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The Average of the Smallest Prime in a Conjugacy Class

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Let C be a conjugacy class of S_n and K an S_n-field. Let $n_{K,C}$ be the smallest prime, which is ramified or whose Frobenius automorphism Frob_p does not belong to C. Under some technical conjectures, we show that the average of $n_{K,C}$ is a constant. We explicitly compute the constant. For S_3- and S_4-fields, our result is unconditional. Let $N_{K,C}$ be the smallest prime for which Frob_p belongs to C. We obtain the average of $N_{K,C}$ under some technical conjectures. When C is the union of all the conjugacy classes not contained in A_n and $n = 3, 4$, our result is unconditional.

1 Introduction

For a fundamental discriminant D, let $\chi_D(\cdot) = (\frac{D}{\cdot})$, and let $N_{D,\pm 1}$ be the smallest prime such that $\chi_D(p) = \pm 1$, resp. Let $n_{D,\pm 1}$ be the smallest prime such that $\chi_D(p) \neq \pm 1$, resp. We can interpret $N_{D,1}$ ($N_{D,-1}$) as the smallest prime that splits completely (inert, resp.) in a quadratic field $\mathbb{Q}(\sqrt{D})$. Under the assumption of the Generalized Riemann Hypothesis (GRH) for $L(s, \chi_D)$, one can show easily that $N_{D,\pm 1}, n_{D,\pm 1} \ll (\log D)^2$. Erdös [16] considered the average of those values over a family $\lim_{X\to\infty} \frac{\sum_{2<p\leq X} N_{p,-1}}{\pi(X)} = \sum_{k=1}^{\infty} \frac{p_k}{2^k} = 3.67464\ldots$, where p runs through primes, and p_k is the k-th prime. Pollack

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[29] generalized Erdös' result to all fundamental discriminants:

$$\lim_{X\to\infty}\frac{\sum_{|D|\le X} N_{D,\pm 1}}{\sum_{|D|\le X} 1}=\sum_q \frac{q^2}{2(q+1)}\prod_{p<q}\frac{p+2}{2(p+1)}=4.98094\ldots,$$

where p and q denote primes. (In [25], the value 4.98094... is misquoted as 4.98085.) Pollack [30] also computed the average of the least inert primes over cyclic number fields of prime degree.

We generalize this problem to the setting of general number fields. We call a number field K of degree n, an S_n-field if its Galois closure $\widehat{K}$ over $\mathbb{Q}$ is an S_n Galois extension. Let $L_n^{(r_2)}(X)$ be the set of S_n-fields K of signature (r_1, r_2) with $|d_K|\le X$, where d_K is the discriminant of K.

Let C be a conjugacy class of S_n. For an unramified prime p, denote by Frob_p, a Frobenius automorphism of p. Define $n_{K,C}$ to be the smallest prime p, which is ramified in K or for which $\mathrm{Frob}_p\notin C$. Similarly define $N_{K,C}$ to be the smallest prime p such that $\mathrm{Frob}_p\in C$. Under GRH, we can show that $n_{K,C}, N_{K,C}\ll(\log|d_K|)^2$. (We have $n_{K,C}, N_{K,C}\ll(\log|d_{\widehat{K}}|)^2$ (cf. [1], [26]), but we have $\log|d_{\widehat{K}}|\asymp\log|d_K|$ (cf. [28, Lemma 3.4]).)

In this paper we consider the average value of $n_{K,C}$ and $N_{K,C}$ over $L_n^{(r_2)}(X)$. First, we prove the following:

Theorem 1.1. Let $n=3,4,5$. When $n=5$, we assume either the strong Artin conjecture, or Conjecture 3.2. Then,

$$\frac{1}{|L_n^{(r_2)}(X)|}\sum_{K\in L_n^{(r_2)}(X)} n_{K,C}=\sum_q \frac{q(1-|C|/|S_n|+f(q))}{1+f(q)}\prod_{p<q}\frac{|C|/|S_n|}{1+f(p)}+O\left(\frac{1}{\log X}\right), \tag{1.1}$$

where p and q denote primes, and $f(p)$ is a certain function of primes that satisfies $f(p)=O(\frac{1}{p})$. See Section 2 for a precise form for each n.

For S_3-fields, Martin and Pollack [25] computed (1.1) for $C=\{1\}$, [(123)]. The main key ingredient was counting S_3-fields with finitely many local conditions, which is a recent result of Taniguchi and Thorne [34]. In [10], we were able to count S_4- and S_5-fields with finitely many local conditions using a result of Belabas, Bhargava, and Pomerance [2], and a result of Shankar and Tsimerman [33]. In Section 2, we review counting number fields with finitely many local conditions, which is the main tool for the proof.

Our key idea is to use the unique quadratic subextension $F=\mathbb{Q}[\sqrt{d_K}]$, which we call the quadratic resolvent. For unconditional bounds on $n_{K,C}$, we use the inequality

$n_{K,C} \le n_{F,1} \le N_{F,-1}$ or $n_{K,C} \le n_{F,-1} \le N_{F,1}$ depending on whether $C \subset A_n$ or $C \not\subset A_n$. We have unconditional bounds of $N_{F,\pm 1}$ by Norton [27] and Pollack [31]. We review this in Section 3.1.

Let $\chi_F = \chi_{d_F}$, where d_F is the discriminant of F. By using the zero-free region of $L(s, \chi_F)$, we can get conditional bounds on $n_{K,C}$. This is done in Section 3.2. Here we need to count the number of S_n-fields with the same quadratic resolvent. For S_3-fields, Cohen and Morra [11] found a formula for the number of S_3-fields having the same quadratic resolvent F. However, the dependence on F of the error term is not explicit, which makes it unsuitable for our purpose. Using a result of Cohen and Thorne [12], we obtain a bound on the number of S_3-fields having the same quadratic resolvent for which the implied constant is independent of F. But for $n \ge 4$, we could not find it in the literature. We state it as Conjecture 3.2. In Section 4, we establish (4.2) (a general version of (1.1) under the counting Conjectures (2.1) and (2.2) and Conjecture 3.2.) See Theorem 4.1 and tables below it for the average values, which were computed by using the computer algebra system PARI/GP.

In Section 5, for $n = 3, 4, 5$, we prove (1.1) without Conjecture 3.2. Instead we use the strong Artin conjecture. Since the strong Artin conjecture is true for $n = 3, 4$, Theorem 1.1 is unconditional for $n = 3, 4$. In this case, we need zero-free regions of different Artin L-functions for each conjugacy class.

In Martin and Pollack [25, Theorem 4.8], the average of $N_{K,C}$ for S_3-fields was studied under GRH. For $N_{K,C}$, we have conditional bounds $N_{K,C} \ll e^{c(\log\log|d_K|)^{\frac{5}{3}+\epsilon}}$ for some constant c [28, Theorem 1.1] under the zero-free region $[\alpha, 1] \times [-(\log|d_{\widehat{K}}|)^2, (\log|d_{\widehat{K}}|)^2]$ of $\frac{\zeta_{\widehat{K}}(s)}{\zeta(s)}$. In Section 6.1, we obtain a better bound $N_{K,C} \ll (A(1-\alpha)\log|d_{\widehat{K}}|)^{\frac{1}{1-\alpha}}$ for some positive constant A. However, we do not have good unconditional bounds for $N_{K,C}$ as we do for $n_{K,C}$. The best bound at the moment is $N_{K,C} \ll |d_{\widehat{K}}|^{40}$ [36]. In Section 6, we compute the average of $N_{K,C}$ under the assumption that $N_{K,C} \ll |d_K|^{\frac{1}{2}-\epsilon_n}$ for some constant $0 < \epsilon_n < 1/2$. Tables for the average values of $N_{K,C}$ for S_n, $n = 3, 4, 5$, are provided.

Theorem 1.2. Let $n = 3, 4, 5$. When $n = 5$, we assume the strong Artin conjecture. When $n = 4, 5$, we assume Conjecture 3.2. Under the assumption that $N_{K,C} \ll |d_K|^{\frac{1}{2}-\epsilon_n}$, we have

$$\frac{1}{|L_n^{(r_2)}(X)|} \sum_{K \in L_n^{(r_2)}(X)} N_{K,C} = \sum_q \frac{q(|C|/|S_n|)}{1+f(q)} \prod_{p<q} \frac{1-|C|/|S_n|+f(p)}{1+f(p)} + O\left(\frac{1}{\log X}\right), \tag{1.2}$$

where p and q denote primes.

In Section 6.4, we obtain an unconditional result for the average of N_{K,C_u}, where C_u is the union of all the conjugacy classes not in A_n, since in this case we have a good unconditional bound. In particular, we have

Theorem 1.3. For S_3-fields and $C = [(12)]$, and S_4-fields and $C_u = [(1234)] \cup [(12)]$, we have an unconditional result for the average of N_{K,C_u}. For S_3-fields and $C = [(12)]$, the average value of $N_{K,C}$ is $5.36802\ldots$. For S_4-fields and $C_u = [(1234)] \cup [(12)]$, the average value of N_{K,C_u} is $5.821569\ldots$.

Under the strong Artin conjecture, it is true for S_5-fields: for $C_u = [(12)(345)] \cup [(12)] \cup [(1234)]$, the average value of N_{K,C_u} is $5.9733589\ldots$.

In the Appendix, using a result of Cohen and Thorne [13], we count the number of S_4-fields with the given cubic resolvent. In Section 5, this result is used to compute the average of $n_{K,C}$, $C = (123)$ for S_4-fields.

2 Counting Number Fields with Local Conditions

Let K be a S_n-field for $n \geq 3$. Let $\mathcal{S} = (\mathcal{LC}_p)$ be a finite set of local conditions; $\mathcal{LC}_p = S_{p,C}$ means that p is unramified and the conjugacy class of Frob_p is C. Define $|S_{p,C}| = \frac{|C|}{|S_n|(1+f(p))}$ for some function $f(p)$, which satisfies $f(p) = O(\frac{1}{p})$. There are also several splitting types of ramified primes, which are denoted by $r_1, r_2, \ldots, r_w$; $\mathcal{LC}_p = S_{p,r_j}$ means that p is ramified and its splitting type is r_j. We assume that there are positive-valued functions $c_1(p)$, $c_2(p)$, $\ldots$, $c_w(p)$ with $\sum_{i=1}^{w} c_i(p) = f(p)$ and define $|S_{p,r_i}| = \frac{c_i(p)}{1+f(p)}$. Let $|\mathcal{S}| = \prod_p |\mathcal{LC}_p|$.

