Math. Proc. Camb. Phil. Soc.: page 1 of 12 © *Cambridge Philosophical Society* 2017
doi:10.1017/S0305004117000019

Extreme residues of Dedekind zeta functions

BY PETER J. CHO†

Department of Mathematical Sciences,
Ulsan National Institute of Science and Technology, Ulsan, Korea.
e-mail: `petercho@unist.ac.kr`

AND HENRY H. KIM‡

Department of Mathematics,
University of Toronto, ON M5S 2E4, Canada,
and Korea Institute for Advanced Study, Seoul, Korea.
e-mail: `henrykim@math.toronto.edu`

(*Received* 28 *September* 2015; *revised* 13 *September* 2016)

Abstract

In a family of S_{d+1}-fields ($d = 2, 3, 4$), we obtain the conjectured upper and lower bounds of the residues of Dedekind zeta functions except for a density zero set. For S_5-fields, we need to assume the strong Artin conjecture. We also show that there exists an infinite family of number fields with the upper and lower bounds, resp.

1. *Introduction*

For a quadratic extension $K = \mathbb{Q}(\sqrt{D})$ with a fundamental discriminant D, $\mathrm{Res}_{s=1}\zeta_K(s) = L(1, \chi_D)$, where $\chi_D = (\frac{D}{\cdot})$ is the quadratic character. In this case, Littlewood [**11**] obtained the bound

$$\left(\frac{1}{2} + o(1)\right) \frac{\zeta(2)}{e^\gamma \log\log|D|} \leqslant L(1, \chi_D) \leqslant (2 + o(1))e^\gamma \log\log|D|$$

under GRH, where γ is the Euler–Mascheroni constant. Under the same hypothesis, he also constructed an infinite family of quadratic fields with $L(1, \chi_D) \geqslant (1 + o(1))e^\gamma \log\log|D|$ and an infinite family of quadratic fields with $L(1, \chi_D) \leqslant (1 + o(1))\frac{\zeta(2)}{e^\gamma \log\log|D|}$. Later, Chowla [**3**] established the latter omega result unconditionally. It has been conjectured that the true upper and lower bounds are $(1 + o(1))e^\gamma \log\log|D|$ and $(1 + o(1))\frac{\zeta(2)}{e^\gamma \log\log|D|}$, resp. In [**12**], Montgomery and Vaughan considered the distribution of $L(1, \chi_D)$ via random variables which take ± 1 with equal probability. They proposed three conjectures which support the expected bounds. In [**5**], some of the conjectures were proved by Granville and Soundararajan.

For a number field K of degree $d+1$, the lower bound and the upper bound of $\mathrm{Res}_{s=1}\zeta_K(s)$

† This work was supported by the 2016 Research Fund(Project Number 1.150147.01) of UNIST(Ulsan National Institute of Science and Technology).
‡ Partially supported by an NSERC grant.

under GRH and the strong Artin conjecture for $\zeta_K(s)/\zeta(s)$ are

$$\left(\frac{1}{2}+o(1)\right)\frac{\zeta(d+1)}{e^\gamma \log\log|D_K|} \leqslant \mathrm{Res}_{s=1}\zeta_K(s) \leqslant (2+o(1))^d (e^\gamma \log\log|D_K|)^d, \qquad (1{\cdot}1)$$

where D_K is the discriminant of a number field K. The proof of (1·1) is given in Section 3 since at least the upper bound is well known but it is hard to find its proof in the literature.

As in the quadratic extension case, we may conjecture that $(1+o(1))(e^\gamma \log\log|D_K|)^d$ and $(1+o(1))\frac{\zeta(d+1)}{e^\gamma \log\log|D_K|}$ are the true upper and lower bounds, resp. In this paper, we show that it is the case except for a density zero set in a family of number fields. A number field K of degree $d+1$ is called a S_{d+1}-field if its Galois closure over $\mathbb{Q}$ is an S_{d+1} Galois extension. For a S_{d+1}-field K, we have a decomposition of $\zeta_K(s)$:

$$\zeta_K(s) = \zeta(s)L(s,\rho,\widehat{K}/Q),$$

where $\widehat{K}$ is the Galois closure of K over $\mathbb{Q}$ and ρ is the standard representation of S_{d+1}. For simplicity, we denote $L(s,\rho,\widehat{K}/Q)$ by $L(s,\rho)$. Hence $\mathrm{Res}_{s=1}\zeta_K(s) = L(1,\rho)$. Then, our first main theorem is

