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Moments of logarithmic derivatives of L-functions

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ABSTRACT

We obtain k-th moments of logarithmic derivatives of two kinds of families of L-functions. For a family of S_{d+1}-fields, where S_{d+1} is the symmetric group on $(d+1)$-letters, it is asymptotic to a constant. For a suitable parametric family, it is asymptotic to $(\log\log X)^k$. In the last section, we obtain k-th moments of logarithm of Artin L-functions. We obtain the distribution of logarithmic derivatives and logarithms of L-functions by the method of moments.

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1. Introduction

This paper is a continuation of [4]. We refer to [4] for unexplained notations: Let K be a number field of degree $d+1$ with discriminant d_K and $\zeta_K(s)$ be the Dedekind zeta function of K. Let $\widehat{K}$ be the Galois closure of K over $\mathbb{Q}$. Then we have $\zeta_K(s) = \zeta(s)L(s,\rho)$ for some d-dimensional complex representation ρ of the Galois group $\mathrm{Gal}(\widehat{K}/\mathbb{Q})$. In [4], we showed that under the Artin conjecture, GRH and certain zero density hypothesis, the upper and lower bounds of $-\frac{L'}{L}(1,\rho)$ are

$$\log\log|d_K| + O(\log\log\log|d_K|), \quad -d\log\log|d_K| + O(\log\log\log|d_K|), \quad \text{resp.} \tag{1.1}$$

We also obtained the first moment of $-\frac{L'}{L}(1,\rho)$ for some parametric families.

In this paper, we study the k-th moments of $\frac{L'}{L}(1,\rho)$ and $\log L(1,\rho)$ for each positive integer k for two kinds of families $L(X)$ of $L(s,\rho)$:

$$\frac{1}{|L(X)|}\sum_{L(s,\rho)\in L(X)}\left(-\frac{L'}{L}(1,\rho)\right)^k, \quad \frac{1}{|L(X)|}\sum_{L(s,\rho)\in L(X)}(\log L(1,\rho))^k.$$

Ihara, Murty and Shimura [11] computed (a,b)-moments of logarithmic derivatives of Dirichlet L-functions. Namely, for a prime m, let X_m denote the set of all non-principal characters χ with conductor m, and $P^{(a,b)}(z) = z^a\overline{z}^b$. Then

$$\frac{1}{|X_m|}\sum_{\chi\in X_m} P^{(a,b)}\left(\frac{L'}{L}(1,\chi)\right) = (-1)^{a+b}\mu^{(a,b)} + O(m^{\epsilon-1}),$$

where $\mu^{(a,b)}$ is a constant which has an explicit expression.

Mourtada and Murty [16] computed moments of $\frac{L'}{L}(1,\chi_D)$, where χ_D is a quadratic character:

$$\sum\nolimits^{*}_{0<\beta D\le X}\left(-\frac{L'}{L}(1,\chi_D)\right)^k = C_k X + O_k(X^{\frac{5}{6}+\epsilon}),$$

where $X > 1, \beta = \pm 1$ and the asterisk indicates that the sum is over fundamental discriminants D, and

$$C_k = \frac{3}{\pi^2}\sum_{n=1}^{\infty}\frac{\Lambda_k(n^2)}{n^2}\prod_{p|n}\left(1+\frac{1}{p}\right)^{-1}, \tag{1.2}$$

where $\Lambda_k(n)$ is defined by $\Lambda_k(n) = \sum_{n=n_1\cdots n_k}\Lambda(n_1)\cdots\Lambda(n_k)$ for $k>0$, and $\Lambda_0(n) = 0$ for all n except for $\Lambda_0(1) = 1$.

Our basic idea is to use a short sum approximation of $-\frac{L'}{L}(1,\rho)$ and $\log L(1,\rho)$ under the assumptions that $L(s,\rho)$ is entire, and is zero-free in the rectangle $[\alpha,1]\times[-(\log N)^m, (\log N)^m]$, where N is the conductor of ρ and $m(1-\frac{\alpha+1}{2}) > 3$ (cf. Daileda [7]).

We use the zero density result of Kowalski–Michel [14] to show that except for $O(X^{\frac{1}{100}})$ L-functions, every L-function in $L(X)$ has the desired zero-free region. For this, we need to assume the strong Artin conjecture for ρ, i.e., ρ is an automorphic representation of $GL(d)$ and assume that the L-functions in $L(X)$ are distinct.

The first family we consider is constructed by G-fields. This is done in Section 3. We say that a degree $d+1$ extension $K/\mathbb{Q}$ is a S_{d+1}-field over $\mathbb{Q}$ if $\mathrm{Gal}(\widehat{K}/\mathbb{Q})$ is isomorphic to the symmetric group S_{d+1}. Define

$$L_{d+1}^{(r_2)}(X) = \left\{ K \mid \frac{X}{2} < |d_K| < X, K : S_{d+1}\text{-field of signature } (r_1, r_2) \right\}.$$

Under the counting conjecture (3.2) and the strong Artin conjecture for ρ, we show that the k-th moment of this family is a constant and we obtain the asymptotic formula (Theorem 3.4). Since the counting conjecture is proved for $G = S_3, S_4, S_5$ [2,20,18,6] and the strong Artin conjecture is known for S_3, S_4 case, our result is unconditional for S_3, S_4. In Section 3.2, we recover the main term (1.2) for quadratic fields. In Section 3.3, we write down an explicit formula for the main term for cubic fields. In Section 3.4, we give an application of k-th moments to the distribution of $-\frac{L'}{L}(1, \rho)$.

The second family is a parametric family defined by a polynomial $f(x, t) \in \mathbb{Q}(t)[x]$ of degree $d+1$ in x. Assume that the splitting field E of $f(x, t)$ over $\mathbb{Q}(t)$ is a regular Galois extension (i.e., $E \cap \overline{\mathbb{Q}} = \mathbb{Q}$.). For a specialization $t \in \mathbb{Z}$, let K_t be the number field obtained by adjoining a root of $f(x, t)$ to the field of rational numbers. Under several assumptions, we showed in [4] that there are infinitely many number fields with the extreme values (1.1) in the family defined by $f(x, t)$, and gave many examples which satisfy the assumptions. Let

$$L(X) = \left\{ \frac{X}{2} < t < X | \, t \equiv s_M \pmod{M}, \, \mathrm{Gal}(\widehat{K_t}/\mathbb{Q}) \simeq G \right\},$$

where s_M and M are carefully chosen so that $-\frac{L'}{L}(1, \rho, t)$ would takes the extreme value $-d \log\log |d_K| + O(\log\log\log |d_K|)$. We show (Theorem 4.8) under the strong Artin conjecture for ρ and a technical assumption that the L-functions in $L(X)$ are distinct, and the estimate (4.6),

$$\frac{1}{|L(X)|} \sum_{L(s,\rho) \in L(X)} \left(-\frac{L'}{L}(1, \rho, t) \right)^k = d^k (\log\log X)^k + O\left((\log\log X)^{k-\frac{1}{2}} \right).$$

In Section 4.3, we give three examples which satisfy the assumptions. So the above holds unconditionally for these examples. In fact, the above holds unconditionally for all the examples from [4] except possibly for A_4 case.

In Section 5, we obtain the asymptotic formula of the k-th moments of $\log L(1, \rho)$ (Proposition 5.3) and its distribution.

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2. Approximation of $\frac{L'}{L}(1,\rho)$ under zero-free region

Let $L(s,\rho)=\sum_{n=1}^{\infty}\lambda_\rho(n)n^{-s}=\prod_p\prod_{i=1}^{d}\left(1-\frac{\alpha_i(p)}{p^s}\right)^{-1}$. Then we have

$$-\frac{L'}{L}(s,\rho)=\sum_{n=1}^{\infty}\Lambda(n)a_\rho(n)n^{-s},\quad a_\rho(p^k)=\sum_{j=1}^{d}\alpha_j(p)^k.$$

Proposition 2.1. *Suppose $L(s,\rho)$ is entire and is zero free in the rectangle $[\alpha,1]\times[-x,x]$, and N is the conductor of ρ. Then*

$$-\frac{L'}{L}(1,\rho)=\sum_{n<x}\frac{a_\rho(n)\Lambda(n)}{n}+O_\alpha\left(\frac{\log N\log x+(\log x)^2}{x^{1-\frac{\alpha+1}{2}}}\right). \tag{2.2}$$

Proof. By Perron's formula,

$$\frac{1}{2\pi i}\int_{c-ix}^{c+ix}-\frac{L'}{L}(1+s,\rho)\frac{x^s}{s}ds=\sum_{n<x}\frac{a_\rho(n)\Lambda(n)}{n}+O\left(\frac{d(\log x)^2}{x}\right),$$

where $c=\frac{1}{\log x}$. We move the contour to $Re(s)=\frac{\alpha+1}{2}-1$ and get the residue $-\frac{L'}{L}(1,\rho)$ at $s=0$. So, the left hand side is $-\frac{L'}{L}(1,\rho)$ plus

$$\frac{1}{2\pi i}\left(\int_{c-ix}^{\frac{\alpha+1}{2}-1-ix}+\int_{\frac{\alpha+1}{2}-1-ix}^{\frac{\alpha+1}{2}-1+ix}+\int_{\frac{\alpha+1}{2}-1+ix}^{c+ix}\right)\left(-\frac{L'}{L}(1+s,\rho)\frac{x^s}{s}\right)ds.$$