Conjecture 2.1. Let $L_n^{(r_2)}(X; \mathcal{S})$ be the set of S_n-fields K of signature (r_1, r_2) with $|d_K| < X$ and the local conditions $\mathcal{S}$. Then

$$|L_n^{(r_2)}(X)| = A(r_2)X + O(X^{\delta}), \tag{2.1}$$

$$|L_n^{(r_2)}(X; \mathcal{S})| = |\mathcal{S}|A(r_2)X + O\left(\left(\prod_{p \in S} p\right)^{\gamma} X^{\delta}\right) \tag{2.2}$$

for some positive constants $A(r_2)$, $\delta < 1$ and γ, and the implied constant is uniformly bounded for p and local conditions at p.

This conjecture is true for S_3-, S_4- and S_5-fields. See below for precise values of $A(r_2)$ for $n = 3, 4, 5$. For S_3-fields, we use a result of Taniguchi and Thorne [34]. Let

$f(p) = p^{-1} + p^{-2}$. Put

$$|S_{p,r_j}| = \frac{p^{-1}}{1+f(p)}, \ \frac{p^{-2}}{1+f(p)},$$

for $r_j = (1^2 1), (1^3)$, respectively. Then

Theorem 2.2. [34] Let $D_0 = \frac{1}{12\zeta(3)}$, $D_1 = \frac{3}{12\zeta(3)}$.

$$|L_3^{(r_2)}(X, \mathcal{S})| = |\mathcal{S}| D_{r_2} X + O\left(\left(\prod_{p \in \mathcal{S}} p \right)^{\frac{16}{9}} X^{\frac{5}{6}} \right).$$

For S_4-fields, take $f(p) = p^{-1} + 2p^{-2} + p^{-3}$. For a conjugacy class C of S_4, let

$$|S_{p,C}| = \frac{|C|}{24(1+f(p))}.$$

Put

$$|S_{p,r_j}| = \frac{1/2 \cdot 1/p}{1+f(p)}, \ \frac{1/2 \cdot 1/p}{1+f(p)}, \ \frac{1/2 \cdot 1/p^2}{1+f(p)}, \ \frac{1/2 \cdot 1/p^2}{1+f(p)}, \ \frac{1/p^2}{1+f(p)}, \text{ and } \frac{1/p^3}{1+f(p)}$$

for $r_j = (1^2 11), (1^2 2), (1^2 1^2), (2^2), (1^3 1), (1^4)$, respectively. By using the results of [2], [35], we showed the following:

Theorem 2.3. [10] Let $D_i = d_i \prod_p (1 + p^{-2} - p^{-3} - p^{-4})$, and $d_0 = \frac{1}{48}$, $d_1 = \frac{1}{8}$, and $d_2 = \frac{1}{16}$.

$$|L_4^{(r_2)}(X, \mathcal{S})| = |\mathcal{S}| D_{r_2} X + O_\epsilon\left(\left(\prod_{p \in \mathcal{S}} p \right)^{2} X^{\frac{143}{144}+\epsilon} \right).$$

For S_5-fields, take $f(p) = p^{-1} + 2p^{-2} + 2p^{-3} + p^{-4}$. For a conjugacy class C of S_5, let

$$|S_{p,C}| = \frac{|C|}{120(1+f(p))}.$$

Put

$$\left|S_{p,r_j}\right| = \frac{1/6 \cdot 1/p}{1+f(p)}, \ \frac{1/2 \cdot 1/p}{1+f(p)}, \ \frac{1/3 \cdot 1/p}{1+f(p)}, \ \frac{1/2 \cdot 1/p^2}{1+f(p)}, \ \frac{1/2 \cdot 1/p^2}{1+f(p)}, \ \frac{1/2 \cdot 1/p^2}{1+f(p)}, \ \frac{1/2 \cdot 1/p^2}{1+f(p)},$$
$$\frac{1/p^3}{1+f(p)}, \ \frac{1/p^3}{1+f(p)}, \text{ and } \frac{1/p^4}{1+f(p)}$$

for $r_j = (1^2111), (1^212), (1^23), (1^21^21), (2^21), (1^311), (1^32), (1^31^2), (1^41), (1^5)$, respectively.

By using the result of [33], we showed

Theorem 2.4. [10] Let $D_i = d_i \prod_p (1 + p^{-2} - p^{-4} - p^{-5})$ and d_0, d_1, d_2 are $\frac{1}{240}$, $\frac{1}{24}$, and $\frac{1}{16}$, respectively.

$$|L_5^{(r_2)}(X, \mathcal{S})| = |\mathcal{S}| D_{r_2} X + O_\epsilon \left(\left(\prod_{p \in \mathcal{S}} p \right)^{2-\epsilon} X^{\frac{199}{200}+\epsilon} \right).$$

3 Bounds on $n_{K,C}$

Recall that $n_{K,C}$ is the smallest prime, which is ramified in K or for which Frob_p does not belong to C. Now $\widehat{K}$ has the quadratic field F fixed by A_n, that is, $F = \mathbb{Q}[\sqrt{d_K}]$. Let d_F be the discriminant of F. Then clearly, $|d_F| \leq |d_K|$. By abuse of language, we call such F the quadratic resolvent of K.

If $C \subset A_n$ and $Frob_p \in C$, then p splits in F. Hence, $n_{K,C} \leq n_{F,1}$.

If $C \not\subset A_n$ and $Frob_p \in C$, then p is inert in F. Hence, $n_{K,C} \leq n_{F,-1}$.

3.1 Unconditional bounds of $n_{K,C}$

By Norton [27], $N_{F,-1} \ll_\epsilon |d_F|^{\frac{1}{4\sqrt{e}}+\epsilon} \ll_\epsilon |d_K|^{\frac{1}{4\sqrt{e}}+\epsilon}$. Since $n_{F,1} \leq N_{F,-1}$,

$$n_{K,C} \ll_\epsilon |d_F|^{\frac{1}{4\sqrt{e}}+\epsilon} \ll_\epsilon |d_K|^{\frac{1}{4\sqrt{e}}+\epsilon} \text{ for } C \subset A_n.$$

By Pollack [31], $N_{F,1} \ll_\epsilon |d_F|^{\frac{1}{4}+\epsilon} \ll_\epsilon |d_K|^{\frac{1}{4}+\epsilon}$. Since $n_{F,-1} \leq N_{F,1}$,

$$n_{K,C} \ll_\epsilon |d_F|^{\frac{1}{4}+\epsilon} \ll_\epsilon |d_K|^{\frac{1}{4}+\epsilon} \text{ for } C \not\subset A_n.$$

3.2 Conditional bounds of $n_{K,C}$

We obtain conditional bounds on $n_{F,C}$ under the zero-free region of $L(s, \chi_F)$, where $\chi_F(p) = (\frac{d_F}{p})$. Suppose $L(s, \chi_F)$ is zero-free on $[\alpha, 1] \times [-(\log|d_F|)^2, (\log|d_F|)^2]$. Then by [8],

$$-\frac{L'}{L}(\sigma, \chi_F) = \sum_{p < (\log|d_F|)^{16/(1-\alpha)}} \frac{\chi_F(p)\log p}{p^\sigma} + O(1),$$

for $1 \leq \sigma \leq 3/2$. Hence, it implies that

$$\left| -\frac{L'}{L}(\sigma, \chi_F) \right| \leq \frac{16}{(1-\alpha)} \log\log|d_F| + O(1).$$

Now for $C \subset A_n$, consider $\zeta_F(s) = \zeta(s)L(s, \chi_F)$. We obtain

$$\sum_p \frac{(1 + \chi_F(p)) \log p}{p^\sigma} = \frac{1}{\sigma - 1} - \frac{L'}{L}(\sigma, \chi_F) + O(1).$$

For a while, we assume that $n_{K,C} \geq 3$. For each prime $p < n_{K,C}$, the prime p splits in the quadratic resolvent F. (i.e., $\chi_F(p) = 1$ for all $p < n_{K,C}$.)

Then

$$\sum_{p < n_{K,C}} \frac{2 \log p}{p^\sigma} \leq \frac{1}{\sigma - 1} - \frac{L'}{L}(\sigma, \chi_F) + O(1).$$

By taking $\sigma - 1 = \frac{\lambda}{\log n_{K,C}}$, we have

$$\frac{1 - 2e^{-\lambda}}{2\lambda} \log n_{K,C} \leq \frac{16}{(1 - \alpha)} \log \log |d_F| + O(1).$$

Hence,

$$n_{K,C} \ll (\log |d_F|)^{\frac{16}{(1-\alpha)A}} \ll (\log |d_K|)^{\frac{16}{(1-\alpha)A}},$$

where $A = \sup_{\lambda \geq 0} \frac{1-2e^{-\lambda}}{\lambda}$, which is 0.37... when $\lambda = 1.678\ldots$. When $n_{K,C} = 2$, it clearly satisfies the above inequality. Hence, we can remove the assumption $n_{K,C} \geq 3$.

Now for $C \not\subset A_n$, consider

$$\sum_p \frac{(1 - \chi_F(p)) \log p}{p^\sigma} = \frac{1}{\sigma - 1} + \frac{L'}{L}(\sigma, \chi_F) + O(1).$$

For each prime $p < n_{K,C}$, the prime p is inert in F. (i.e., $\chi_F(p) = -1$ for all $p < n_{K,C}$.) We have the inequality

$$\sum_{p < n_{K,C}} \frac{2 \log p}{p^\sigma} \leq \frac{1}{\sigma - 1} + \frac{L'}{L}(\sigma, \chi_F) + O(1).$$

Hence,

$$n_{K,C} \ll (\log |d_F|)^{\frac{16}{(1-\alpha)A}} \ll (\log |d_K|)^{\frac{16}{(1-\alpha)A}}. \tag{3.1}$$

(Here we implicitly assume that $n_{K,C} \geq 3$ and remove the restriction after we obtain the upper bound.)