THEOREM 1·1. *Let $L(X)$ be a set of S_{d+1}-fields with $X/2 \leqslant |D_K| \leqslant X$, $d+1 = 3, 4$ and 5. For S_5-fields, we assume the strong Artin conjecture for $L(s,\rho)$. Then, except for $O(Xe^{-c'\frac{\log X}{\log\log X}\log\log\log X})$ L-functions for some constant $c' > 0$,*

$$(1+o(1))\frac{\zeta(d+1)}{e^\gamma \log\log|D_K|} \leqslant L(1,\rho) \leqslant (1+o(1))(e^\gamma \log\log|D_K|)^d.$$

where $o(1) = O\left(\frac{1}{(\log\log|D_K|)^{1/2}}\right)$.

Secondly, under the same hypothesis, we construct an infinite family of S_{d+1}-fields with extreme residue values.

THEOREM 1·2. *Let $d+1 = 3, 4$, and 5. For $d+1 = 5$, we assume the strong Artin conjecture. Then*:

(i) *the number of S_{d+1}-fields K of signature (r_1, r_2) with $\frac{X}{2} \leqslant |D_K| \leqslant X$ for which*

$$L(1,\rho) = (e^\gamma \log\log|D_K|)^d\left(1+O\left(\frac{1}{(\log\log|D_K|)^{1/2}}\right)\right)$$

is $\geqslant A(r_2)X\exp\left(-\log|S_{d+1}|\cdot\frac{\log X}{\log\log X} - \log\log\log X\right)$;

(ii) *the number of S_{d+1}-fields K of signature (r_1, r_2) with $X/2 \leqslant |D_K| \leqslant X$ for which*

$$L(1,\rho) = \frac{\zeta(d+1)}{e^\gamma \log\log|D_K|}\left(1+O\left(\frac{1}{(\log\log X)^{1/2}}\right)\right)$$

is $\geqslant A(r_2)X\exp\left(-\log\frac{|S_{d+1}|}{d+1}\cdot\frac{\log X}{\log\log X} - \log\log\log X\right)$,

where $A(r_2)$ is a constant which occurs in Conjecture 2·1. (Note that Conjecture 2·1 is proved when $d+1 = 3, 4, 5$.)

We also construct an infinite family of S_{d+1}-fields with bounded residues.

THEOREM 1·3. *Let $d+1 = 3, 4$, and 5. For $d+1 = 5$, we assume the strong Artin conjecture.*

Then the number of S_{d+1}-fields K of signature (r_1, r_2) with $X/2 \leqslant |D_K| \leqslant X$ for which

$$L(1,\rho) = \begin{cases} \zeta(2)^{\frac{d}{2}}(1+o(1)), & \text{if } d \text{ is even,} \\ \zeta(2)^{\frac{d-3}{2}}\zeta(3)(1+o(1)), & \text{if } d \geqslant 3 \text{ is odd.} \end{cases}$$

is $\geqslant A(r_2)X \exp\left(-\log\frac{|S_{d+1}|}{|C|}\cdot\frac{\log X}{\log\log X} - \log\log\log X\right)$*, where*

$$C = \begin{cases} (1,2)(3,4)\cdots(d-1,d), & \text{if } d \text{ is even,} \\ (1,2)(3,4)\cdots(d-4,d-3)(d-2,d-1,d), & \text{if } d \text{ is odd.} \end{cases}$$

This work is motivated by the work of Lamzouri [**9**, **10**], who constructed primitive characters χ with large values of $L(1,\chi)$. Basically, we follow [**5**, **9**, **10**, **12**]. The arguments in [**9**] are easily extended. However, obtaining an analogue of [**9**, proposition 2·4] is a main obstacle to extend his method. It is resolved in Proposition 4·2.