Let $\Omega=[\frac{\alpha+1}{2},2]\times[-x,x]$ be a rectangular region in the complex plane. Then for $s\in\Omega$, $\left|-\frac{L'}{L}(s)\right|\ll_\alpha \log N+\log(|s|+1)$. (See page 236 in [7].) Hence the integral is

$$\ll_\alpha \frac{\log N+\log x}{x}+\frac{\log N\log x+(\log x)^2}{x^{1-\frac{\alpha+1}{2}}}\ll\frac{\log N\log x+(\log x)^2}{x^{1-\frac{\alpha+1}{2}}},$$

and the claim follows. $\square$

By setting $x=(\log N)^m$ with $m(1-\frac{\alpha+1}{2})>3$, we have

$$\begin{aligned}-\frac{L'}{L}(1,\rho)&=\sum_{n<(\log N)^m}\frac{a_\rho(n)\Lambda(n)}{n}+O\left(\frac{1}{\log N}\right)\\&=\sum_{p<(\log N)^m}\frac{a_\rho(p)\log p}{p}+O\left(1\right).\end{aligned} \tag{2.3}$$

Due to lack of GRH, we cannot use the above result directly. In [4], we extended the zero density result of Kowalski–Michel [14] to isobaric automorphic representations of $GL(n)$. By applying Theorem 3.4 of [4] to $L(X)$, we can show that every automorphic L-function in $L(X)$ excluding exceptional $O(X^{1/100})$ L-functions has a zero-free region $[\alpha, 1] \times [-(\log|d_{K_t}|)^m, (\log|d_{K_t}|)^m]$, where $m(1-\frac{\alpha+1}{2}) > 3$. Let us denote by $\widehat{L}(X)$, the set of the automorphic L-functions with the zero-free region.

3. Moments of $\frac{L'}{L}(1,\rho)$ over S_{d+1}-fields

Let $L_{d+1}^{(r_2)}(X)$ be the set of S_{d+1}-fields K of signature (r_1, r_2) with $\frac{X}{2} < |d_K| < X$. Let $\mathcal{S} = (LC_p)$ be a finite set of local conditions, namely, $LC_p = \mathcal{S}_{p,C}$ means that the conjugacy class of p is C. Let $|\mathcal{S}_{p,C}| = \frac{|C|}{|G|(1+f(p))}$ for some function $f(p)$ which satisfies $f(p) = O(\frac{1}{p})$. There are also several splitting types of ramified primes, which are denoted by $r_1, \cdots, r_w$. Then $LC_p = \mathcal{S}_{p,r_i}$ means that the splitting type of the ramified prime p is r_i. Then there are positive valued functions $c_1(p), c_2(p), \cdots, c_w(p)$ with $\sum_{i=1}^{w} c_i(p) = f(p)$, such that $|\mathcal{S}_{p,r_i}| = \frac{c_i(p)}{1+f(p)}$. Let $|\mathcal{S}| = \prod_{p\in\mathcal{S}} |LC_p|$.

Let $L_{d+1}^{(r_2)}(X, \mathcal{S})$ be the set of S_{d+1}-fields K of signature (r_1, r_2) with $\frac{X}{2} < |d_K| < X$, and the local condition $\mathcal{S}$.

By abuse of notation, we denote $L_{d+1}(X)^{r_2}$ as a set of L-functions $L(s,\rho)$ for $K \in L_{d+1}^{(r_2)}(X)$. Here we need care in order to ensure one to one correspondence between two sets. Two number fields K_1 and K_2 are said to be arithmetically equivalent if $\zeta_{K_1}(s) = \zeta_{K_2}(s)$. If two number fields K_1 and K_2 are conjugate, then they are arithmetically equivalent. The converse is not always true. A number field K_1 is called arithmetically solitary if $\zeta_{K_1}(s) = \zeta_{K_2}(s)$ implies that K_1 and K_2 are conjugate. It is known that S_{d+1}-fields and A_{d+1}-fields are arithmetically solitary. See [13, Chap. II].

We have

Conjecture 3.1.

$$|L_{d+1}^{(r_2)}(X)| = A(r_2)X + O(X^{\delta}) \tag{3.2}$$

$$|L_{d+1}^{(r_2)}(X, \mathcal{S})| = |\mathcal{S}|A(r_2)X + O\left(\Big(\prod_{p\in\mathcal{S}} p\Big)^{\gamma} X^{\delta}\right)$$

for some positive constant $A(r_2)$, $\delta < 1$ and γ, and the implied constant is uniformly bounded for p and local conditions at p.

This conjecture is true when $G = S_3, S_4$ and S_5 [20,2,21,18,6]. We give explicit constants in the case of S_3, following Taniguchi and Thorne [20]. Let $L(X)^{\pm}$ be the set of cubic fields K with $0 < \pm d_K < X$. Then

$$|L(X)^{\pm}| = \frac{A^{\pm}}{12\zeta(3)}X + B^{\pm}\frac{4\zeta(1/3)}{5\Gamma(2/3)^3\zeta(5/3)}X^{5/6} + O(X^{7/9+\epsilon}),$$

where $A^+ = 1$, $A^- = 3$, $B^+ = 1$, and $B^- = \sqrt{3}$. Here, we count only one cubic field from three conjugate fields. Let TS_p, PS_p, and IN_p be the local conditions of p which means that p is totally split, partially split and inert respectively. Let $S = \{LC_{p_i} | i = 1, 2, \cdots, u\}$ be a set of local conditions at p_i. Then

$$|LC_p| = \begin{cases} \frac{1/6}{1+1/p+1/p^2} & \text{if } LC_p = TS_p, \\ \frac{3/6}{1+1/p+1/p^2} & \text{if } LC_p = PS_p, \\ \frac{2/6}{1+1/p+1/p^2} & \text{if } LC_p = IN_p, \\ \frac{1/p}{1+1/p+1/p^2} & \text{if } p \text{ is partially ramified}, \\ \frac{1/p^2}{1+1/p+1/p^2} & \text{if } p \text{ is totally ramified}, \end{cases} \tag{3.3}$$

and $|L(X,\mathcal{S})^{\pm}| = |\mathcal{S}|\frac{C^{\pm}}{12\zeta(3)}X + O(E_{\mathcal{S}}(X))$, where

$$E_{\mathcal{S}}(X) = \begin{cases} X^{\frac{5}{6}}, & \text{if } \prod_{i=1}^{u} p_i^{e_i} < X^{\frac{1}{18}-\epsilon}, \\ (\prod_{i=1}^{u} p_i^{e_i})X^{\frac{7}{9}+\epsilon}, & \text{if } X^{\frac{1}{18}-\epsilon} \le \prod_{i=1}^{u} p_i^{e_i} \le X^{\frac{2}{9}-\epsilon}, \end{cases}$$

and $e_i = \frac{8}{9}$ for unramified p_i and $e_i = \frac{16}{9}$ for ramified p_i. For explicit constants $A(r_2), LC_p$ in the cases of S_4 and S_5, see [6].

For simplicity, we write $L_{d+1}^{(r_2)}(X)$ by $L(X)$, and $L(s,\rho) \in L_{d+1}^{(r_2)}(X)$ by $\rho \in L(X)$. Then, we have the following k-th moments theorem.

Theorem 3.4. *Assume (3.2) and the strong Artin conjecture for ρ. Then*

$$\begin{aligned} &\frac{1}{|L(X)|}\sum_{\rho\in L(X)}\left(-\frac{L'}{L}(1,\rho)\right)^k \\ &= \sum_{\substack{q_1,\dots,q_{u+v} \\ \mathbf{a}_{u+j},\mathbf{e}_{u+j},\, j=1,\dots,v}} \gamma(q_1,...,q_u,(q_{u+1},\mathbf{a}_{u+1},\mathbf{e}_{u+1}),...,(q_{u+v},\mathbf{a}_{u+v},\mathbf{e}_{u+v})) \\ &\quad + O\left(\frac{1}{\log X}\right), \end{aligned} \tag{3.5}$$

where $q_1, \cdots, q_{u+v}$ run over distinct primes and $\mathbf{e}_{u+j}, \mathbf{a}_{u+j}$ run through sets of positive integers such that $u + |\mathbf{e}_{u+1}| + \cdots |\mathbf{e}_{u+v}| = k$, $|\mathbf{a}_{u+j}\mathbf{e}_{u+j}| \ge 2$, $j = 1, ..., v$. In particular, for $k = 1$,

$$\frac{1}{|L(X)|}\sum_{\rho\in L(X)}\left(-\frac{L'}{L}(1,\rho)\right) = \sum_{p,\, u\ge 2}\frac{(A(p,u,1)+f_1(p,u,1))\log p}{p^u} + \sum_{p}\frac{f_1(p)\log p}{p} + O\left(\frac{1}{\log X}\right).$$

The precise definition of the function γ is given in (3.12). Since (3.2) is valid for S_3, S_4, S_5, and the strong Artin conjecture is valid for S_3, S_4, we have Theorem 3.4 unconditionally for S_3, S_4.