Because we lack the GRH, we cannot use the above bound directly. In [7], we extended the result of Kowalski and Michel [22] to isobaric automorphic representations of $GL(n)$.

Let $n = n_1 + n_2 + \cdots + n_r$, and let $S(X)$ be a set of isobaric representations $\pi = \pi_1 \boxplus \pi_2 \boxplus \cdots \boxplus \pi_r$, where each π_j is a cuspidal automorphic representation of $GL(n_j)/\mathbb{Q}$ and satisfies the Ramanujan-Petersson conjecture at the finite places. Suppose $S(X)$ satisfies the following conditions:

(1) There exists $e > 0$ such that for $\pi = \pi_1 \boxplus \pi_2 \boxplus \cdots \boxplus \pi_r \in S(X)$, $Cond(\pi_1) \cdots Cond(\pi_r) \leq X^e$;
(2) There exists $d > 0$ such that $|S(X)| \leq X^d$;
(3) The Γ-factors of π_j are of the form $\prod_{k=1}^{n_j} \Gamma(s/2 + \alpha_k)$ where $\alpha_k \in \mathbb{R}$;
(4) Given two representations $\pi, \pi' \in S(X)$, for each j, π_j is not equivalent to any π'_k if $n_j = n_k$.

For $\alpha \geq 3/4$ and $T \geq 2$, let

$$N(\pi; \alpha, T) = |\{\rho : L(\rho, \pi) = 0, Re(\rho) \geq \alpha, |Im(\rho)| \leq T\}|.$$

Then, clearly $N(\pi; \alpha, T) = N(\pi_1; \alpha, T) + \cdots + N(\pi_r; \alpha, T)$.

Theorem 3.1. ([7, Theorem 3.4]) Let $S(X)$ be as above. Then for some $B \geq 0$,

$$\sum_{\pi \in S(X)} N(\pi; \alpha, T) \leq T^B X^{c_0(1-\alpha)/(2\alpha-1)}.$$

One can choose any $c_0 > c'_0$, where $c'_0 = 5n'e/2 + d$ with $n' = \max\{n_i\}_{1 \leq i \leq r}$.

In application of Theorem 3.1 to a family $S(X)$, it is very important to estimate the size of the set of L-functions $L(s, \pi)$ coming from $S(X)$. Our L-functions in consideration are Artin L-functions associated with S_n-fields. To show that they are all distinct, arithmetic equivalence of number fields is used. Two number fields K and F are arithmetically equivalent if $\zeta_K(s) = \zeta_F(s)$. We say that a number field K is arithmetically solitary if $\zeta_K(s) = \zeta_F(s)$ implies that K and F are conjugate. It is known that S_n-fields are arithmetically solitary [20, Chapter II].

Let

$$L(X)^{\pm} = \{F | F : \text{quadratic field}, \pm d_F \leq X\}.$$

We may treat $L(X)^{\pm}$ as families of quadratic Dirichlet L-functions $L(s, \chi_F)$. We apply Theorem 3.1 to $L(X)^{\pm}$ with $d = e = 1$, $T = \log X$, and $q = X$. Since distinct fundamental discriminants give distinct L-functions $L(s, \chi_F)$, $|L(X)^{\pm}| = c_{\pm} X + O(\sqrt{X})$ for some positive constants $c_{\pm}$. For any $\epsilon > 0$, by choosing α close to 1 so that $c_0(1-\alpha)/(2\alpha-1) < \epsilon$. Then we can show that every L-function in $L(X)^{\pm}$ is zero-free on $[\alpha, 1] \times [-(\log X)^2, (\log X)^2]$ except for $O(X^{\epsilon})$ L-functions. In Section 5, we will apply Theorem 3.1 to various families of L-functions.

In order to use this result, we need to count the number of S_n-fields with the common quadratic resolvent. Let F be a quadratic field, and let $QR_n(X,F)$ be the set of S_n-fields with the common quadratic resolvent F and the absolute value of the discriminant bounded by X. Since $d_K = d_F m^2$ for some $m \in \mathbb{Z}^+$, by the counting Conjecture (2.1), it is expected that $|QR_n(X,F)| \ll (X/|d_F|)^{1/2}$. For our purpose, a weaker bound is enough.

Conjecture 3.2. There exist constants $\alpha_n > 0, 0 < \beta_n < 1/2$ such that

$$|QR_n(X,F)| \ll X^{\frac{1}{2}} (\log X)^{\alpha_n} |d_F|^{-\frac{1}{2}+\beta_n},$$

where the implied constant is independent of F.

When $n = 3$, Cohen and Morra [11] found a formula for the number of S_3-fields having the same quadratic resolvent F. However, in their result, the dependence on F of the error term is not explicit, which makes it unsuitable for our purpose. We use a result of Cohen and Thorne [12]: given a quadratic field F, let

$$\Phi_F(s) = \frac{1}{2} + \sum_{K \in \mathcal{F}(F)} \frac{1}{f(K)^s},$$

where $d_K = d_F f(K)^2$, and $\mathcal{F}(F)$ is the set of all cubic fields K with the quadratic resolvent field F. Let D be a fundamental discriminant, and $D^* = -3D$ if $3 \nmid D$, and $D^* = -\frac{D}{3}$ if $3|D$. Define $\mathcal{L}_3(D) = \mathcal{L}_{D^*} \cup \mathcal{L}_{-27D}$, where $\mathcal{L}_N$ is the set of cubic fields of discriminant N. By Theorem 2.5 in [12],

$$\Phi_F(s) = \sum_{i=1}^{|\mathcal{L}_3(d_F)+1|} \Phi_i(s), \quad \Phi_i(s) = \sum_{n=1}^{\infty} \frac{a_i(n)}{n^s},$$

and $a_i(n) \le 2^{\omega(n)} \ll 2^{\frac{\log n}{\log\log n}} \ll n^{\epsilon}$. Hence, each $\Phi_i(s)$ is absolutely convergent for $Re(s) > 1$. We also have $\Phi_i(1+c+it) \ll (\frac{\zeta(1+c)}{\zeta(2+2c)})^2 \ll \frac{1}{c^2}$ with an absolute implied constant. Now we apply Perron's formula to each $\Phi_i(s)$. For $c = \frac{1}{\log x}$,

$$\sum_{n<x} a_i(n) = \int_{1+c-ix}^{1+c+ix} \Phi_i(1+c+it)\frac{x^s}{s}\, ds + O\left(x^{\epsilon}\right),$$

with an absolute implied constant. Since $\Phi_i(1+c+it) \ll (\log x)^2$, the integral is majorized by $x(\log x)^3$. So $\sum_{n<x} a_i(n) \ll x(\log x)^3$. By [15], $|\mathcal{L}_N| \ll |N|^{\frac{1}{3}+\epsilon}$. Hence,

$$|\{K \in \mathcal{F}(F) | f(K) \le x\}| = \sum_{i=1}^{|\mathcal{L}_3(d_F)+1|} \sum_{n<x} a_i(n) \ll |d_F|^{\frac{1}{3}+\epsilon} x(\log x)^3.$$

Therefore,

$$|\{K \in \mathcal{F}(F) \,|\, f(K) \le x\}| \ll x(\log x)^3 |d_F|^{\frac{1}{3}+\epsilon}.$$

Hence, we have proved

Proposition 3.3.

$$|QR_3(X,F)| \ll X^{\frac{1}{2}}(\log X)^3 |d_F|^{-\frac{1}{6}+\epsilon},$$

where the implied constant is independent of F.

Remark 3.4. Martin and Pollack [25] obtained a bound $|QR_3(X,F)| \ll_{\epsilon} X^{\frac{5}{6}+\epsilon}|d_F|^{-\frac{1}{2}}$, which is enough in computing the average of $n_{K,C}$ for S_3-fields, but not enough for $N_{K,C}$.

Remark 3.5. A recent preprint by Pierce, Turnage-Butterbaugh, and Wood [28] generalizes Theorem 3.1. Essentially what they do is to replace the condition (4) with the following weaker condition [28, (6.6)]: for any i and any $\pi \in S(X)$,

$$\left|\{\pi' \in S(X) \mid \pi_i' \text{ is equivalent to } \pi_i\}\right| \ll X^{\tau} \tag{3.2}$$

for some positive constant $\tau < d$. For $K \in L_n^{(r_2)}(X)$, write $\frac{\zeta_{\widehat{K}}(s)}{\zeta(s)} = L(s,\chi_F)\prod_{\psi} L(s,\psi)^{\psi(1)}$, where F is the quadratic resolvent of K, and ψ runs over the irreducible characters of S_n such that $\psi(1) > 1$. When we consider

$$S(X) = \left\{\frac{\zeta_{\widehat{K}}(s)}{\zeta(s)} \,|\, K \in L(X)\right\},$$

Conjecture 3.2 is a refinement of Condition (3.2) when $\pi_i = \chi_F$.

4 Average Value of $n_{K,C}$

In this section, we prove Theorem 4.1 (Theorem 1.1) under Conjecture 3.2, and the counting Conjectures (2.1) and (2.2).