2. *Counting number fields with local conditions*

Let K be a S_{d+1}-field of signature (r_1, r_2) for $d+1 \geqslant 3$. We assume that we can count S_{d+1}-fields with finitely many local conditions. Namely, let $\mathcal{S} = (\mathcal{LC}_p)$ be a finite set of local conditions: $\mathcal{LC}_p = \mathcal{S}_{p,C}$ means that p is unramified and the conjugacy class of Frob_p is C. Define $|\mathcal{S}_{p,C}| = \frac{|C|}{|S_n|(1+f(p))}$ for some positive valued function $f(p)$ which satisfies $f(p) = O(1/p)$. More explicitly [**2**], we have $f(p) = p^{-1} + p^{-2}$ if $d+1 = 3$; $f(p) = p^{-1}+2p^{-2}+p^{-3}$ if $d+1 = 4$; $f(p) = p^{-1}+2p^{-2}+2p^{-3}+p^{-4}$ if $d+1 = 5$. There are also several splitting types of ramified primes, which are denoted by $r_1, r_2, \ldots, r_w$: $\mathcal{LC}_p = \mathcal{S}_{p,r_j}$ means that p is ramified and its splitting type is r_j. We assume that there are positive valued functions $c_1(p), c_2(p), \ldots, c_w(p)$ with $\sum_{i=1}^{w} c_i(p) = f(p)$ and define $|\mathcal{S}_{p,r_i}| = \frac{c_i(p)}{1+f(p)}$. We define the local condition $\mathcal{LC}_p = S_{p,r}$ which means that p is ramified, i.e, $r = r_j$ for some j. Define $|\mathcal{S}_{p,r}| = \frac{f(p)}{1+f(p)}$. Let $|\mathcal{S}| = \prod_p |\mathcal{LC}_p|$.

Let $L(X)^{r_2}$ be the set of S_{d+1}-fields K of signature (r_1, r_2) with $X/2 < |D_K| < X$, and let $L(X;\mathcal{S})^{r_2}$ be the set of S_{d+1}-fields K of signature (r_1, r_2) with $X/2 < |D_K| < X$ and the local conditions $\mathcal{S}$. It is noted that we pick up only one number field K from $d+1$ conjugate number fields.

Then we have:

CONJECTURE 2·1. *For some positive constants $\delta < 1$ and κ,*

$$|L(X)^{r_2}| = A(r_2)X + O(X^{\delta}), \tag{2·1}$$
$$|L(X;\mathcal{S})^{r_2}| = |\mathcal{S}|A(r_2)X + O\left(\left(\prod_{p\in S} p\right)^{\kappa} X^{\delta}\right),$$

where the implied constant is uniformly bounded for p and local conditions at p, and see [**2**] *for the precise value of $A(r_2)$ when $d+1 = 3, 4, 5$.*

It is worth noting here that we can control only all the primes up to $c\log X$, where $c < (1-\delta)/\kappa$. If we impose local conditions for all $p \leqslant c'\log X$ with $c' \geqslant (1-\delta)/\kappa$, the error term in Conjecture 2·1 would be larger than the size of $L(X)^{r_2}$.

For S_3-fields, the conjecture was shown by Taniguchi and Thorne [**14**]. In [**2**][1] we proved that Conjecture 2·1 is true for S_4 and S_5-fields.

By abuse of notation, we denote $L(X)^{r_2}$ and $L(X;\mathcal{S})^{r_2}$ as sets of L-functions $L(s,\rho,\widehat{K}/\mathbb{Q}) = \zeta_K(s)/\zeta(s)$. Here we need care in order to ensure one to one correspondence between two sets. Two number fields K_1 and K_2 are said to be arithmetically equivalent if $\zeta_{K_1}(s) = \zeta_{K_2}(s)$. If two number fields K_1 and K_2 are conjugate, then they are arithmetically equivalent. The converse is not always true. A number field K_1 is called arithmetically solitary if $\zeta_{K_1}(s) = \zeta_{K_2}(s)$ implies that K_1 and K_2 are conjugate. It is known that S_n-fields and A_n-fields are arithmetically solitary. See [**7**, chapter II].

To ease the notations, throughout the article, we denote $L(X)^{r_2}$, $L(s,\rho,\widehat{K}/\mathbb{Q})$ by $L(X)$, $L(s,\rho)$ resp. if there is no danger of confusion.