Recall $\widehat{L}(X)$ at the end of Section 2: the set of L-functions with the desired zero-free regions. If $\rho \in \widehat{L}(X)$, by (2.3), we have

$$-\frac{L'}{L}(1,\rho) = \sum_{p \le x} \frac{a_\rho(p)\log p}{p} + C_\rho, \quad \text{for } x = (\log X)^m,$$

where $C_\rho = O_\alpha(1)$.

If $L(s,\rho)$ may not have the desired zero-free region, we use the following trivial bound:

Lemma 3.6.

$$\left|\frac{L'}{L}(1,\rho)\right| \ll_\epsilon \begin{cases} |d_K|^\epsilon, & \textit{if } K \textit{ contains a quadratic subfield} \\ (\log |d_K|)^2, & \textit{if } K \textit{ does not contain a quadratic subfield} \end{cases}.$$

Proof. By [19], Lemma 10, if K does not contain a quadratic subfield, $L(s,\rho)$ does not have a Siegel zero, i.e., no zero in the region $1-(16\log|d_K|)^{-1} \le Re(s) \le 1$. By [12], page 103, $|\frac{L'}{L}(1,\rho)| \ll \log|d_K| + \sum_{|1-\varrho|<1} \frac{1}{1-\varrho}$. By [12], page 102, the number of zeros such that $|1-\varrho|<1$ is $O(\log|d_K|)$. Hence $|\frac{L'}{L}(1,\rho)| \ll (\log|d_K|)^2$.

If K contains a quadratic subfield F, then any possible Siegel zero of $\zeta_K(s)$ is that of $\zeta_F(s)$. But the Siegel zero β satisfies $\beta < 1-c(\epsilon)|d_F|^{-\epsilon}$ for any $\epsilon>0$ (ineffective implied constant) ([8], page 126). Since $|d_F| \le |d_K|$, we have $|\frac{L'}{L}(1,\rho)| \ll_\epsilon |d_K|^\epsilon$. $\square$

We note that S_{d+1}-fields and A_{d+1}-fields ($d+1 \ge 3$) do not contain quadratic subfields. If K does not contain a quadratic subfield, the sum over the exceptional set is $\ll X^{\frac{1}{100}+\epsilon}$ for any k. When K contains a quadratic subfield, given k, we choose ϵ such that $\frac{1}{100}+k\epsilon<1$. Then the sum over the exceptional set is $\ll X^{\frac{1}{100}+k\epsilon}$. Hence we can replace the sum $\sum_{\rho\in L(X)}$ by $\sum_{\rho\in\widehat{L}(X)}$.

3.1. *Proof of Theorem 3.4*

We first show that (3.5) is bounded above by a constant. Consider

$$\left(-\frac{L'}{L}(1,\rho)\right)^k = \sum_{r=0}^k \binom{k}{r} (\Sigma_\rho)^r C_\rho^{k-r},$$

where $\Sigma_\rho = \sum_{p\le x} \frac{a_\rho(p)\log p}{p}$. By Cauchy–Schwartz inequality

$$\sum_{\rho\in L(X)} (\Sigma_\rho)^r C_\rho^{k-r} \ll \left(\sum_{\rho\in L(X)} (\Sigma_\rho)^{2r}\right)^{\frac12} \left(\sum_{\rho\in L(X)} C_\rho^{2(k-r)}\right)^{\frac12}. \tag{3.7}$$

Hence we only need to estimate

$$\sum_{\rho\in L(X)}\left(\sum_{p\le x}\frac{a_\rho(p)\log p}{p}\right)^r=\sum_{p_1,\ldots,p_r}\frac{(\log p_1)\cdots(\log p_r)}{p_1\cdots p_r}\sum_{\rho\in L(X)}a_\rho(p_1)\cdots a_\rho(p_r). \quad (3.8)$$

Now by combining the same primes, we need to consider, for $q_1,\cdots,q_u$ distinct primes,

$$\sum_{\rho\in L(X)}a_\rho(q_1)^{e_1}\cdots a_\rho(q_u)^{e_u},$$

where $e_1+\cdots+e_u=r$. Let N be the number of conjugacy classes of S_{d+1}, w the number of splitting types of ramified primes. Partition the sum $\sum_{\rho\in L(X)}$ into $(N+w)^u$ sums, namely, given $(LC_1,...,LC_u)$, where LC_i is either $\mathcal{S}_{q_i,C_i}$ or $\mathcal{S}_{q_i,r_j}$, we consider the set of $\rho\in L(X)$ with the local conditions LC_i for each i. Note that in each such partition, $a_\rho(q_1)^{e_1}\cdots a_\rho(q_u)^{e_u}$ remains a constant.

Consider the case where all $e_i>1$. Then, in (3.8), we would have $X\prod_{i=1}^u\left(\sum_{q_i\le x}\frac{1}{q_i^{e_i}}\right)$, which is $O(X)$. Now assume that $e_i=1$ and q_i is unramified for some i, say $i=1$. Fix the splitting types of $q_2,\cdots,q_u$, and let Frob_{q_1} runs through the conjugacy classes of G. Then by (3.2), the sum of such N partitions is

$$\sum_C\left(\frac{|C|a_\rho(q_1)}{|G|(1+f(q_1))}A(LC_2,...,LC_u)X+O((q_1\cdots q_u)^\gamma X^\delta)\right),$$

for a constant $A(LC_2,...,LC_u)$. Let χ_ρ be the character of ρ. Then $a_\rho(p)=\chi_\rho(g)$, where $g=\mathrm{Frob}_p$. By orthogonality of characters, $\sum_C|C|a_\rho(q_1)=\sum_{g\in G}\chi_\rho(g)=0$. Hence the above sum is $O((q_1\cdots q_u)^\gamma X^\delta)$, and it contributes $O(X)$.

Now we are reduced to the case $e_1=e_2=\cdots=e_i=1$, $e_{i+1}>1,...,e_u>1$, and $q_1,\cdots,q_i$ are all ramified. Then, the bound

$$\sum_{\rho\in L(X)}a_\rho(q_1)\cdots a_\rho(q_i)a_\rho(q_{i+1})^{e_{i+1}}\cdots a_\rho(q_u)^{e_u}\ll\frac{X}{q_1\cdots q_i}$$

implies that the contribution of this case to (3.8) is $O(X)$. Therefore, we have proved that

$$(3.5)=O(1).$$

Now we obtain the asymptotic formula for (3.5). By (2.3), we have

$$-\frac{L'}{L}(1,\rho)=\Sigma_\rho+C_\rho,$$

where $\Sigma_\rho=\sum_{n<(\log X)^m}\frac{a_\rho(n)\Lambda(n)}{n}$ and $C_\rho=O\left(\frac{1}{\log X}\right)$.

Then

$$\left(-\frac{L'}{L}(1,\rho)\right)^k = (\Sigma_\rho)^k + \sum_{r=0}^{k-1} \binom{k}{r} (\Sigma_\rho)^r C_\rho^{k-r}.$$

By (3.7), it is enough to consider

$$\frac{1}{|L(X)|} \sum_{\rho\in L(X)} (\Sigma_\rho)^r, \text{ for } r = 2, 4, \cdots, 2k-2, \text{ and } k. \tag{3.9}$$

Here

$$\sum_{\rho\in L(X)} (\Sigma_\rho)^r = \sum_{n_i<(\log X)^m, i=1,\cdots,r} \frac{\Lambda(n_1)\cdots\Lambda(n_r)}{n_1\cdots n_r} \alpha(n_1,...,n_r), \tag{3.10}$$

where $\alpha(p_1^{a_1},...,p_r^{a_r}) = \sum_{\rho\in L(X)} a_\rho(p_1^{a_1})\cdots a_\rho(p_r^{a_r})$. Note that $|\alpha(p_1^{a_1},...,p_r^{a_r})| \le d^r|L(X)|$.