For simplicity of notation, we denote $L_n^{(r_2)}(X)$ by $L(X)$. Take $y = \frac{1-\delta}{4\gamma}\log X$, where δ and γ are the constants in (2.1) and (2.2). Then

$$\sum_{K\in L(X)} n_{K,C} = \sum_{K\in L(X),\, n_{K,C}\le y} n_{K,C} + \sum_{K\in L(X),\, n_{K,C}>y} n_{K,C}.$$

Here $n_{K,C} = q$ means that for all primes $p < q$, $\mathrm{Frob}_p \in C$ and q is ramified or $\mathrm{Frob}_q \notin C$. By the counting conjectures, there are

$$\frac{1-|C|/|S_n|+f(q)}{1+f(q)}\prod_{p<q}\frac{|C|/|S_n|}{1+f(p)}A(r_2)X + O\left(X^{\frac{1+3\delta}{4}}\right)$$

such number fields in $L(X)$. Hence,

$$\begin{aligned}
\sum_{K\in L(X),\, n_{K,C}\le y} n_{K,C} &= \sum_{q\le y} q \sum_{K\in L(X),\, n_{K,C}=q} 1\\
&= A(r_2)X\sum_{q\le y}\frac{q(1-|C|/|S_n|+f(q))}{1+f(q)}\prod_{p<q}\frac{|C|/|S_n|}{1+f(p)} + O\left(y^2 X^{\frac{1+3\delta}{4}}\right)\\
&= A(r_2)X\sum_{q}\frac{q(1-|C|/|S_n|+f(q))}{1+f(q)}\prod_{p<q}\frac{|C|/|S_n|}{1+f(p)}\\
&\quad + O\left(X\sum_{q>y} q\prod_{p<q}(|C|/|S_n|) + (\log X)^2 X^{\frac{1+3\delta}{4}}\right).
\end{aligned}$$

Since

$$\sum_{q>y} q\prod_{p<q}(|C|/|S_n|) \le \sum_{q>y} q\left(\frac{|C|}{|S_n|}\right)^{\pi(q)} \ll \frac{1}{\log X},$$

we have

$$\sum_{K\in L(X),\, n_{K,C}\le y} n_{K,C} = A(r_2)X\sum_{q}\frac{q(1-|C|/|S_n|+f(q))}{1+f(q)}\prod_{p<q}\frac{|C|/|S_n|}{1+f(p)} + O\left(\frac{X}{\log X}\right).$$

Now we divide the sum $\sum_{|d_K|\le X,\, n_{K,C}>y} n_{K,C}$ into two subsums. Let $E(X)$ be the set of S_n-fields K in $L(X)$ for which the quadratic L-function $L(s,\chi_F)$, where F is the quadratic resolvent of K, may not have the desired zero-free region $[\alpha, 1] \times [-(\log X)^2, (\log X)^2]$. If $K \notin E(X)$, $L(s, \chi_F)$ has no zeros in $[\alpha, 1] \times [-(\log |d_F|)^2, (\log |d_F|)^2]$.

$$\sum_{K \in L(X),\, n_{K,C} > y} n_{K,C} = \sum_{n_{K,C} > y,\, K \notin E(X)} n_{K,C} + \sum_{n_{K,C} > y,\, K \in E(X)} n_{K,C}. \tag{4.1}$$

To handle the first sum, we use the conditional bound on $n_{K,C}$ in Section 3.2.

$$\begin{aligned}\sum_{n_{K,C} > y,\, K \notin E(X)} n_{K,C} &\ll (\log X)^{\frac{16}{(1-\alpha)A}} \sum_{n_{K,C} > y} 1 \\ &\ll (\log X)^{\frac{16}{(1-\alpha)A}} \left(X \prod_{p<y} (|C|/|S_n|) + X^{\delta} \left(\prod_{p<y} p \right)^{\gamma} \right) \\ &\ll X (\log X)^{\frac{16}{(1-\alpha)A}} \left(\frac{|C|}{|S_n|} \right)^{\pi(y)} + X^{\frac{1+\delta}{2}}.\end{aligned}$$

We use the fact that $(\frac{|C|}{|S_n|})^{\pi(y)} \ll e^{-\frac{\log X}{\log\log X}} \ll (\log X)^{-k}$ for any k.

Let's deal with the second sum. From the unconditional bound in Section 3.1, $n_{K,C} \ll |d_F|^{\frac{1}{4\sqrt{e}}+\epsilon}$ or $|d_F|^{\frac{1}{4}+\epsilon}$ depending on whether $C \subset A_n$ or $C \not\subset A_n$. In any case $n_{K,C} \ll |d_F|^{\frac{1}{4}+\epsilon}$. By Conjecture 3.2, we have at most $O(X^{\frac{1}{2}} (\log X)^{\alpha_n} |d_F|^{-\frac{1}{2}+\beta_n})$ S_n-fields with the same quadratic resolvent F. In Section 3.2, we showed that there are at most X^{ϵ} quadratic fields F, which may not have the desired zero-free region. Hence, $|E(X)| \ll X^{\frac{1}{2}+2\epsilon}$. Therefore,

$$\sum_{n_{K,C} > y,\, K \in E(X)} n_{K,C} \ll X^{\frac{1}{2}+2\epsilon} X^{\frac{1}{4}+\epsilon} \ll X^{\frac{3}{4}+3\epsilon}.$$

Our discussion is summarized as follows:

Theorem 4.1. Let $L_n^{(r_2)}(X)$ be the set of S_n-fields K of signature (r_1, r_2) with $|d_K| < X$. Assume the counting Conjectures (2.1) and (2.2) and Conjecture 3.2. Let C be a conjugacy class of S_n and $n_{K,C}$ be the least prime with $\mathrm{Frob}_p \notin C$. Then,

$$\frac{1}{|L_n^{(r_2)}(X)|} \sum_{K \in L_n^{(r_2)}(X)} n_{K,C} = \sum_{q} \frac{q(1 - |C|/|S_n| + f(q))}{1 + f(q)} \prod_{p<q} \frac{|C|/|S_n|}{1+f(p)} + O\left(\frac{1}{\log X} \right), \tag{4.2}$$

where the sum and the product are over primes.

For S_3-fields, the counting Conjectures (2.1) and (2.2) and Conjecture 3.2 are true. Hence, the above theorem holds unconditionally. For S_4- and S_5-fields, the counting Conjectures (2.1) and (2.2) are true. Hence, under Conjecture 3.2, Theorem 4.1 holds for S_4- and S_5-fields.

The tables below show average values of $n_{K,C}$ for S_3-, S_4-, and S_5-fields. The computations are done by PARI.

S_3	Average of $n_{K,C}$
$\{1\}$	2.1211027...
$[(12)]$	2.6719625...
$[(123)]$	2.3192802...

S_4	Average of $n_{K,C}$
$\{1\}$	2.0206694...
$[(12)(34)]$	2.0691556...
$[(1234)]$	2.1653006...
$[(12)]$	2.1653006...
$[(123)]$	2.2516575...

S_5	Average of $n_{K,C}$
$\{1\}$	2.0036404...
$[(12)(34)]$	2.0632551...
$[(123)]$	2.0891619...
$[(12)(345)]$	2.0891619...
$[(12)]$	2.0399630...
$[(1234)]$	2.1505010...
$[(12345)]$	2.1120340...

Remark 4.2. From the tables above, we can see that the average value of $n_{K,C}$ is close to 2 and $n_{K,C} < n_{K,C'}$ if $|C| < |C'|$. In fact, it is expected from the formula for the average value of $n_{K,C}$. The probability for $n_{K,C}$ to be 2 is $\frac{1-|C|/|S_n|+f(2)}{1+f(2)}$, which happens to most of the number fields. For example, for S_5-fields, the probability for $n_{K,[e]}$ to be 2 is 0.996396....

Remark 4.3. Let $L(X)^{\pm}$ be the set of real/complex quadratic extension F with $\pm d_F \leq X$. For the sake of completeness, we record the average of $n_{F,\pm 1}$. It is easy to check that the probabilities for a prime p is to ramify, split, or be inert are $\frac{1}{p+1}$, $\frac{p}{2(p+1)}$, or $\frac{p}{2(p+1)}$, respectively. Hence,

$$\lim_{X\to\infty} \frac{\sum_{\pm d_F \leq X} n_{F,\pm 1}}{|L(X)^{\pm}|} = \sum_q \frac{q^2+2q}{2(q+1)} \prod_{p<q} \frac{p}{2(p+1)} = 2.83264\ldots.$$

5 Alternative Proof of Theorem 1.1 without Conjecture 3.2

In this section, we show how we can avoid using Conjecture 3.2, which is necessary to estimate (4.1). Instead we assume the strong Artin conjecture and use various Artin L-functions, depending on the conjugacy class. We consider S_n-fields for $n = 3, 4, 5$. Since the strong Artin conjecture is known for S_3- and S_4-fields [6], our result is unconditional for S_3- and S_4-fields. We show, by a case by case analysis on each C,

$$\sum_{K\in L(X),\, n_{K,C}>y} n_{K,C} = O\left(\frac{X}{\log X}\right).$$

We still divide it into two subsums

$$\sum_{K\in L(X),\, n_{K,C}>y} n_{K,C} = \sum_{n_{K,C}>y,\, K\notin E(X)} n_{K,C} + \sum_{n_{K,C}>y,\, K\in E(X)} n_{K,C}.$$

However, the exceptional set $E(X)$ will be different for each C, since we consider zero-free regions of different Artin L-functions. The second sum is estimated by using the unconditional bounds of $n_{K,C}$ in Section 3.1. For the first sum, we need conditional bounds, conditional on zero-free regions of various Artin L-functions. We use the following formula as in [24]: for a conjugacy class C of S_n, define, for $\sigma > 1$,

$$F_C(\sigma) = -\frac{|C|}{|S_n|}\sum_{\psi}\overline{\psi}(C)\frac{L'}{L}(\sigma,\psi,\widehat{K}/\mathbb{Q}), \tag{5.1}$$

where ψ runs over the irreducible characters of S_n and $L(s,\psi,\widehat{K}/\mathbb{Q})$ is the Artin L-function attached to the character ψ. By orthogonality of characters,

$$F_C(\sigma) = \sum_p \sum_{m=1}^{\infty} \frac{\theta(p^m)\log p}{p^{m\sigma}}, \tag{5.2}$$

where for a prime p unramified in $\widehat{K}$,

$$\theta(p^m) = \begin{cases} 1 & \text{if } (\mathrm{Frob}_p)^m \in C, \\ 0 & \text{otherwise.} \end{cases}$$

and $0 \le \theta(p^m) \le 1$ if p ramifies in $\widehat{K}$.

5.1 S_4-fields

Here, we follow the notations in [14] for characters of S_4: χ_4 is a degree 3 representation, and $\chi_5 = \chi_4 \otimes \chi_2$, where χ_2 is the sign character.