3. *Formula for $L(1,\rho)$ under a certain zero-free region*

In this paper, we assume the strong Artin conjecture, namely that the Artin L-function $L(s,\rho)$ is an automorphic representation of GL_d. This is true for S_3-fields and S_4-fields. It implies the Artin conjecture, namely, $L(s,\rho)$ is entire. For this section, we only need the Artin conjecture. However, in Section 4, we need the strong Artin conjecture in order to use Kowalski–Michel zero density theorem [**8**]. We find an expression of $L(1,\rho)$ as a product over small primes under the assumption that $L(s,\rho)$ has a certain zero-free region. Here all the implicit constants only depend on the degree d of $L(s,\rho)$.

For $Re(s) > 1$, $L(s,\rho)$ has the Euler product:

$$L(s,\rho) = \prod_p \prod_{i=1}^{d} \left(1 - \frac{\alpha_i(p)}{p^s}\right)^{-1}.$$

Then, for $Re(s) > 1$,

$$\log L(s,\rho) = \sum_{n=2}^{\infty} \frac{\Lambda(n) a_\rho(n)}{n^s \log n},$$

where $a_\rho(p^k) = \alpha_1(p)^k + \cdots + \alpha_d(p)^k$. First, we show that when $L(s,\rho)$ has a certain zero-free region, the value $\log L(1,\rho)$ is determined by a short sum.

PROPOSITION 3·1. *If $L(s,\rho)$ is entire and is zero-free in the rectangle* $[\alpha, 1] \times [-x, x]$, *where $x = (\log N)^\beta$, $\beta(1-\alpha) > 2$, and N is the conductor of ρ, then*

$$\log L(1,\rho) = \sum_{n<x} \frac{\Lambda(n) a_\rho(n)}{n \log n} + O((\log N)^{-1}). \tag{3·1}$$

Proof. By Perron's formula (cf. [**13**, theorem 4·1·4]), if x is not an integer,

$$\frac{1}{2\pi i}\int_{c-ix}^{c+ix} \log L(1+s,\rho)\,\frac{x^s}{s}\,ds = \sum_{n<x} \frac{\Lambda(n) a_\rho(n)}{n\log n} + O\left(\sum_{n=1}^{\infty} \left(\frac{x}{n}\right)^c \left|\frac{\Lambda(n)a_\rho(n)}{n\log n}\right| \min\left(1, \frac{1}{x\left|\log \frac{x}{n}\right|}\right)\right)$$

[1] In [**2**], we used the Greek letter γ in place of κ. However, γ is taken for the Euler–Mascheroni constant in this article.

where $c = 1/\log x$. If x is an integer, in the above error term, the sum is over $n \neq x$ and we add an error term $O(\frac{1}{x\log x})$. In any case, we can show the error term is $O(\frac{\log x}{x})$ by controlling the terms with $x/2 \leqslant n \leqslant 3x/2$ and the other terms separately. See [**13**, theorem 4·2·9] for the details.

Now move the contour to $Re(s) = \alpha - 1 + 1/\log x$. We get the residue $\log L(1, \rho)$ at $s = 0$. So the left hand side is $\log L(1, \rho)$ plus

$$\frac{1}{2\pi i}\left(\int_{c-ix}^{\alpha-1+c-ix} + \int_{\alpha-1+c-ix}^{\alpha-1+c+ix} + \int_{\alpha-1+c+ix}^{c+ix}\right) \log L(1+s, \rho)\frac{x^s}{s}\, ds.$$

In order to estimate $|\log L(s, \rho)|$ for $\alpha + c \leqslant Re(s) \leqslant 1 + c$, we follow [**6**, lemma 8·1]: consider the circles with centre $2 + it$ and radii $r = 2 - \sigma < R = 2 - \alpha$. By the assumption, $\log L(s, \rho)$ is holomorphic inside the larger circle. By Daileda [**4**, page 222], for $1/2 < Re(s) \leqslant 3/2$, $|L(s, \rho)| \leqslant N^{\frac{1}{2}}(|s| + 1)^{\frac{d}{2}}$. Hence $Re \log L(s, \rho) = \log|L(s, \rho)| \ll \log N + \log(|s| + 1)$. Clearly, if $Re(s) \geqslant 3/2$, $|\log L(s, \rho)| = O(1)$. By the Borel–Carathéodory theorem,

$$\begin{aligned}|\log L(s, \rho)| &\leqslant \frac{2r}{R-r}\max_{|z-(2+it)|=R} Re \log L(z, \rho) + \frac{R+r}{R-r}|\log L(2+it, \rho)| \\ &\ll (\log x)(\log N + \log(|s| + 1)).\end{aligned}$$