By combining the same prime powers, we can write (3.10) as follows: We write $\mathbf{a} = (a_1,...,a_t), \mathbf{e} = (e_1,...,e_t)$, where a_i's and e_i's are positive integers. Denote $|\mathbf{e}| = e_1 + \cdots + e_t$ and $|\mathbf{ae}| = a_1e_1 + \cdots + a_te_t$. Let

$$\begin{gathered} q^{\mathbf{a}} = \max\{q^{a_1},...,q^{a_t}\}, \mathbf{e}q^{\mathbf{a}} = e_1q^{a_1} + \cdots + e_tq^{a_t}, \text{ and} \\ a_\rho(q,\mathbf{a},\mathbf{e}) = a_\rho(q^{a_1})^{e_1}\cdots a_\rho(q^{a_t})^{e_t}. \end{gathered} \tag{3.11}$$

Then

$$\text{(3.10)} = \sum_{\substack{q_1,\dots,q_u,q_{u+1},\dots,q_{u+v} \\ \mathbf{e}_{u+j},\, \mathbf{a}_{u+j},\, j=1,\dots,v}} \prod_{i=1}^{u} \frac{\log q_i}{q_i} \prod_{j=1}^{v} \frac{(\log q_{u+j})^{|\mathbf{e}_{u+j}|}}{q_{u+j}^{|\mathbf{a}_{u+j}\mathbf{e}_{u+j}|}}$$
$$\cdot\beta(q_1,...,q_u,(q_{u+1},\mathbf{a}_{u+1},\mathbf{e}_{u+1}),...,(q_{u+v},\mathbf{a}_{u+v},\mathbf{e}_{u+v})),$$

where $q_1,\cdots,q_{u+v}$ run over distinct primes such that $q_i < (\log X)^m$, $i = 1,...,u$, $q_{u+j}^{\mathbf{a}_{u+j}} < (\log X)^m$, $j = 1,...,v$ and $\mathbf{e}_{u+j}, \mathbf{a}_{u+j}$ run through sets of positive integers such that $u + |\mathbf{e}_{u+1}| + \cdots |\mathbf{e}_{u+v}| = r$, $|\mathbf{a}_{u+j}\mathbf{e}_{u+j}| \ge 2$, $j = 1,...,v$, and

$$\begin{aligned} &\beta(q_1,...,q_u,(q_{u+1},\mathbf{a}_{u+1},\mathbf{e}_{u+1}),...,(q_{u+v},\mathbf{a}_{u+v},\mathbf{e}_{u+v})) \\ &\quad= \sum_{\rho\in L(X)} a_\rho(q_1)\cdots a_\rho(q_u)a_\rho(q_{u+1},\mathbf{a}_{u+1},\mathbf{e}_{u+1})\cdots a_\rho(q_{u+v},\mathbf{a}_{u+v},\mathbf{e}_{u+v}). \end{aligned}$$

Now as before, we partition $\sum_{\rho\in L(X)}$ into $(N+w)^{u+v}$ sums. We obtain that

$$\beta(q_1, ..., q_u, (q_{u+1}, \mathbf{a}_{u+1}, \mathbf{e}_{u+1}), ..., (q_{u+v}, \mathbf{a}_{u+v}, \mathbf{e}_{u+v}))$$
$$= \sum_P \prod_{i=1}^u a_\rho(q_i) \prod_{j=1}^v a_\rho(q_{u+j}, \mathbf{a}_{u+j}, \mathbf{e}_{u+j}) \left(\sum_{\substack{\rho \in L(X) \\ LC_1,...,LC_{u+v}}} 1 \right),$$

where the sum $\sum_P$ is over $(N+w)^{u+v}$ partitions. By (3.2),

$$\sum_{\substack{\rho \in L(X) \\ LC_1,...,LC_{u+v}}} 1 = \prod_{i=1}^{u+v} |LC_i||L(X)| + O((q_1 \cdots q_{u+v})^\gamma X^\delta).$$

The error term contributes $O\left(X^{\delta+\epsilon-1}\right)$ to (3.9). Now $a_\rho(q_i)$ and $a_\rho(q_{u+j}, \mathbf{a}_{u+j}, \mathbf{e}_{u+j})$ remain constants in the partition and depend only on q_i's. If $g = \mathrm{Frob}_q$, then we can write them as $\chi(q_i)$, $\chi(q_{u+j}, \mathbf{a}_{u+j}, \mathbf{e}_{u+j})$, resp. Then

$$\sum_P \chi(q_1) \cdots \chi_\rho(q_u) \chi_\rho(q_{u+1}, \mathbf{a}_{u+j}, \mathbf{e}_{u+j}) \cdots \chi_\rho(q_{u+v}, \mathbf{a}_{u+j}, \mathbf{e}_{u+j}) \prod_{i=1}^{u+v} |LC_i|$$
$$= \prod_{i=1}^u \left(\sum_C \frac{\chi_\rho(q_i)|C|}{|G|(1+f(q_i))} + \sum_{l=1}^w \frac{\chi_\rho(q_i) c_l(q_i)}{1+f(q_i)} \right)$$
$$\cdot \prod_{j=1}^v \left(\sum_C \frac{\chi_\rho(q_{u+j}, \mathbf{a}_{u+j}, \mathbf{e}_{u+j})|C|}{|G|(1+f(q_{u+j}))} + \sum_{l=1}^w \frac{\chi_\rho(q_{u+j}, \mathbf{a}_{u+j}, \mathbf{e}_{u+j}) c_l(q_{u+j})}{1+f(q_{u+j})} \right).$$

By orthogonality of characters, $\sum_C |C|\chi_\rho(q) = 0$. Recall that $\sum_{l=1}^w c_l(q) = f(q)$. Here $\sum_{l=1}^w \frac{\chi_\rho(q_i)c_l(q_i)}{1+f(q_i)}$, $\sum_{l=1}^w \frac{\chi_\rho(q_{u+j}, \mathbf{a}_{u+j}, \mathbf{e}_{u+j})c_l(q_{u+j})}{1+f(q_{u+j})}$ and $\sum_C \frac{\chi_\rho(q_{u+j}, \mathbf{a}_{u+j}, \mathbf{e}_{u+j})|C|}{|G|(1+f(q_{u+j}))}$ are independent of ρ. We denoted them by $f_1(q_i)$, $f_1(q_{u+j}, \mathbf{a}_{u+j}, \mathbf{e}_{u+j})$ and $A(q_{u+j}, \mathbf{a}_{u+j}, \mathbf{e}_{u+j})$, resp. Note that $f_1(q) = O(\frac{1}{q})$. We showed in [6] that $\sum_C |C|\chi_\rho(q)^2 = |G|$ for each prime q. Similarly, $\sum_C |C|\chi_\rho(q^2) = |G|$.

Hence

$$(3.9) = \sum_{\substack{q_1,...,q_{u+v} \\ \mathbf{e}_{u+j}, \mathbf{a}_{u+j}, j=1,...,v}} \gamma(q_1, ..., q_u, (q_{u+1}, \mathbf{a}_{u+1}, \mathbf{e}_{u+1}), ..., (q_{u+v}, \mathbf{a}_{u+v}, \mathbf{e}_{u+v}))$$
$$+ O(X^{\delta+\epsilon-1}),$$

where $q_1, \cdots, q_{u+v}$ run over distinct primes such that $q_i < (\log X)^m$, $i = 1, ..., u$, $q_{u+j}^{\mathbf{a}_{u+j}} < (\log X)^m$, $j = 1, ..., v$, and

$$\gamma(q_1, ..., q_u, (q_{u+1}, \mathbf{a}_{u+1}, \mathbf{e}_{u+1}), ..., (q_{u+v}, \mathbf{a}_{u+v}, \mathbf{e}_{u+v})) \tag{3.12}$$
$$= \prod_{i=1}^u \frac{\log q_i}{q_i} f_1(q_i) \prod_{j=1}^v \frac{(\log q_{u+j})^{|\mathbf{e}_{u+j}|}}{q_{u+j}^{|\mathbf{a}_{u+j}\mathbf{e}_{u+j}|}} \left(A(q_{u+j}, \mathbf{a}_{u+j}, \mathbf{e}_{u+j}) + f_1(q_{u+j}, \mathbf{a}_{u+j}, \mathbf{e}_{u+j}) \right),$$

Therefore, (3.9) is bounded by a constant independent of r and it implies that

$$\frac{1}{|L(X)|}\sum_{\rho\in L(X)}\left(-\frac{L'}{L}(1,\rho)\right)^k=\frac{1}{|L(X)|}\sum_{\rho\in L(X)}(\Sigma_\rho)^k+O_k\left(\frac{1}{\log X}\right).$$

Note that $\sum_{p>(\log X)^m}\frac{\log p}{p^2}\ll\frac{1}{(\log X)^m}\ll\frac{1}{\log X}$, and $\sum_{p>(\log X)^m}\frac{(\log p)^2}{p^2}\ll\frac{m\log\log X}{(\log X)^m}\ll\frac{1}{\log X}$. This implies that the tail of the convergent infinite series

$$\sum_{\substack{q_1,\ldots,q_{u+v}\\ \mathbf{e}_{u+j},\,\mathbf{a}_{u+j},\,j=1,\ldots,v}}\gamma(q_1,\ldots,q_u,(q_{u+1},\mathbf{a}_{u+1},\mathbf{e}_{u+1}),\ldots,(q_{u+v},\mathbf{a}_{u+v},\mathbf{e}_{u+v})),$$

is at most $O(\frac{1}{\log X})$. Hence we proved the theorem.