5.1.1 Case 1. C=[(1234)]

From (5.2),

$$\sum_{p\in C} \frac{\log p}{p^\sigma} = \frac{1}{4}\cdot\frac{1}{\sigma-1} - \frac{1}{4}\left(-\frac{L'}{L}(s,\chi_2) - \frac{L'}{L}(\sigma,\chi_4) + \frac{L'}{L}(\sigma,\chi_5)\right) + O(1).$$

Here

$$-\frac{L'}{L}(\sigma,\chi_2) - \frac{L'}{L}(\sigma,\chi_4) = \sum_p \frac{\chi_2(p)+\chi_4(p)}{p^\sigma} + O(1) \geq -2\sum_{p\in C}\frac{\log p}{p^\sigma} + O(1).$$

Hence, we have

$$\frac{1}{2}\sum_{p\in C}\frac{\log p}{p^\sigma} \leq \frac{1}{4}\cdot\frac{1}{\sigma-1} - \frac{1}{4}\frac{L'}{L}(\sigma,\chi_5) + O(1). \tag{5.3}$$

Since $\chi_5 = \chi_4 \otimes \chi_2$, the conductor of χ_5 is at most $|d_K|^3$. Since χ_4 is modular [6], χ_5 is modular, that is, $L(s,\chi_5)$ is a cuspidal automorphic L-function of $GL_3/\mathbb{Q}$. Consider a family of Artin L-functions:

$$\tilde{L}(X) = \{L(s,\chi_5)\,|\,K \in L(X)\}.$$

Then all L-functions in $\tilde{L}(X)$ are distinct because $L(s,\chi_4)$'s are distinct since K is arithmetically solitary. Hence, the size of $\tilde{L}(X)$ is the same with that of $L(X)$. By applying Theorem 3.1 to $\tilde{L}(X)$ with $T = (\log X^3)^2$ and α close to 1, every L-function in the family is zero-free on $[\alpha,1]\times[-(3\log X)^2,(3\log X)^2]$ except for $O(X^\epsilon)$ L-functions.

For a L-function with such a zero-free region,

$$-\frac{L'}{L}(\sigma,\chi_5) \leq \frac{16\cdot 3}{(1-\alpha)}\log\log d_K + O(1), \tag{5.4}$$

for $1 \leq \sigma \leq 3/2$. (See (5.1) in [8]). Plugging (5.4) into (5.3), and taking $\sigma = 1 + \frac{\lambda}{\log n_{K,C}}$, we obtain $\frac{1-2e^{-\lambda}}{4\lambda}\log n_{K,C} \leq \frac{12}{(1-\alpha)}\log\log|d_K| + O(1)$. Hence,

$$n_{K,C} \ll (\log|d_K|)^{\frac{48}{(1-\alpha)A}}, \tag{5.5}$$

where $A = \sup_{\lambda\geq 0}\frac{1-2e^{-\lambda}}{\lambda}$.

Let $E_{\chi_5}(X)$ be the set of $K \in L(X)$ such that $L(s, \chi_5)$ may not have the desired zero-free region. Then $|E_{\chi_5}(X)| \ll X^{\epsilon}$. Then

$$\begin{aligned}\sum_{K \in L(X),\, n_{K,C} > y} n_{K,C} &= \sum_{K \notin E_{\chi_5}(X),\, n_{K,C} > y} n_{K,C} + \sum_{K \in E_{\chi_5}(X),\, n_{K,C} > y} n_{K,C} \\ &\ll (\log X)^{\frac{48}{(1-\alpha)A}} \sum_{K \in L(X)^{r_2},\, n_{K,C} > y} 1 + X^{1/4+\epsilon} \cdot X^{\epsilon} \\ &\ll (\log X)^{\frac{48}{(1-\alpha)A}} \left(X \left(\frac{|C|}{|S_4|} \right)^{\pi(y)} + X^{\frac{1+\delta}{2}} \right) \ll \frac{X}{\log X}.\end{aligned}$$

5.1.2 Case 2. C=[(12)(34)]

From (5.2)

$$\sum_{p \in C} \frac{\log p}{p^{\sigma}} = \frac{1}{8} \cdot \frac{1}{\sigma - 1} - \frac{1}{8} \left(\frac{L'}{L}(\sigma, \chi_2) + 2\frac{L'}{L}(\sigma, \chi_3) - \frac{L'}{L}(\sigma, \chi_4) - \frac{L'}{L}(\sigma, \chi_5) \right) + O(1).$$

Since

$$-\frac{L'}{L}(\sigma, \chi_2) - 2\frac{L'}{L}(\sigma, \chi_3) \leq 5 \sum_{p} \frac{\log p}{p^{\sigma}} + O(1) = \frac{5}{\sigma - 1} + O(1),$$

we have

$$\sum_{p \in C} \frac{\log p}{p^{\sigma}} \leq \frac{6}{8} \cdot \frac{1}{\sigma - 1} + \frac{1}{8} \left(\frac{L'}{L}(\sigma, \chi_4) + \frac{L'}{L}(\sigma, \chi_5) \right) + O(1).$$

Now consider a family of Artin L-functions:

$$L'(X) = \{ L(s, \chi_4) \mid K \in L(X) \}.$$

Since S_4-fields are arithmetically solitary, $|L'(X)| = |L(X)|$. By applying Theorem 3.1 to $L'(X)$, we obtain the exceptional set $E_{\chi_4}(X)$, which is the set of $K \in L(X)$ such that $L(s, \chi_4)$ may not have the desired zero-free region. Let $E(X)$ be the union of $E_{\chi_4}(X)$ and $E_{\chi_5}(X)$. Then if $K \notin E(X)$, $L(s, \chi_4)$ and $L(s, \chi_5)$ are simultaneously zero-free on $[\alpha, 1] \times [-(3 \log |d_K|)^2, (3 \log |d_K|)^2]$. So we have

$$\left| -\frac{L'}{L}(\sigma, \chi_4) \right|, \quad \left| -\frac{L'}{L}(\sigma, \chi_5) \right| \leq \frac{16 \cdot 3}{(1 - \alpha)} \log \log |d_K| + O(1).$$

With this conditional bound, as we did in Case 1, we can show

$$\sum_{K \in L(X),\, n_{K,C} > y} n_{K,C} = O\left(\frac{X}{\log X} \right).$$

5.1.3 Case 3. C=[(12)]

From (5.2),

$$\sum_{p\in C}\frac{\log p}{p^{\sigma}}=\frac{1}{4}\cdot\frac{1}{\sigma-1}-\frac{1}{4}\left(-\frac{L'}{L}(\sigma,\chi_2)+\frac{L'}{L}(\sigma,\chi_4)-\frac{L'}{L}(\sigma,\chi_5)\right)+O(1).$$

Here

$$-\frac{L'}{L}(\sigma,\chi_2)-\frac{L'}{L}(\sigma,\chi_5)\geq -2\sum_{p\in C}\frac{\log p}{p^{\sigma}}+O(1).$$

Hence, we have

$$\frac{1}{2}\sum_{p\in C}\frac{\log p}{p^{\sigma}}\leq\frac{1}{4}\cdot\frac{1}{\sigma-1}-\frac{1}{4}\cdot\frac{L'}{L}(\sigma,\chi_4)+O(1).$$

In Case 2, we showed that every $L(s,\chi_4)$ except for $O(X^{\epsilon})$ L-functions satisfies

$$\left|-\frac{L'}{L}(\sigma,\chi_4)\right|\leq\frac{16\cdot 3}{(1-\alpha)}\log\log|d_K|+O(1).$$

From this we have

$$\sum_{K\in L(X),\, n_{K,C}>y} n_{K,C}=O\left(\frac{X}{\log X}\right).$$

5.1.4 Case 4. C = [(123)]

From (5.2),

$$\sum_{p\in C}\frac{\log p}{p^{\sigma}}=\frac{1}{3}\cdot\frac{1}{\sigma-1}-\frac{1}{3}\left(\frac{L'}{L}(\sigma,\chi_2)-\frac{L'}{L}(\sigma,\chi_3)\right)+O(1).$$

Since

$$-\frac{L'}{L}(\sigma,\chi_2)\leq\frac{1}{\sigma-1}+O(1),$$

we have,

$$\sum_{p\in C}\frac{\log p}{p^{\sigma}}\leq\frac{2}{3}\cdot\frac{1}{\sigma-1}+\frac{1}{3}\cdot\frac{L'}{L}(\sigma,\chi_3)+O(1).$$

Here $L(s,\chi_3)=\frac{\zeta_M(s)}{\zeta(s)}$, where M is the cubic resolvent of K. Note that M is an S_3-field. Let

$$L_{CR}(X)=\{L(s,\chi_3)\,|\,M\text{: } S_3\text{-field, } |d_M|\leq X\}.$$

By arithmetically solitary property, $|L_{CR}(X)|$ is the same as the number of S_3-fields for which $|d_M|\leq X$. By applying Theorem 3.1 to $L_{CR}(X)$ with $T=(\log X)^2$, we can see that every $L(s,\chi_3)$ in $L_{CR}(X)$ except for $O(X^\epsilon)$ L-functions satisfies

$$\left|-\frac{L'}{L}(\sigma,\chi_3)\right| \le \frac{16\cdot 3}{(1-\alpha)}\log\log|d_K| + O(1),$$

and with this bound, we have $n_{K,C} \ll (\log|d_K|)^{\frac{16}{(1-\alpha)A}}$, where $A = \sup_{\lambda\ge 0}\frac{1-3e^{-\lambda}}{3\lambda} = 0.10...$ when $\lambda = 2.29...$.

