameter

$$\{\varepsilon_p(\beta_p\alpha_p)^{\pm 1},\ \varepsilon_p(\beta_p\alpha_p^{-1})^{\pm 1},\ (b_p p^3)^{\pm 1},\ (b_p p^2)^{\pm 1},\ (b_p p)^{\pm 1},\ b_p^{\pm 1}\},$$

where $\varepsilon_p = \pm 1$ and $b_p \in \mathbb{C}^\times$.

Now we assume Conjecture 6.1. Let $\Pi_{f,h}$ be an irreducible component of the cuspidal representation of $G'(\mathbb{A})$ generated by $\mathcal{F}_{f,h}$. Then it is unramified at every prime p. Let Π be the transfer of $\Pi_{f,h}$ to $GL_{12}(\mathbb{A})$. Then Π is unramified at all p by the property of Langlands functoriality. By the classification of automorphic representations of GL_N [21], Π is the Langlands quotient of

$$\sigma_1|\det|^{r_1} \boxplus \cdots \boxplus \sigma_k|\det|^{r_k} \boxplus \sigma_{k+1} \boxplus \cdots \boxplus \sigma_{k+l} \boxplus \tilde{\sigma}_k|\det|^{-r_k} \boxplus \cdots \boxplus \tilde{\sigma}_1|\det|^{-r_1},$$

where $r_1 \geq r_2 \geq \cdots \geq r_k > 0$, and $\sigma_1, \dots, \sigma_{k+l}$ are unitary (irreducible) cuspidal representations of $GL_{n_i}(\mathbb{A})$. Note also that if $\{c_{1p}, \dots, c_{mp}\}$ is the multiset of Satake p-parameters of a cuspidal representation π of $GL_m(\mathbb{A})$, $p^{-\frac{1}{2}} < |c_{ip}| < p^{\frac{1}{2}}$ for each i (cf. [21], page 554).

Now we recall the classification of spherical unitary representations of $GL_N(\mathbb{Q}_p)$ [36]: For an unramified unitary character χ, let $\chi(\det_n)$ be the representation $g \longmapsto \chi(\det_n(g))$ of $GL_n(\mathbb{Q}_p)$. It is a quotient of $Ind_B^{GL_n} \chi|\cdot|^{\frac{n-1}{2}} \otimes \chi|\cdot|^{\frac{n-1}{2}-1} \otimes \cdots \otimes \chi|\cdot|^{-\frac{n-1}{2}}$. Let $\pi(\chi(\det_n), \alpha)$ be the representation of $GL_{2n}(\mathbb{Q}_p)$ induced from $\chi(\det_n)|\det|^{\alpha} \otimes \chi(\det_n)|\det|^{-\alpha}$, where $0 < \alpha < \frac{1}{2}$. Then any spherical unitary representation of $GL_N(\mathbb{Q}_p)$ is induced from

$$\chi_1(\det_{n_1}) \otimes \cdots \otimes \chi_q(\det_{n_q}) \otimes \pi(\mu_1(\det_{m_1}), \alpha_1) \otimes \cdots \otimes \pi(\mu_r(\det_{m_r}), \alpha_r),$$

where $n_1 + \cdots + n_q + 2(m_1 + \cdots + m_r) = N$, $0 < \alpha_1, \dots, \alpha_r < \frac{1}{2}$, and $\chi_1, \dots, \chi_q, \mu_1, \dots, \mu_r$ are unramified unitary characters. Hence by comparing the Satake parameters, we can see that $b_p = 1$ for all p, or $b_p = p^{-1}$ for all p.

Therefore, Π should be either $\Pi_1 \boxplus 1_{GL_7} \boxplus 1$, or $\Pi_1 \boxplus 1_{GL_5} \boxplus 1_{GL_3}$, where Π_1 is an automorphic representation of GL_4, and 1_{GL_n} denotes the trivial representation of GL_n.

Now we can see easily that $\wedge^2 \Pi_1 = \mathrm{Sym}^2(\pi_f) \oplus \mathrm{Sym}^2(\pi_h)$. Then by [2], Π_1 is of the form $\Pi_1 = \sigma_1 \boxtimes \sigma_2$ for σ_1, σ_2, cuspidal representations of $GL_2(\mathbb{A})$. Since $\wedge^2(\sigma_1 \boxtimes \sigma_2) = Ad(\sigma_1) \otimes \omega_{\sigma_1}\omega_{\sigma_2} \boxplus Ad(\sigma_2) \otimes \omega_{\sigma_1}\omega_{\sigma_2}$, $\omega_{\sigma_1}\omega_{\sigma_2} = 1$, $Ad(\sigma_1) = Ad(\pi_f)$ and $Ad(\sigma_2) = Ad(\pi_h)$. By [33], $\sigma_1 = \pi_f \otimes \chi_1$ and $\sigma_2 = \pi_h \otimes \chi_2$ for some characters χ_1, χ_2. Hence $\Pi_1 = (\pi_f \boxtimes \pi_h) \otimes \chi_1\chi_2$. Since the central character of Π is trivial, $\chi_1\chi_2 = 1$. Therefore, $\Pi_1 = \pi_f \boxtimes \pi_h$, and $\epsilon_p = 1$ for all p. This shows that $\Pi = (\pi_f \boxtimes \pi_h) \boxplus 1_{GL_7} \boxplus 1$, or $(\pi_f \boxtimes \pi_h) \boxplus 1_{GL_5} \boxplus 1_{GL_3}$. However by the classification in [8, Section 7.2.2], the latter case cannot happen under Conjecture 6.2.