3.2. Quadratic fields

We can recover the main term for quadratic fields as in (1.2). For simplicity, we consider only real quadratic fields. Define

$$L(X)=\left\{\frac{X}{2}<d<X\,|\,d\text{: fundamental discriminant}\right\}.$$

It is well-known that $|L(X)|=\frac{3}{2\pi^2}X+O(X^{1/2})$. (See Section 4.3.1.)

For a fundamental discriminant d with $d\equiv 0 \pmod 4$, 2 is ramified in $\mathbb{Q}(\sqrt{d})$. Hence, the local density for a ramified 2 is 1/3. For a fundamental discriminant d with $d\equiv 1 \pmod 4$, 2 is unramified in $\mathbb{Q}(\sqrt{d})$. If $d\equiv 1 \pmod 8$, $\chi_d(2)=\left(\frac{d}{2}\right)=1$. If $d\equiv 5 \pmod 8$, $\chi_d(2)=\left(\frac{d}{2}\right)=-1$. Hence, the local densities for a totally split 2 and for an inert 2 are both $\frac{1}{3}=\frac{1/2}{1+1/2}$.

For an odd prime p, there are $\frac{p-1}{2}$ non-zero quadratic residues and $\frac{p-1}{2}$ non-zero quadratic non-residues. From Section 4.3.1, we can see that the local densities for a totally split prime p and the local density for an inert prime p are the same, and they are given by $\frac{p-1}{2}\cdot(1-p^{-2})^{-1}\cdot\frac{1}{p}=\frac{1/2}{1+1/p}$. Hence the local density for a ramified prime p is $\frac{1/p}{1+1/p}$.

With these local densities, we have

$$|L(X,\mathcal{S})|=|S|\frac{3}{2\pi^2}X+O(X^{1/2}).$$

For $K=\mathbb{Q}(\sqrt{d})$, $\rho=\chi_d=\left(\frac{d}{\cdot}\right)$, which is completely multiplicative. If $|\mathbf{a}_j\mathbf{e}_j|$ is odd, then $\sum_C\chi_\rho(q_j,\mathbf{a}_j,\mathbf{e}_j)|C|=0$. For a ramified prime p, $\chi_\rho(p)=\chi_\rho(p,\mathbf{a},\mathbf{e})=0$. Note that for even $|\mathbf{ae}|$, $\sum_C\frac{\chi_\rho(q,\mathbf{a},\mathbf{e})|C|}{|G|(1+f(q))}=\frac{1/2}{1+1/p}+\frac{1/2}{1+1/p}=\left(1+\frac{1}{p}\right)^{-1}$. Hence, the main term in Theorem 3.4 becomes

$$\sum_{\substack{q_1,\cdots,q_v \\ \mathbf{e}_j,\mathbf{a}_j,j=1,\dots,v \\ |\mathbf{a}_j\mathbf{e}_j|:even}} \prod_{j=1}^{v} \frac{(\log q_j)^{|\mathbf{e}_j|}}{q_j^{|\mathbf{a}_j\mathbf{e}_j|}} \cdot \frac{1}{1+\frac{1}{q_j}} = \sum_{n=1}^{\infty} \frac{\Lambda_k(n^2)}{n^2} \prod_{p|n} \left(1+\frac{1}{p}\right)^{-1},$$

which is (1.2).

3.3. Cubic fields

For cubic fields, by (3.3),

$$f_1(q) = \sum_{l=1}^{w} \frac{\chi_\rho(q)c_l(q)}{1+f(q)} = \frac{1}{q+1+q^{-1}}, \quad f_1(q,\mathbf{a},\mathbf{e}) = \sum_{l=1}^{w} \frac{\chi_\rho(q,\mathbf{a},\mathbf{e})}{1+f(q)} = \frac{1}{q+1+q^{-1}}.$$

It is left to find a simpler expression for $A(q,\mathbf{a},\mathbf{e}) = \sum_C \frac{\chi_\rho(q,\mathbf{a},\mathbf{e})|C|}{|G|(1+f(q))}$ to determine the main term.

First, we consider a totally split prime q. In this case $a_\rho(q^a) = 2$ for any $a \geq 1$. Hence, it contributes $\frac{2^{|\mathbf{e}|}}{6(1+q^{-1}+q^{-2})}$. For a partially split prime q, $a_\rho(q^a) = 0$ for an odd a and $a_\rho(q^a) = 2$ for an even a. For $\mathbf{a} = (a_1, a_2, \cdots, a_t)$, if some of a_i is odd, then it gives 0. If all of a_i's are even, then it gives $\frac{3\cdot 2^{|\mathbf{e}|}}{6(1+q^{-1}+q^{-2})}$. Denote it by $\frac{3\cdot 2^{|\mathbf{e}|}\cdot\delta_{\mathbf{a}\equiv 0(2)}}{6(1+q^{-1}+q^{-2})}$, where $\delta_{\mathbf{a}\equiv 0(2)}$ is 1 if all of a_i's are even and 0 otherwise.

For an inert prime q, $a_\rho(q^a) = -1$ for $a \equiv 1,2 \pmod 3$ and $a_\rho(q^a) = 2$ for $a \equiv 0 \pmod 3$. For $\mathbf{a} = (a_1, a_2, \cdots, a_t)$ and $\mathbf{e} = (e_1, e_2, \cdots, e_t)$, define $Mod_i = \{e_j | a_j \equiv i \pmod 3\}$ for $i = 0,1,2$. Let $|Mod_i| = \sum_{e_j \in Mod_i} e_j$. Then this case contributes $\frac{2\cdot(-1)^{|Mod_1|+|Mod_2|}\cdot 2^{|Mod_0|}}{6(1+q^{-1}+q^{-2})}$.

For example, if all a_i's are congruent to 0 modulo 6, then

$$\sum_C \frac{\chi_\rho(q,\mathbf{a},\mathbf{e})|C|}{|G|(1+f(q))} = \frac{2^{|\mathbf{e}|}}{1+q^{-1}+q^{-2}}.$$

Hence, we have

$$\gamma(q_1,\cdots,q_u,q_{u+1},\cdots,q_{u+v}) = \prod_{i=1}^{u} \frac{\log q_i}{1+q_i+q_i^2} \prod_{j=1}^{v} \frac{(\log q_{u+j})^{|\mathbf{e}_{u+j}|}}{q_{u+j}^{|\mathbf{a}_{u+j}\mathbf{e}_{u+j}|}}$$
$$\cdot \left(\frac{2^{|\mathbf{e}_{u+j}|} + 3\cdot 2^{|\mathbf{e}_{u+j}|}\cdot\delta_{\mathbf{a}_{u+j}\equiv 0(2)} + 2\cdot(-1)^{|Mod_1|+|Mod_2|}\cdot 2^{|Mod_0|}}{6(1+q_{u+j}^{-1}+q_{u+j}^{-2})} + \frac{1}{q_{u+j}+1+q_{u+j}^{-1}} \right).$$

Define a set B_k of ordered pairs of positive integers:

$$B_k = \left\{ (n,m) \in \mathbb{N}\times\mathbb{N} \,\middle|\, \begin{smallmatrix} \gcd(n,m)=1,\ n,m\text{: square-free, } \omega(nm)\leq k; \\ \text{if } m=1,\ n \text{ is the product of } k \text{ distinct primes} \end{smallmatrix} \right\},$$

where $\omega(t)$ is the number of prime divisors of t. We need the restriction on n for $m = 1$ due to the condition $u + |\mathbf{e}_{u+1}| + \cdots + |\mathbf{e}_{u+v}| = k$. Then, we have

$$\sum_{\substack{q_1,\ldots,q_{u+v} \\ \mathbf{a}_{u+j},\mathbf{e}_{u+j},\, j=1,\ldots,v}} \gamma(q_1,\ldots,q_u,(q_{u+1},\mathbf{a}_{u+1},\mathbf{e}_{u+1}),\ldots,(q_{u+v},\mathbf{a}_{u+v},\mathbf{e}_{u+v}))$$

$$= \sum_{(n,m)\in B_k} \prod_{p|n} \frac{\log p}{1+p+p^2}$$

$$\cdot \Bigg[\sum_{\substack{m=q_1q_2\cdots q_v \\ \mathbf{a}_{q_1},\cdots,\mathbf{a}_{q_v} \\ \mathbf{e}_{q_1},\cdots,\mathbf{e}_{q_v}}} \prod_{i=1}^{v} \frac{(\log q_i)^{|\mathbf{e}_{q_i}|}}{q_i^{|\mathbf{a}_{q_i}\mathbf{e}_{q_i}|}} \Bigg(\frac{2^{|\mathbf{e}_{q_i}|}(1+3\cdot\delta_{\mathbf{a}_{q_i}\equiv 0(2)}) + 2\cdot(-1)^{|Mod_1|+|Mod_2|}\cdot 2^{|Mod_0|}}{6(1+q_i^{-1}+q_i^{-2})}$$

$$+ \frac{1}{q_i+1+q_i^{-1}} \Bigg) \Bigg],$$

where $|\mathbf{e}_{q_1}| + \cdots + |\mathbf{e}_{q_v}| = k - \nu(n)$ and $|\mathbf{a}_{q_i}\mathbf{e}_{q_i}| \geq 2$ for all i.