Let $E_{\chi_3}(X)$ be the set of $K \in L(X)$ such that $L(s,\chi_3)$ may not have the desired zero-free region. Note that there are at most $O(X^{\frac{1}{2}+\epsilon})$ S_4-fields in $L(X)$ that have the common cubic resolvent M. (See the Appendix.) Hence, $|E_{\chi_3}(X)| \ll X^\epsilon X^{\frac{1}{2}+\epsilon}$. Then

$$\begin{aligned}\sum_{K\in L(X),\, n_{K,C}>y} n_{K,C} &= \sum_{K\notin E_{\chi_3}(X),\, n_{K,C}>y} n_{K,C} + \sum_{K\in E_{\chi_3}(X),\, n_{K,C}>y} n_{K,C}\\ &\ll (\log X)^{\frac{16}{(1-\alpha)A}} \sum_{K\in L(X),\, n_{K,C}>y} 1 + X^{1/4+\epsilon}\cdot X^\epsilon\cdot X^{\frac{1}{2}+\epsilon} = O\left(\frac{X}{\log X}\right).\end{aligned}$$

5.1.5 *Case 5. $C = \{1\}$*

In [8], we showed that if $L(s,\chi_4) = \zeta_K(s)/\zeta(s)$ is entire and zero-free on $[\alpha,1]\times[-(\log|d_K|)^2, (\log|d_K|)^2]$, then $n_{K,e} \ll (\log|d_K|)^{\frac{16}{(1-\alpha)A}}$, where $A = \sup_{\lambda\ge 0}\frac{1-\frac{4}{3}e^{-\lambda}}{\lambda} = 0.5...$ when $\lambda = 0.96...$ (there is a typographical error in [8, Theorem 1.1]). Since every field K in $L(X)$ except for $O(X^\epsilon)$ fields has such upper bound,

$$\sum_{K\in L(X),\, n_{K,C}>y} n_{K,C} = O\left(\frac{X}{\log X}\right).$$

5.2 S_5-fields

We assume the strong Artin conjecture for S_5 fields, and follow notations in [14] for characters of S_5. Then $L(s,\chi_3) = \zeta_K(s)/\zeta(s)$, and $L(s,\chi_5) = \zeta_H(s)/\zeta(s)$, where H is the sextic resolvent of K. For the sign character χ_2,

$$\chi_4 = \chi_3\otimes\chi_2, \quad \chi_6 = \chi_5\otimes\chi_2, \text{ and } \chi_7 = \wedge^2\chi_3.$$

Then, the conductor of $L(s,\chi_4)$ is bounded by $|d_K||d_K|^4 = |d_K|^5$. The conductor of $L(s,\chi_5)$ is $|d_H| = (16|d_K|)^3$. (See page 76 in [3].) The conductor of $L(s,\chi_6)$ is bounded by $|d_H||d_K|^5 \le 2^{12}|d_K|^8$. The conductor of $L(s,\chi_7)$ is bounded by $|d_K|^7$. ([4]; G. Henniart noted in a private communication that it can be improved to $|d_K|^{\frac{3}{2}}$.)

5.2.1 Case 1. C = [(12345)]

From (5.2),

$$\sum_{p\in C}\frac{\log p}{p^\sigma}=\frac{1}{5}\cdot\frac{1}{\sigma-1}-\frac{1}{5}\left(\frac{L'}{L}(\sigma,\chi_2)-\frac{L'}{L}(\sigma,\chi_3)-\frac{L'}{L}(\sigma,\chi_4)+\frac{L'}{L}(\sigma,\chi_7)\right)+O(1).$$

Since

$$\frac{L'}{L}(\sigma,\chi_3)+\frac{L'}{L}(\sigma,\chi_4)\leq 2\sum_{p\in C}\frac{\log p}{p^\sigma}+O(1),\quad -\frac{L'}{L}(\sigma,\chi_2)\leq\frac{1}{\sigma-1}+O(1),$$

we have

$$\frac{3}{5}\sum_{p\in C}\frac{\log p}{p^\sigma}\leq\frac{2}{5}\cdot\frac{1}{\sigma-1}-\frac{1}{5}\cdot\frac{L'}{L}(s,\chi_7)+O(1).$$

Define $L_1(X)=\{L(s,\chi_7)\,|\,K\in L(X)\}$. We show that every L-function in $\tilde{L}(X)$ is distinct.

Lemma 5.1. Let $L(s,\chi_7)=L(s,\chi_7,\widehat{K}/\mathbb{Q})$ and $L(s,\chi_7')=L(s,\chi_7',\widehat{K}'/\mathbb{Q})$. Suppose $L(s,\chi_7)=L(s,\chi_7')$. Then K and K' are conjugate.

Proof. Recall the following from [5]: $\chi_3=As(\sigma)$, the Asai lift of σ, where σ is a degree 2 representation of $\tilde{A}_5$ over $F=\mathbb{Q}[\sqrt{d_K}]$. Let π be the cuspidal representation of GL_2/F corresponding to σ, and let Π be the cuspidal representation of $GL_4/\mathbb{Q}$ corresponding to χ_3 by the strong Artin conjecture. Then $\wedge^2\Pi\simeq I_F^{\mathbb{Q}}(Sym^2\pi)$.

Let π',Π' be defined by K'. Suppose $L(s,\Pi,\wedge^2)=L(s,\Pi',\wedge^2)$. Then $L(s,Sym^2(\pi))=L(s,Sym^2(\pi'))$. By Ramakrishnan [32], $\pi'\simeq\pi\otimes\chi$ for a quadratic character of F. Then by Krishnamurthy [23], $As(\pi)\simeq As(\pi')$. Hence, $\Pi\simeq\Pi'$. Therefore, $L(s,\chi_3,K/\mathbb{Q})=L(s,\chi_3,K'/\mathbb{Q})$. Since K is arithmetically solitary, K and K' are conjugate. ■

Hence, $|L_1(X)|=|L(X)|$. By applying Theorem 3.1 to $L_1(X)$ with $T=(\log X^7)^2$, we can show that every $L(s,\chi_7)$ in $L_1(X)$ except for $O(X^\epsilon)$ L-function satisfies

$$\left|-\frac{L'}{L}(\sigma,\chi_7,\widehat{K}/\mathbb{Q})\right|\leq\frac{16\cdot 28}{(1-\alpha)}\log\log|d_K|+O(1).$$

Hence,

$$\sum_{K\in L(X),\, n_{K,C}>Y}n_{K,C}=O\left(\frac{X}{\log X}\right).$$

5.2.2 Case 2. C=[(1234)]

From (5.2),

$$\sum_{p\in C}\frac{\log p}{p^\sigma}=\frac{1}{4}\cdot\frac{1}{\sigma-1}-\frac{1}{4}\left(-\frac{L'}{L}(\sigma,\chi_2)+\frac{L'}{L}(\sigma,\chi_5)-\frac{L'}{L}(\sigma,\chi_6)\right)+O(1).$$

Since

$$\frac{L'}{L}(\sigma,\chi_2)+\frac{L'}{L}(\sigma,\chi_6)\le 2\sum_{p\in C}\frac{\log p}{p^\sigma},$$

we have

$$\frac{1}{2}\sum_{p\in C}\frac{\log p}{p^\sigma}\le\frac{1}{4}\cdot\frac{1}{\sigma-1}-\frac{1}{4}\cdot\frac{L'}{L}(\sigma,\chi_5).$$

Define $L_2(X)=\{L(s,\chi_5)\,|\,K\in L(X)\}$. We show that every L-function in $L_2(X)$ is distinct.

Lemma 5.2. Let $L(s,\chi_5)=L(s,\chi_5,\widehat{K}/\mathbb{Q})$ and $L(s,\chi_5')=L(s,\chi_5',\widehat{K}'/\mathbb{Q})$. Suppose $L(s,\chi_5)=L(s,\chi_5')$. Then K and K' are conjugate.

Proof. It is easy to see (cf. [17, p. 28]) $\wedge^2\chi_5=\chi_3\otimes\chi_2\oplus\wedge^2\chi_3$. Now let χ_3',χ_5' be defined by K', and suppose $\chi_5\simeq\chi_5'$. Then $\wedge^2\chi_5\simeq\wedge^2\chi_5'$. Hence, $\chi_3\otimes\chi_2\oplus\wedge^2\chi_3\simeq\chi_3'\otimes\chi_2'\oplus\wedge^2\chi_3'$. By strong multiplicity one, $\wedge^2\chi_3\simeq\wedge^2\chi_3'$. Hence, by Lemma 5.1, $\chi_3\simeq\chi_3'$. ■

Since $L(s,\chi_5)$'s in $L_2(X)$ are all distinct, we can conclude that $|L_2(X)|=|L(X)|$. By applying Theorem 3.1 to $L_2(X)$ with $T=(\log(16X)^4)^2$, we proceed as in Case 1.

5.2.3 Case 3. $C=[(12)(345)]$

From (5.2),

$$\sum_{p\in C}\frac{\log p}{p^\sigma}=\frac{1}{6}\cdot\frac{1}{\sigma-1}-\frac{1}{6}\left(-\frac{L'}{L}(\sigma,\chi_2)-\frac{L'}{L}(\sigma,\chi_3)+\frac{L'}{L}(\sigma,\chi_4)-\frac{L'}{L}(\sigma,\chi_5)+\frac{L'}{L}(\sigma,\chi_6)\right)+O(1).$$

Since

$$\frac{L'}{L}(\sigma,\chi_2)+\frac{L'}{L}(\sigma,\chi_3)+\frac{L'}{L}(\sigma,\chi_5)\le 3\sum_{p\in C}\frac{\log p}{p^\sigma},$$

we have

$$\frac{1}{2}\sum_{p\in C}\frac{\log p}{p^\sigma}\le\frac{1}{6}\cdot\frac{1}{\sigma-1}-\frac{1}{6}\cdot\frac{L'}{L}(\sigma,\chi_4)-\frac{1}{6}\frac{L'}{L}(\sigma,\chi_6).$$

Since $\chi_6 = \chi_5 \otimes \chi_2$ and L-functions $L(s, \chi_5)$ are all distinct, $L(s, \chi_6)$ are also all distinct. Since the dimensions of χ_4 and χ_6 are different, the isobaric sums $\chi_4 \boxplus \chi_6$ coming from $L(X)$ satisfy the conditions for Theorem 3.1. Apply Theorem 3.1 to $L_3(X) = \{L(s, \chi_4)L(s, \chi_6) \mid K \in L(X)\}$, and we proceed as in Case 1.