The Satake parameters at p behave uniformly and it follows from this that $\mathcal{H}(G'(\mathbb{A}_f)) \cdot \mathcal{F}_{f,h}$ is isotypic. Since it is generated by the class one vector $\mathcal{F}_{f,h}$, it is irreducible. It follows that $\mathcal{F}_{f,h}$ is a Hecke eigenform and gives rise to a cuspidal representation $\Pi_{f,h}$ of $G'(\mathbb{A})$. We have also shown that the degree 12 standard L-function of $\Pi_{f,h}$ is

$$L(s, \Pi_{f,h}) = L(s, \pi_f \times \pi_h)\zeta(s)^2\zeta(s \pm 1)\zeta(s \pm 2)\zeta(s \pm 3),$$

or

$$L(s, \Pi_{f,h}) = L(s, \pi_f \times \pi_h)\zeta(s)^2\zeta(s \pm 1)^2\zeta(s \pm 2),$$

where the first L-function is the Rankin–Selberg L-function.

8 Remark on non-vanishing hypothesis

Recall

$$\mathcal{F}_{f,h}(Z) = \int_{SL_2(\mathbb{Z})\backslash\mathbb{H}} F_f \begin{pmatrix} Z & 0 \\ 0 & \tau \end{pmatrix} \overline{h(\tau)} (\mathrm{Im}\tau)^{2k+6}\, d\tau.$$

We consider the question of nonvanishing of $\mathcal{F}_{f,h}$. We have a Fourier–Jacobi expansion of F_f;

$$F_f \begin{pmatrix} Z & w \\ {}^t w & \tau \end{pmatrix} = \sum_S \mathcal{F}_S(\tau, w) e^{2\pi Tr(ZS)\sqrt{-1}}, \tag{8.1}$$

where $S \in \mathfrak{J}_2(\mathbb{Z})_+$ and $\mathcal{F}_S$ is a Fourier–Jacobi coefficient of index S as in [24]. Here

$$\mathcal{F}_S(\tau, w) = \sum_{\lambda \in \Lambda} \theta_{[\lambda]}(S; \tau, w)\mathcal{F}_{S,\lambda}(\tau),$$

where $\theta_{[\lambda]}(S; \tau, w)$ is a theta series and $\mathcal{F}_{S,\lambda}(\tau)$ is a vector-valued modular form, which is obtained from a suitable compatible family of Eisenstein series. (See Section 8 in [24].)

Lemma 8.1 *We have the estimate:*

$$|\mathcal{F}_S(\tau, 0)| \ll y^{-(2k+8)} Tr(S)^{2(2k+8)}, \quad y = \mathrm{Im}(\tau).$$

Proof Let $Z = X + Y\sqrt{-1}$. Then

$$\mathcal{F}_S(\tau, 0)e^{-2\pi Tr(YS)} = \int F_f \begin{pmatrix} Z & 0 \\ 0 & \tau \end{pmatrix} e^{-2\pi i Tr(XS)}\, dX,$$

where the integral is over $\mathfrak{J}_2(\mathbb{R})/\mathfrak{J}_2(\mathbb{Z})$. Set $Y = \frac{1}{Tr(S)} I_2$. Then

$$|\mathcal{F}_S(\tau, 0)| \ll y^{-(2k+8)} Tr(S)^{2(2k+8)}.$$

□

By the above lemma, for a fixed Z,

$$\sum_S |\mathcal{F}_S(\tau, 0)e^{2\pi Tr(ZS)\sqrt{-1}}| = y^{-(2k+8)} \sum_S Tr(S)^{2(2k+8)} e^{-2\pi Tr(YS)}.$$

Now $Tr(YS) \geq c_Y Tr(S)$ for a constant $c_Y > 0$. Hence

$$\sum_S Tr(S)^{2(2k+8)} e^{-2\pi Tr(YS)} \leq \sum_S Tr(S)^{2(2k+8)} e^{-2\pi c_Y Tr(S)},$$

which is bounded. Therefore

$$\mathcal{F}_{f,h}(Z) = \int_{SL_2(\mathbb{Z})\backslash\mathbb{H}} \left(\sum_S \mathcal{F}_S(\tau, 0)e^{2\pi Tr(ZS)\sqrt{-1}} \right) \overline{h(\tau)} \mathrm{Im}(\tau)^{2k+6}\, d\tau.$$

converges absolutely. So we can interchange the sum and integral. We write

$$\mathcal{F}_{f,h}(Z) = \sum_S A_S\, e^{2\pi Tr(ZS)\sqrt{-1}},$$

where

$$A_S = \int_{SL_2(\mathbb{Z})\backslash\mathbb{H}} \mathcal{F}_S(\tau, 0)\overline{h(\tau)} \mathrm{Im}(\tau)^{2k+6}\, d\tau.$$

Here $\mathcal{F}_S(\tau, 0)$ is a modular form of weight $2k + 8$. Hence A_S is the Petersson inner product of $\mathcal{F}_S(\tau, 0)$ and h. In order that $\mathcal{F}_{f,h}$ is identically zero, we need $A_S = 0$ for all S. As S runs over $\mathfrak{J}_2(\mathbb{Z})_+$, it is very likely that $\mathcal{F}_S(\tau, 0)$ will span the whole space $S_{2k+8}(SL_2(\mathbb{Z}))$. So we expect that $A_S \neq 0$ for some S, and $\mathcal{F}_{f,h}$ is not identically zero.

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