3.4. Method of moments and distribution of $-\frac{L'}{L}(1,\rho)$

In this section, we briefly review the method of moments from [10], page 59 or [1, Lemma 5.1, Lemma 5.7] in order to motivate the study of k-th moments: A distribution function F is a real-valued, non-decreasing function such that $\lim_{x\to-\infty} F(x) = 0$ and $\lim_{x\to\infty} F(x) = 1$. Its characteristic function is $\phi(t) = \int_{-\infty}^{\infty} e^{itx}dF(x)$. Let $\{F_n(x)\}$ be a sequence of distribution functions, and let $\alpha_r = \lim_{n\to\infty}\int_{-\infty}^{\infty} x^r dF_n(x)$ for each positive integer r. Then there exists a subsequence $\{F_{n_j}(x)\}$ which converges weakly to a limiting distribution $F(x)$ for which $\alpha_r = \int_{-\infty}^{\infty} x^r dF(x)$ for each r. In that case, $\phi(t) = \sum_{r=0}^{\infty} \alpha_r \frac{(it)^r}{r!}$ is the characteristic function of $F(x)$.

By taking counting measures, we have the following theorem:

Theorem 3.13. *Let L be a family of objects $\rho \in L$ with invariant $d_\rho \in \mathbb{Z}_+$. For each $\rho \in L$, we are given $\alpha(\rho) \in \mathbb{R}$. Let $L(X) = \{\rho \in L \,|\, d_\rho \leq X\}$. Suppose for each positive integer k, $\frac{1}{|L(X)|}\sum_{\rho\in L(X)} \alpha(\rho)^k \to r(k)$ as $X \to \infty$. Then*

$$\lim_{X\to\infty} \frac{1}{|L(X)|} \#\{\rho \in L(X) \,|\, \alpha(\rho) \leq x\} = F(x),$$

at each point of continuity of $F(t)$, where $F(t)$ is a distribution function whose characteristic function is given by

$$f(t) = \sum_{k=0}^{\infty} \frac{r(k)}{k!}(it)^k.$$

By Theorem 3.4,

$$\lim_{X\to\infty}\frac{1}{|L(X)|}\sum_{\rho\in L(X)}\left(-\frac{L'}{L}(1,\rho)\right)^k = r(k).$$

If $r(k) \ll c^{k\log\log k}$ for some absolute constant $c > 1$, the corresponding characteristic function $f(t)$ is convergent for any t. Then

Corollary 3.14. *Assume (3.2) and the strong Artin conjecture for ρ. Assume that $r(k) \ll c^{k\log\log k}$ for some absolute constant $c > 1$. Then,*

$$\lim_{X\to\infty}\frac{1}{|L(X)|}\#\left\{\rho\in L(X)\,\Big|\, -\frac{L'}{L}(1,\rho)\le x\right\} = F(x),$$

at each point of continuity of $F(t)$, where $F(t)$ is a distribution function whose characteristic function is given by

$$f(t) = \sum_{k=0}^{\infty}\frac{r(k)}{k!}(it)^k.$$

4. Moments of $\frac{L'}{L}(1,\rho)$ over parametric families

4.1. Regular extensions and their Galois representations

A finite extension E of the rational function field $\mathbb{Q}(t)$ is called regular if $E\cap\overline{\mathbb{Q}} = \mathbb{Q}$. Suppose $f(x,t) \in \mathbb{Q}(t)[x]$ is an irreducible polynomial in x of degree $d+1$ with coefficients in $\mathbb{Q}(t)$, and gives rise to a regular Galois extension over $\mathbb{Q}(t)$ with the Galois group G. Let K_t be a field obtained by adjoining to $\mathbb{Q}$ a root of $f(x,t)$ with a specialization $t\in\mathbb{Z}$ and let $\widehat{K_t}$ be the Galois closure of K_t over $\mathbb{Q}$. Let C be any conjugacy class of G. Recall

Theorem 4.1 *(Serre [17], Section 4.6). There is a constant $c_f > 0$ depending on $f(x,t)$ such that for any prime $p \ge c_f$, there is $t_C \in \mathbb{Z}$ so that for any $t \equiv t_C \pmod p$, p is unramified in $\widehat{K_t}/\mathbb{Q}$, and $Frob_p \in C$.*

Let $L(s,\rho,t) = \sum_{n=1}^{\infty}\lambda(n,t)n^{-s} = \zeta_{K_t}(s)/\zeta(s)$ be the Artin L-function attached to the number field K_t. Note that the conductor of $L(s,\rho,t)$ is $|d_{K_t}|$, and $\lambda(p,t) = N(p,t)-1$, where $N(p,t)$ is the number of distinct solutions of $f(x,t)\equiv 0 \pmod p$. Hence $-1\le\lambda(p,t)\le d$.

By Theorem 4.1, for any prime $p \ge c_f$, we can choose an integer s_p so that for any $t\equiv s_p \pmod p$, the Frobenius element of p is the identity in G. For $X > 0$, let $y = \frac{\log X}{\log\log X}$ and $M = \prod_{c_f\le p\le y} p$. Let s_M be an integer such that $s_M \equiv s_p \pmod p$ for all $c_f \le p\le y$. So if $t\equiv s_M \pmod M$, for all $c_f\le p\le y$, p splits completely in $\widehat{K_t}$, and $\lambda(p,t) = d$.

Assume that the discriminant of $f(x,t)$ is a polynomial in t of degree D. Then there is a constant A such that $|d_{K_t}| \le At^D$. We define a set $L(X)$ of positive integers:

$$L(X) = \left\{ \frac{X}{2} < t < X \mid t \equiv s_M \pmod{M},\ \mathrm{Gal}(\widehat{K_t}/\mathbb{Q}) \simeq G \right\}.$$

By the construction of $L(X)$, each t in $L(X)$ corresponds to an Artin L-function $L(s,\rho,t)$ with $\lambda(p,t) = d$ for $c_f \le p \le y$. By abuse of notation, we denote by $L(X)$, the set of Artin L-functions $L(s,\rho,t)$. Now we assume the strong Artin conjecture.

But it is possible that different $t \in L(X)$ correspond to the same automorphic L-function, namely, $\zeta_{K_{t_1}}(s) = \zeta_{K_{t_2}}(s)$. We assume that it does not happen. See the discussion at the beginning of Section 3. In section 4.3, we give three parametric polynomials which satisfy these assumptions.

Now we consider the k-th moments

$$\sum_{L(s,\rho,t)\in L(X)} \left(-\frac{L'}{L}(1,\rho,t)\right)^k.$$

Recall the definition of $\widehat{L}(X)$ at the end of Section 2: the set of L-functions with the desired zero-free regions. For those $L(s,\rho,t)$ which may not have the desired zero-free region, we use the trivial bound (Lemma 3.6) as in the proof of Theorem 3.4, and we may replace the sum $L(s,\rho,t) \in L(X)$ by $L(s,\rho,t) \in \widehat{L}(X)$.

For $L(s,\rho,t) \in \widehat{L}(X)$, by applying Proposition 2.1 with $x = (\log AX^D)^m$ and $y = \frac{\log X}{\log\log X}$, we have ((4.4) of [4])

$$-\frac{L'}{L}(1,\rho,t) = d\log\log X + \sum_{y<p\le x} \frac{\lambda(p,t)\log p}{p} + C_t, \tag{4.2}$$

where $C_t = O(\log\log\log X)$. Here we use the fact that $\lambda(p,t) = d$ for all $c_f \le p \le y = \frac{\log X}{\log\log X}$. Hence

$$\left(-\frac{L'}{L}(1,\rho,t)\right)^k = d^k(\log\log X)^k + \sum_{i=0}^{k-1}\binom{k}{i}(d\log\log X)^i \sum_{r=0}^{k-i}\binom{k-i}{r}(\Sigma_t)^r C_t^{k-r-i},$$

where $\Sigma_t = \sum_{y<p\le x} \frac{\lambda(p,t)\log p}{p}$.