5.2.4 Case 4. $C = (12)(34)$

From (5.2),

$$\sum_{p \in C} \frac{\log p}{p^\sigma} = \frac{1}{8} \cdot \frac{1}{\sigma - 1} - \frac{1}{8}\left(\frac{L'}{L}(\sigma, \chi_2) + \frac{L'}{L}(\sigma, \chi_5) + \frac{L'}{L}(\sigma, \chi_6) - 2\frac{L'}{L}(\sigma, \chi_7)\right) + O(1).$$

Since

$$\frac{L'}{L}(\sigma, \chi_7) \leq 2\sum_{p \in C} \frac{\log p}{p^\sigma} + O(1), \quad -\frac{L'}{L}(\sigma, \chi_2) \leq \frac{1}{\sigma - 1} + O(1),$$

we have

$$\frac{1}{2}\sum_{p \in C} \frac{\log p}{p^\sigma} \leq \frac{1}{4} \cdot \frac{1}{\sigma - 1} - \frac{1}{8}\left(\frac{L'}{L}(\sigma, \chi_5) + \frac{L'}{L}(\sigma, \chi_6)\right).$$

Define $L_4(X) = \{L(s, \chi_6) \mid K \in L(X)\}$. Then every L-function in $L_4(X)$ is distinct, and $|L_4(X)| = |L(X)|$. Applying Theorem 3.1 to $L_4(X)$, every L-function $L(s, \chi_6)$ in $L_4(X)$ except for $O(X^\epsilon)$ has the desired zero-free region. Hence, together with $L_2(X)$ in Case 2, we see that $L(s, \chi_5)$ and $L(s, \chi_6)$ are simultaneously zero-free in the desired region except for $O(X^\epsilon)$ fields. Then we proceed as in Case 1.

5.2.5 Case 5. $C = [(12)]$.

We use the L-function $L(s, \chi_3)$:

$$-\frac{L'}{L}(\sigma, \chi_3) = \sum_p \frac{\chi_3(p)\log p}{p^\sigma} + O(1).$$

Note that $\chi_3(p) = 2$ if $\mathrm{Frob}_p \in C$, and $1 + \chi_3(p) \geq 1$. Then,

$$3\sum_{p < n_{K,C}} \frac{\log p}{p^\sigma} \leq -\frac{\zeta'}{\zeta}(\sigma) - \frac{L'}{L}(\sigma, \chi_3) + O(1).$$

By applying Theorem 3.1 to the set $L_5(X) = \{L(s, \chi_3) \mid K \in L(X)\}$, we see that every $L(s, \chi_3)$ in $L_5(X)$ except for $O(X^\epsilon)$ L-function satisfies

$$-\frac{L'}{L}(\sigma, \chi_3) \leq \frac{64}{1 - \alpha}\log\log|d_K| + O(1).$$

Hence, by taking $\sigma - 1 = \frac{\lambda}{\log n_{K,C}}$, we have $n_{K,C} \le (\log |d_K|)^{\frac{64}{(1-\alpha)A}}$, where $A = \sup_{\lambda>0} \frac{2-3e^{-\lambda}}{\lambda}$. We obtain

$$\sum_{K \in L(X),\, n_{K,C} > Y} n_{K,C} = O\left(\frac{X}{\log X}\right).$$

5.2.6 Case 6. C = [(123)]

Since $\chi_3(p) = 1$ if $\mathrm{Frob}_p \in C$, we can use $L(s, \chi_3)$. This case is similar to the case $C = (12)$.

5.2.7 Case 7. C = {1}

Since $\chi_3(p) = 4$ if $\mathrm{Frob}_p \in C$, we can use $L(s, \chi_3)$. This case is similar to the case $C = (12)$.

5.3 S_3-fields

For the sake of completeness, we include the case of S_3. Here, we follow the notations in [14] for characters of S_3.

5.3.1 Case 1. C = [(123)]

From (5.2),

$$\sum_{p \in C} \frac{\log p}{p^\sigma} = \frac{1}{3} \cdot \frac{1}{\sigma - 1} - \frac{1}{3}\left(\frac{L'}{L}(\sigma, \chi_2) - \frac{L'}{L}(\sigma, \chi_3)\right) + O(1).$$

Then

$$\sum_{p < n_{K,C}} \frac{\log p}{p^\sigma} \le \frac{2}{3} \cdot \frac{1}{\sigma - 1} + \frac{1}{3} \cdot \frac{L'}{L}(\sigma, \chi_3) + O(1).$$

Since $L(s, \chi_3) = \frac{\zeta_K(s)}{\zeta(s)}$, $L(s, \chi_3)$ is modular, i.e., $L(s, \chi_3)$ is a cuspidal automorphic L-function of $GL_2/\mathbb{Q}$. This case is similar to S_4, $C = (1234)$.

5.3.2 Case 2. C = [(12)]

From (5.2),

$$\sum_{p \in C} \frac{\log p}{p^\sigma} = \frac{1}{2} \cdot \frac{1}{\sigma - 1} + \frac{1}{2} \cdot \frac{L'}{L}(\sigma, \chi_2) + O(1).$$

This case was done in Section 3.2.

5.3.3 Case 3. C = {1}

This case is similar to S_4, $C = \{1\}$.

We summarize our discussions as

Theorem 5.3. Theorem 1.1 holds unconditionally for S_3- and S_4-fields. Under the strong Artin conjecture, Theorem 1.1 holds for S_5-fields.

Remark 5.1. The above method can be generalized to S_n-fields and a special conjugacy class: Let K be an S_n-field, and let $L(s,\chi) = \frac{\zeta_K(s)}{\zeta(s)}$. Let C be a conjugacy class of S_n such that $\chi(C) \geq 1$. Then under the counting conjectures (2.1)–(2.2) and the strong Artin conjecture for $L(s,\chi)$, we have

$$\frac{1}{|L_n^{(r_2)}(X)|} \sum_{K \in L_n^{(r_2)}(X)} n_{K,C} = \sum_q \frac{q(1-|C|/|S_n|+f(q))}{1+f(q)} \prod_{p<q} \frac{|C|/|S_n|}{1+f(p)} + O\left(\frac{1}{\log X}\right).$$

6 Average Value of $N_{K,C}$

In this section, we compute the average of $N_{K,C}$.

6.1 Conditional bounds of $N_{K,C}$

For $N_{K,C}$, we have conditional bounds $N_{K,C} \ll e^{c(\log\log|d_K|)^{\frac{5}{3}+\epsilon}}$ for some constant c by [28, Theorem 1.1] under the zero-free region $[\alpha,1] \times [-(\log|d_{\widehat{K}}|)^2, (\log|d_{\widehat{K}}|)^2]$. We obtain a better bound.

Proposition 1.20. Let C be any conjugacy class. We assume that for all nontrivial irreducible character ψ, the strong Artin conjecture holds for $L(s,\psi) = L(s,\psi,\widehat{K}/\mathbb{Q})$ and it is zero-free in $[\alpha,1] \times [-(\log|d_{\widehat{K}}|)^2, (\log|d_{\widehat{K}}|)^2]$. Then there exists an absolute constant A and a prime p such that $\mathrm{Frob}_p \in C$, and $p \ll (A(1-\alpha)\log|d_{\widehat{K}}|)^{\frac{1}{1-\alpha}}$. Hence, $N_{K,C} \ll (A(1-\alpha)\log|d_{\widehat{K}}|)^{\frac{1}{1-\alpha}}$.

Proof. We follow [26, p. 140] and the proof of [9, Proposition 4.2]: For $1 < x < y$, $k(s,x,y) = \left(\frac{y^{s-1}-x^{s-1}}{s-1}\right)^2$, consider

$$\frac{1}{2\pi i}\int_{(2)} F_C(s)k(s,x,y)\,\mathrm{d}s,$$

where $F_C(s)$ is in (5.1). Then, its inverse Mellin transform is given by

$$\hat{k}(u,x,y) = \frac{1}{2\pi i}\int_{(2)} k(s,x,y)u^{-s}ds = \begin{cases} 0 & \text{if } u > y^2\,, \\ u^{-1}\log\frac{y^2}{u} & \text{if } xy < u < y^2, \\ u^{-1}\log\frac{u}{x^2} & \text{if } x^2 < u < xy, \\ 0 & \text{if } u < x^2. \end{cases}$$

By (5.2), it is

$$\sum_{Frob_p \in C} \log p \cdot \hat{k}(p,x,y) + O\left(\frac{\log y}{x \log x} \log \frac{y}{x}\right).$$

On the other hand, by moving contours to the left, it is

$$\frac{|C|}{|G|}\left(\log \frac{y}{x}\right)^2 - \frac{|C|}{|G|}\sum_{\psi} \overline{\psi(C)} \sum_{\rho_\psi} k(\rho_\psi, x, y),$$

where ρ_ψ runs over all zeros of $L(s,\psi)$. In the proof of [9, Proposition 4.2], we showed that, for $N = |d_{\widehat{K}}|$,

$$k(\rho_\psi, x, y) \ll x^{-2}(\log N)^2 + x^{-2(1-\alpha)}(1-\alpha)^{-1}\log N,$$

when $L(s,\psi,\widehat{K}/\mathbb{Q})$ has the above zero-free region in the proposition.

Let's assume that $Frob_p \notin C$ for all primes $p < y^2$. Then, we obtain that

$$\left(\log \frac{y}{x}\right)^2 \ll x^{-2(1-\alpha)}(1-\alpha)^{-1}\log N + x^{-2}(\log N)^2 + \frac{\log y}{x \log x}\log \frac{y}{x}.$$

For B with $B(1-\alpha)\log N \geq 1$, take $x = (B^3(1-\alpha)\log N)^{\frac{1}{2}(1-\alpha)^{-1}}$ and $y = B^{\frac{1}{2}(1-\alpha)^{-1}}x$. Then $y \leq x^{3/2}$ and $\log y/\log x \leq \log(y/x)$ for such y and x. With our choice of x and y, for sufficiently large B, it is easy to show that the above inequality is not consistent. ■

Remark 6.2. The proposition also implies the conditional bounds (3.1): $n_{K,C} \ll_\alpha (\log|d_{\widehat{K}}|)^{\frac{1}{1-\alpha}}$.

6.2 Unconditional bounds of $N_{K,C}$

We do not have good unconditional bounds on $N_{K,C}$. Currently the best bound is $N_{K,C} \ll |d_{\widehat{K}}|^{40}$ [36].