Hence, in order to compute the k-th moment, we need to deal with the sum, for each integer r, $0 \le r \le k-i$,

$$\sum_{L(s,\rho,t)\in \widehat{L}(X)} (\Sigma_t)^r C_t^{k-i-r}. \tag{4.3}$$

By Cauchy–Schwartz inequality,

$$(4.3) \ll \left(\sum_{L(s,\rho,t) \in \widehat{L}(X)} (\Sigma_t)^{2r} \right)^{\frac{1}{2}} \left(\sum_{L(s,\rho,t) \in \widehat{L}(X)} C_t^{2(k-i-r)} \right)^{\frac{1}{2}}.$$

Here $\sum_{L(s,\rho,t)\in\widehat{L}(X)} C_t^{2(k-i-r)} \ll |\widehat{L}(X)|(\log\log\log X)^{2(k-i-r)}$. Now consider

$$E(r,X) = \sum_{L(s,\rho,t) \in \widehat{L}(X)} (\Sigma_t)^r = \sum_{L(s,\rho,t) \in \widehat{L}(X)} \left(\sum_{y<p\le x} \frac{\lambda(p,t)\log p}{p} \right)^r \tag{4.4}$$

$$= \sum_{p_1,\dots,p_r} \frac{(\log p_1)\cdots(\log p_r)}{p_1\cdots p_r} \sum_{L(s,\rho,t)\in\widehat{L}(X)} \lambda(p_1,t)\cdots\lambda(p_r,t).$$

If $p_i = p_j$ for some $i \neq j$, then by the trivial estimate, the contribution of such case to (4.4) is $\ll |\widehat{L}(X)|(\log\log X)^{r-1}$.

Now we assume that $p_i \neq p_j$ for each $i \neq j$. To deal with this case, recall the following [4]: Suppose $f(x,t)$ gives rise to a regular Galois extension over $\mathbb{Q}(t)$. For a fixed prime p, consider the equation $f(x,t) \equiv 0 \pmod{p}$. Let $A_{i,p}$ be the number of $t \pmod{p}$ such that $\lambda(p,t) = i$, i.e., $f(x,t) \equiv 0 \pmod{p}$ has $i+1$ roots. Then we have $\sum_{i=-1}^{n-1} A_{i,p} = p + O(1)$, where $O(1)$ is bounded by D, the degree of discriminant of $f(x,t)$. We proved in [4, (5.2)],

$$\sum_{i=-1}^{d} iA_{i,p} = O(\sqrt{p}). \tag{4.5}$$

Now define $Q_i = \{\frac{X}{2} < t < X \mid t \in L(X) \text{ and } t \equiv i \,(\mathrm{mod}\ p_1\cdots p_r)\}$ and write $L(X) = \bigcup_i Q_i$.

Let R be a subset of $\{0,1,2,\cdots,p_1p_2\cdots p_r - 1\}$ for which $k \in R$ if and only if at least one of $p_1, p_2, \cdots, p_r$ is ramified for $t \in Q_k$. Note that $|R| < D^r$. Now we assume

$$|Q_i| = c_{p_1\cdots p_r} \frac{|L(X)|}{p_1\cdots p_r} + O\left(\frac{|L(X)|}{p_1\cdots p_r(\log X)^{\frac{1}{2}}} \right) \quad \text{for } i \notin R, \tag{4.6}$$

where $c_{p_1\cdots p_r}$ is a constant close to 1, independent of i. We give in Section 4.3 several examples which satisfy (4.6). Then we have

Proposition 4.7. *Assume the estimate (4.6). For distinct r primes p_i with $y < p_i \le x$, $i = 1, \cdots, r$,*

$$\sum_{L(s,\rho,t)\in\widehat{L}(X)} \lambda(p_1,t)\cdots\lambda(p_r,t) \ll \frac{|\widehat{L}(X)|}{\sqrt{p_1\cdots p_r}} + \frac{|\widehat{L}(X)|}{(\log X)^{\frac{1}{2}}},$$

where the implied constant is independent of $p_1, \cdots, p_r$.

Note that in [4], we showed Proposition 4.7 for a single prime p.

Proof. Since $|\widehat{L}(X)| = |L(X)| + O(X^{1/100})$, it is enough to show that

$$\sum_{L(s,\rho,t)\in L(X)} \lambda(p_1,t)\cdots\lambda(p_r,t) \ll \frac{|L(X)|}{\sqrt{p_1\cdots p_r}} + \frac{|L(X)|}{(\log X)^{\frac{1}{2}}}.$$

If $k\in R$,

$$\left|\sum_{L(s,\rho,t)\in Q_k} \lambda(p_1,t)\cdots\lambda(p_r,t)\right| \le d^r\frac{|L(X)|}{p_1\cdots p_r} + O(1).$$

If $k\notin R$, p_i is unramified for all $t\in Q_k$, and $\lambda(p_i,t)=j(k_i)$ for a unique $j(k_i)$. In that case,

$$\sum_{L(s,\rho,t)\in Q_k} \lambda(p_1,t)\cdots\lambda(p_r,t) = j(k_1)\cdots j(k_r)c_{p_1\cdots p_r}\frac{|L(X)|}{p_1\cdots p_r} + O\left(\frac{|L(X)|}{p_1\cdots p_r(\log X)^{\frac{1}{2}}}\right).$$

Hence

$$\begin{aligned}\sum_{k\notin R}\sum_{L(s,\rho,t)\in Q_k} \lambda(p_1,t)\cdots\lambda(p_r,t) &= \sum_{k\notin R} j(k_1)\cdots j(k_r)|Q_k| \\ &= c_{p_1\cdots p_r}\frac{|L(X)|}{p_1\cdots p_r}\sum_{k\notin R} j(k_1)\cdots j(k_r) + O\left(\frac{|L(X)|}{(\log X)^{\frac{1}{2}}}\right) \\ &= c_{p_1\cdots p_r}\frac{|L(X)|}{p_1\cdots p_r}\prod_{u=1}^{r}\left(\sum_{i=-1}^{d} iA_{i,p_u}\right) + O\left(\frac{|L(X)|}{(\log X)^{\frac{1}{2}}}\right).\end{aligned}$$

By (4.5),

$$\sum_{L(s,\rho,t)\in L(X)} \lambda(p_1,t)\cdots\lambda(p_r,t) \ll \frac{|L(X)|}{\sqrt{p_1\cdots p_r}} + \frac{|L(X)|}{(\log X)^{\frac{1}{2}}}. \qquad \square$$

By Proposition 4.7 and (4.4) we have

$$E(r,X) \ll |\widehat{L}(X)|(\log\log X)^{r-1}.$$

We summarize our discussion as

Theorem 4.8. *Assume the strong Artin conjecture for $L(s,\rho,t)$, and the L-functions in $L(X)$ are distinct. Assume the estimate (4.6). Then*

$$\frac{1}{|L(X)|}\sum_{L(s,\rho,t)\in L(X)}\left(-\frac{L'}{L}(1,\rho,t)\right)^k = d^k(\log\log X)^k + O\left((\log\log X)^{k-\frac{1}{2}}\right).$$

4.2. Families with different moments

In the previous sections, we chose the trivial conjugacy class $[e]$ so that $\lambda(p,t) = d$ for $c_f \le p \le y$. But for any conjugacy class C of G, by Theorem 4.1, there is a constant c_f depending on f such that we can choose, for any prime $p \ge c_f$, an integer $i_{p,C}$ so that for any $t \equiv i_{p,C} \pmod p$, Frob_p belongs to C. Let $i_{M,C}$ be an integer such that $i_{M,C} \equiv i_{p,C} \pmod p$ for all $c_f \le p \le y$. So if $t \equiv i_{M,C} \pmod M$, for all $c_f \le p \le y$, Frob_p belongs to C and $\lambda(p,t) = \chi(C)$, where χ is the trace of the representation ρ.

As we did before, we define a set $L(X)$ given by

$$L(X) = \left\{\frac{X}{2} < t < X \mid t \equiv i_{M,C} \pmod M,\ \mathrm{Gal}(\widehat{K_t}/\mathbb{Q}) \simeq G\right\}.$$

Then the k-th moment of $-\frac{L'}{L}(1,\rho,t)$ is

$$\frac{1}{|L(X)|}\sum_{\rho\in\widehat{L}(X)}\left(-\frac{L'}{L}(1,\rho,t)\right)^k = \chi(C)^k(\log\log X)^k + O\left((\log\log X)^{k-\frac{1}{2}}\right).$$

4.3. Examples

We recall concrete examples from [4,5].

4.3.1. Quadratic fields

Consider $K_t = \mathbb{Q}[\sqrt{t}]$ for t square free and $t \equiv 1 \pmod 4$. For $M = 4\prod_{3\le p\le y} p$, we define

$$L(X) = \left\{\frac{X}{2} < t < X |\ t \text{ square-free and } t \equiv s_M \pmod M\right\}.$$

For $t \in L(X)$, since the conductor for $L(s,\rho,t)$ is t, all L-functions in $L(X)$ is distinct. In this case,

$$Q_i = \left\{\frac{X}{2} < t < X |\ t \text{ square-free},\ t \equiv s_M \pmod M,\ t \equiv i \pmod{p_1p_2\cdots p_r}\right\}.$$

Since $p_j > y$ for $j = 1, \cdots, r$, $(p_1p_2\cdots p_r, M) = 1$. If $i \not\equiv 0 \pmod{p_j}$ for all j, then by [7], p. 248, we have

$$|Q_i| = \frac{3}{\pi^2}\prod_{q|M}(1-q^{-2})^{-1}\frac{X}{M}\left[\prod_{j=1}^{r}(1-p_j^{-2})^{-1}\frac{1}{p_j}\right] + O(X^{1/2})$$

$$= c_{p_1\cdots p_r}\frac{|L(X)|}{p_1\cdots p_r} + O(X^{1/2}),$$

where $c_{p_1\cdots p_r} = \prod_{j=1}^{r}(1-p_j^{-2})^{-1}$, which is independent of i. Since $p_j \ll (\log X)^m$, it follows that $X^{1/2} \ll \frac{|L(X)|}{p_1\cdots p_r(\log X)^{1/2}}$ and (4.6) is verified.

4.3.2. A_5 quintic fields

Consider the polynomial $f(x,t) = x^5 + 5(5t^2-1)x - 4(5t^2-1)$, where $5t^2-1$ is square free. Here the discriminant of $f(x,t)$ is $\mathrm{disc}(f(x,t)) = 2^8 5^6 t^2 (5t^2-1)^4$.

Let

$$L(X) = \left\{ \frac{X}{2} < t < X \mid 5t^2-1 \text{ square-free}, t \text{ even}, t \equiv i_M \pmod{M} \right\}.$$

In [4], we showed that the splitting field of $f(x,t)$ over $\mathbb{Q}(t)$ is an A_5 regular extension and $L(s,\rho,t)$ is a cuspidal automorphic L-functions of $GL_4/\mathbb{Q}$. Also it is shown that $L(s,\rho,t)$'s are distinct. Here

$$Q_i = \left\{ \frac{X}{2} < t < X |\ 5t^2-1 \text{ square-free},\ t \text{ even}, \right.$$
$$\left. t \equiv s_M \pmod{M},\ t \equiv i \pmod{p_1\cdots p_r} \right\}.$$

Let R be the set of solutions $t \pmod{p_1\cdots p_r}$ for $disc(f(x,t)) \equiv 0 \pmod{p_1\cdots p_r}$. Since $p_j > y$ for $j = 1,\cdots,r$, $(p_1p_2\cdots p_r, M) = 1$. For $i \notin R$, by [9], we have

$$|Q_i| = \prod_{q\nmid M}\left(1-\left(1+\left(\frac{5}{q}\right)\right)q^{-2}\right)\frac{X}{2M}\left[\prod_{j=1}^{r}\left(1-\left(1+\left(\frac{5}{q}\right)\right)p_j^{-2}\right)^{-1}\frac{1}{p_j}\right]$$
$$+ O(X^{2/3}\log X)$$
$$= c_{p_1\cdots p_r}\frac{|L(X)|}{p_1\cdots p_r} + O(X^{2/3}\log X),$$

where $c_{p_1\cdots p_r} = \prod_{j=1}^{r}\left(1-\left(1+\left(\frac{5}{q}\right)\right)p_j^{-2}\right)^{-1}$, which is independent of i. Since $p_j \ll (\log X)^m$, it follows that $X^{2/3}\log X \ll \frac{|L(X)|}{p_1\cdots p_r(\log X)^{1/2}}$, and (4.6) is verified.

4.3.3. S_5 quintic fields

In [5], we considered a parametric polynomial $f(x,t) = x^5 + tx + t$ such that $\mathrm{disc}(f(x,t)) = t^4(256t + 3125)$. We showed that the splitting field of $f(x,t)$ over $\mathbb{Q}(t)$ is a regular S_5 extension. Let

$$L(X) = \left\{ \frac{X}{2} < t < X \mid t \text{ square-free}, t \equiv 1 \pmod{5}, t \equiv i_M \pmod{M} \right\}.$$

The congruence condition $t \equiv 1 \pmod 5$ is required to show that $L(s, \rho, t)$ is a cuspidal automorphic L-function of $GL_4/\mathbb{Q}$ using a result of Calegari [3]. We showed that all $L(s, \rho, t)$'s are distinct, and (4.6) can be verified as in the case of quadratic fields.

Remark 4.9. Theorem 4.8 is valid for all the examples in [4] except possibly for A_4 examples.

5. Moments of $\log L(1, \rho)$

Suppose $L(s, \rho)$ is entire and is zero free in the rectangle $[\alpha, 1] \times [-x, x]$, and N is the conductor of ρ. Then as in (2.3), we have

$$\log L(1, \rho) = \sum_{2 \le n < x} \frac{\Lambda(n) a_\rho(n)}{n \log n} + O\left(\frac{\log N \log x + (\log x)^2}{x^{1 - \frac{\alpha+1}{2}}} \right). \tag{5.1}$$

This implies the following approximation of $\log L(1, \rho)$ as a sum over a short interval:

$$\log L(1, \rho) = \sum_{p \le (\log N)^m} \frac{a_\rho(p)}{p} + O_\alpha(1), \quad m\left(1 - \frac{\alpha+1}{2}\right) > 3. \tag{5.2}$$

For those $L(s, \rho)$ which may not have the desired zero-free region, we use an unconditional bound given by Louboutin [15]: $L(1, \rho) \ll (\log |d_K|)^d$.

Applying (5.2) to $L(s, \rho, t) \in \widehat{L}(X)$ in Section 4.1, we have

$$\log L(1, \rho, t) = d \log \log \log X + \sum_{y \le p < (\log X)^m} \frac{a_\rho(p)}{p} + O(\log \log \log \log X).$$

As we did in section 4, we can show that the k-th moment for this family is

$$\frac{1}{|L(X)|} \sum_{L(s,\rho,t) \in L(X)} (\log L(1, \rho, t))^k = d^k (\log \log \log X)^k + O((\log \log \log X)^{k-1/2}).$$

For G_{d+1}-fields, we can show as in Sections 3,

$$\frac{1}{|L(X)|} \sum_{\rho \in L(X)} (\log L(1, \rho))^k = O(1).$$

In order to obtain the asymptotic formula, we use (5.1). Write

$$\log L(1, \rho) = \Sigma_\rho + C_\rho, \quad \Sigma_\rho = \sum_{n < (\log X)^m} \frac{\Lambda(n) a_\rho(n)}{n \log n},$$

and $C_\rho = O\left(\frac{1}{\log X}\right)$. Then

$$\sum_{\rho\in L(X)} (\log L(1,\rho))^k = \sum_{\rho\in L(X)} (\Sigma_\rho)^k + \sum_{\rho\in L(X)} \sum_{r=0}^{k-1} \binom{k}{r} (\Sigma_\rho)^r C_\rho^{k-r}.$$

As before, we can show

Proposition 5.3. *Let $L(X)$ be the set of S_{d+1}-fields as in Section 3. Let $f_1(q), f_1(q,\mathbf{a},\mathbf{e})$, and $A(q,\mathbf{a},\mathbf{e})$ be as in Theorem 3.4. Then*

$$\frac{1}{|L(X)|}\sum_{\rho\in L(X)} \log L(1,\rho) = \sum_{p,\, u\geq 2} \frac{A(p,u,1)+f_1(q,u,1)}{up^u} + \sum_p \frac{f_1(p)}{p} + O\left(\frac{1}{\log X}\right).$$

For $\mathbf{a}=(a_1,\dots,a_t)$ and $\mathbf{e}=(e_1,\dots,e_t)$, define $\mathbf{a}^{\mathbf{e}} = a_1^{e_1}a_2^{e_2}\cdots a_t^{e_t}$. Then,

$$\begin{aligned}&\frac{1}{|L(X)|}\sum_{\rho\in L(X)} (\log L(1,\rho))^k \\ &= \sum_{\substack{q_1,\dots,q_{u+v}\\ \mathbf{e}_{u+j},\mathbf{a}_{u+j},\, j=1,\dots,v}} \prod_{i=1}^{u} \frac{f_1(q_i)}{q_i} \prod_{j=1}^{v} \frac{(A(q_{u+j},\mathbf{a}_{u+j},\mathbf{e}_{u+j}) + f_1(q_{u+j},\mathbf{a}_{u+j},\mathbf{e}_{u+j}))}{\mathbf{a}_{u+j}^{\mathbf{e}_{u+j}} q_{u+j}^{|\mathbf{a}_{u+j}\mathbf{e}_{u+j}|}} \\ &+ O\left(\frac{1}{\log X}\right),\end{aligned}$$

where $q_1,\cdots,q_{u+v}$ are as in Theorem 3.4.

We write

$$\lim_{X\to\infty}\frac{1}{|L(X)|}\sum_{\rho\in L(X)} (\log L(1,\rho))^k = \tilde{r}(k).$$

Corollary 5.4. *Assume (3.2) and the strong Artin conjecture for ρ. Assume that $\tilde{r}(k) \ll c^{k\log\log k}$ for some absolute constant $c>1$. Then,*

$$\lim_{X\to\infty}\frac{1}{|L(X)|}\#\{\rho\in L(X) \mid \log L(1,\rho)\leq x\} = \tilde{F}(x),$$

at each point of continuity of $\tilde{F}(t)$, where $\tilde{F}(t)$ is a distribution function whose characteristic function is given by

$$\tilde{f}(t) = \sum_{k=0}^{\infty} \frac{\tilde{r}(k)}{k!}(it)^k.$$

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