We have the following bound on $d_{\widehat{K}}$ by [28, Lemma 3.4]: $c_1|d_K|^{(n-1)!} \leq |d_{\widehat{K}}| \leq c_2|d_K|^{\frac{n!}{2}}$ for some constants c_1, c_2. Hence, $\log|d_{\widehat{K}}| \asymp \log|d_K|$. (In Section 5, we showed $|d_{\widehat{K}}| \ll |d_K|^{a_n}$ for some a_n for $n \leq 5$ by case by case analysis, using the conductor-discriminant formula, $\log|d_{\widehat{K}}| = \sum_\psi \psi(1)\log A_\psi$, where A_ψ is the conductor of $L(s,\psi)$.) We make the following conjecture:

Conjecture 6.3. $N_{K,C} \ll |d_K|^{\frac{1}{2}-\epsilon_n}$ for some constant $0 < \epsilon_n < 1/2$.

Remark 6.4. The referee brought to our attention a recent result of Ge, Milinovich, and Pollack [18]. They show in our notations that under subconvexity bounds of $\zeta_{\widehat{K}}(s)$, $\zeta_{\widehat{K}}(s) \ll |s|^A |d_{\widehat{K}}|^{\frac{1}{4}-\theta}$ for $Re(s) = \frac{1}{2}$, one has $N_{\widehat{K},C} \ll |d_{\widehat{K}}|^{\frac{1}{2}-2\theta+\epsilon}$ for $C = \{1\}$. By using the well known fact that p splits completely in K if and only if p splits completely in $\widehat{K}$ (cf. [19]), we have $N_{\widehat{K},C} = N_{K,C}$. Since $|d_{\widehat{K}}| \ll |d_K|^{\frac{n!}{2}}$, we have $N_{K,C} \ll |d_K|^{\frac{n!}{4}-n!\theta+\epsilon}$. In particular, when $n = 3$, we have $\theta = \frac{1}{1889}$ [18, Example 1]. Hence, $N_{K,C} \ll |d_K|^{\frac{3}{2}-\frac{6}{1889}+\epsilon}$ for $C = \{1\}$.

6.3 Proof of Theorem 1.2

Since the idea of proof is similar to the case of $n_{K,C}$, we omit some details. Consider

$$\sum_{K\in L(X)} N_{K,C} = \sum_{K\in L(X),\, N_{K,C}\le y} N_{K,C} + \sum_{K\in L(X),\, N_{K,C}>y} N_{K,C}.$$

Here $N_{K,C} = q$ means that for all primes $p < q$, $\mathrm{Frob}_p \notin C$ and $\mathrm{Frob}_q \in C$. By the counting conjectures, there are

$$\frac{|C|/|S_n|}{1+f(q)} \prod_{p<q} \frac{1-|C|/|S_n|+f(p)}{1+f(p)} A(r_2)X + O\left(X^{\frac{1+3\delta}{4}}\right).$$

such number fields in $L(X)$. Hence,

$$\begin{aligned}\sum_{K\in L(X),\, N_{K,C}\le y} N_{K,C} &= \sum_{q\le y} q \sum_{K\in L(X),\, N_{K,C}=q} 1\\ &= A(r_2)X \sum_{q\le y} \frac{q(|C|/|S_n|)}{1+f(q)} \prod_{p<q} \frac{1-|C|/|S_n|+f(p)}{1+f(p)} + O\left(y^2 X^{\frac{1+3\delta}{4}}\right)\\ &= A(r_2)X \sum_{q} \frac{q(|C|/|S_n|)}{1+f(q)} \prod_{p<q} \frac{1-|C|/|S_n|+f(p)}{1+f(p)} + O\left(\frac{X}{\log X}\right).\end{aligned}$$

In order to estimate the second sum $\sum_{K\in L(X),\, N_{K,C}>y} N_{K,C}$, we divide the sum into two subsums:

$$\sum_{K\in L(X),\, N_{K,C}>y} N_{K,C} = \sum_{N_{K,C}>y,\, K\notin E(X)} N_{K,C} + \sum_{N_{K,C}>y,\, K\in E(X)} N_{K,C}, \tag{6.1}$$

where $E(X)$ is the union of $E_\psi(X)$, and $E_\psi(X)$ is the exceptional set in which $L(s,\psi)$ may not have the desired zero-free region in $[\alpha,1]\times[-(\log|d_{\widehat{K}}|)^2, (\log|d_{\widehat{K}}|)^2]$.

We proceed as in $n_{K,C}$ case; the first sum is estimated by using conditional bounds in Section 6.1. For the second sum, $\sum_{N_{K,C}>Y,\, K\in E(X)} N_{K,C} \ll X^{\frac{1}{2}-\epsilon_n}|E(X)|$. Hence, we need to estimate $|E(X)|$. We consider case by case, using the same notations as in Section 5:

$n=3$: We can show $|E_{\chi_3}(X)| \ll X^{\epsilon_n}$, and $|E_{\chi_2}(X)| \ll X^{\frac{1}{2}+\frac{1}{2}\epsilon_n}$.

$n=4$: We showed that $|E_{\chi_4}(X)|, |E_{\chi_5}(X)| \ll X^{\epsilon_n}$, and $|E_{\chi_3}(X)| \ll X^{\frac{1}{2}+\frac{1}{2}\epsilon_n}$. By Conjecture (3.2), $|E_{\chi_2}(X)| \ll X^{\frac{1}{2}+\frac{1}{2}\epsilon_n}$.

$n=5$: We showed that $|E_{\chi_i}(X)| \ll X^{\epsilon_n}$ for $i=3,\dots,7$. By Conjecture 3.2, $|E_{\chi_2}(X)| \ll X^{\frac{1}{2}+\frac{1}{2}\epsilon_n}$.

Hence, we have proved Theorem 1.2.

The tables below show average values of $N_{K,C}$ for S_3-, S_4-, and S_5-fields. The computations are done by PARI. The average values of $N_{K,C}$ for S_3 are given in [25].

S_3	Average of $N_{K,C}$
$\{1\}$	19.79522...
$[(12)]$	5.36802...
$[(123)]$	8.54472...

S_4	Average of $N_{K,C}$
$\{1\}$	108.71075...
$[(12)(34)]$	28.96178...
$[(1234)]$	12.69279...
$[(12)]$	12.69279...
$[(123)]$	9.098479...

S_5	Average of $N_{K,C}$
$\{1\}$	716.34521...
$[(12)(34)]$	29.19651...
$[(123)]$	20.75158...
$[(12)(345)]$	20.75158...
$[(12)]$	47.44681...
$[(1234)]$	12.88664...
$[(12345)]$	16.72312...

6.4 Unconditional result on N_{K,C_u}

Let C_u be the union of all the conjugacy classes not contained in A_n and N_{K,C_u} be the smallest prime for which $\mathrm{Frob}_p \in C_u$. Then we have an unconditional bound for N_{K,C_u}: Since $d_K = d_F m^2$ for some integer m, if $\mathrm{Frob}_p \in C_u$, then p is inert in F. (i.e., $(\frac{d_F}{p}) = (\frac{d_K}{p}) = -1$.) Conversely, if p is inert in F and $p \nmid d_K$ (i.e., $(\frac{d_K}{p}) = -1$), then Frob_p is in C_u. Hence, N_{K,C_u} is the smallest prime such that $(\frac{d_K}{p}) = -1$. Hence, by Norton [27],

$$N_{K,C_u} \ll |d_K|^{\frac{1}{4\sqrt{e}}+\epsilon}.$$

(Norton's result is valid for imprimitive characters.)

Hence, combining with the conditional bound in Section 6.1, we have the following:

Theorem 6.5. Assume the counting conjectures (2.1)–(2.2). Assume the strong Artin conjecture. Then

$$\frac{1}{|L_n^{(r_2)}(X)|}\sum_{K\in L_n^{(r_2)}(X)} N_{K,C_u} = \sum_q \frac{\frac{1}{2}q}{1+f(q)}\prod_{p<q}\frac{\frac{1}{2}+f(p)}{1+f(p)} + O\left(\frac{1}{\log X}\right).$$

Since the assumptions in Theorem 6.5 hold for S_3- and S_4-fields, we have Theorem 1.3.

7 Appendix: S_4-fields with the Same Cubic Resolvent

Given a noncyclic cubic field M, let

$$\Phi_M(s) = 1 + \sum_{K\in\mathcal{F}(M)}\frac{1}{f(K)^s},$$

where $d_K = d_M f(K)^2$, and $\mathcal{F}(M)$ is the set of all S_4-fields K with the cubic resolvent field M. Let $\mathcal{L}(M,n^2)$ be the set of quartic fields whose cubic resolvents are isomorphic to M and whose discriminants are $n^2 d_M$, and $\mathcal{L}_{tr}(M,64)$ the subset of $\mathcal{L}(M,64)$, where 2 is totally ramified. Define $\mathcal{L}_2(M) = \mathcal{L}(M,1)\cup\mathcal{L}(M,4)\cup\mathcal{L}(M,16)\cup\mathcal{L}_{tr}(M,64)$. By Klüners [21], $|\mathcal{L}(M,n)| \ll (n^2|d_M|)^{\frac{1}{2}+\epsilon}$, hence $|\mathcal{L}_2(M)| \ll |d_M|^{\frac{1}{2}+\epsilon}$.

By Theorem 1.4 in Cohen and Thorne [13],

$$\Phi_M(s) = \sum_{i=1}^{|\mathcal{L}_2(M)+1|}\Phi_i(s), \quad \Phi_i(s) = \sum_{n=1}^{\infty}\frac{a_i(n)}{n^s},$$

and $a_i(n) \le 3^{\omega(n)} \ll 3^{\frac{\log n}{\log\log n}} \ll n^{\epsilon}$. We also have

$$\Phi_i(1+c+it) \ll \left(\frac{\zeta(1+c)}{\zeta(2+2c)}\right)^3 \ll \frac{1}{c^3}.$$

By applying Perron's formula to each $\Phi_i(s)$ for $i = 1,2,\dots,|\mathcal{L}_2(M)+1|$, we can obtain that

$$|\{K\in\mathcal{F}(M)\,|\, f(K)\le x\}| \ll x(\log x)^4|d_M|^{\frac{1}{2}+\epsilon}.$$

Hence, we have proved the following:

Proposition 7.1. Let $CR_4(X,M)$ be the set of S_4-fields K with the given cubic resolvent M and $|d_K| \le X$. Then

$$|CR_4(X,M)| \ll X^{\frac{1}{2}}(\log X)^4 |d_M|^{\epsilon},$$

where the implied constant is independent of M.

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