Hence the integral is majorized by $x^{\alpha-1}(\log N)(\log x)^2$. Since $\beta(1 - \alpha) > 2$, $x^{\alpha-1}(\log N)(\log x)^2 \ll (\log N)^{-1}$.

Remark 3·2. *Assume that $L(s, \rho)$ satisfies GRH. Take $\alpha = 1/2 + \epsilon^2$ and $\beta = 2 + \epsilon$. Then, from the above proof, we can see that*

$$\log L(1, \rho) = \sum_{n<(\log N)^{2+\epsilon}} \frac{\Lambda(n)a_\rho(n)}{n\log n} + O\left(\frac{\log\log N}{(\log N)^{\frac{\epsilon}{2}-(2\epsilon^2+\epsilon^3)}}\right),$$

for any $\epsilon > 0$.

Now, using Proposition 3·1, we express $L(1, \rho)$ as a product over small primes. We omit p from $\alpha_i(p)$ for simplicity.

$$\sum_{n<x}\frac{\Lambda(n)a_\rho(n)}{n\log n} = \sum_{k,\, p^k<x}\frac{\alpha_1^k + \cdots + \alpha_d^k}{kp^k} = \sum_{p<x}\sum_{i=1}^{d}\sum_{k<\frac{\log x}{\log p}}\frac{1}{k}(\alpha_i p^{-1})^k. \tag{3·2}$$

Here

$$\sum_{k<\frac{\log x}{\log p}}\frac{1}{k}(\alpha_i p^{-1})^k = -\log(1 - \alpha_i p^{-1}) + A_p,$$

where

$$|A_p| \leqslant \sum_{k\geqslant\frac{\log x}{\log p}}\frac{1}{k}p^{-k} \leqslant \frac{\log p}{\log x}\cdot\frac{p^{-\frac{\log x}{\log p}}}{1 - p^{-1}}.$$

Here $p^{\frac{\log x}{\log p}} = x$. Hence

$$(3·2) = -\sum_{p<x}\sum_{i=1}^{d}\log(1 - \alpha_i p^{-1}) + d\sum_{p<x}A_p.$$

Here

$$\sum_{p<x} |A_p| \leqslant \frac{1}{x\log x} \sum_{p<x} \frac{\log p}{1-p^{-1}} \leqslant \frac{2}{\log x}.$$

Therefore, it is summarized as follows:

PROPOSITION 3·3. *If $L(s,\rho)$ is entire and is zero-free in the rectangle $[\alpha,1]\times[-x,x]$, where $x=(\log N)^\beta$, $\beta(1-\alpha)>2$, and N is the conductor of ρ, then*

$$L(1,\rho) = \prod_{p<x}\prod_{i=1}^{d}(1-\alpha_i p^{-1})^{-1}\left(1+O\left(\frac{1}{\log x}\right)\right). \tag{3·3}$$

Furthermore, if $L(s,\rho)$ satisfies GRH, then

$$L(1,\rho) = \prod_{p<(\log N)^{2+\epsilon}}\prod_{i=1}^{d}(1-\alpha_i p^{-1})^{-1}\left(1+O\left(\frac{1}{\log\log N}\right)\right).$$

In order to find the upper and lower bound of $L(1,\rho)$, we examine the Euler product: Let C be a conjugacy class of S_{d+1}, and let C be a product of $d_1,\ldots,d_k$ cycles, where $d_i\geqslant 1$ for all i and $d_1+\cdots+d_k=d+1$. Then if $\mathrm{Frob}_p\in C$, $(1-X)\prod_{i=1}^d(1-\alpha_i X) = (1-X^{d_1})\cdots(1-X^{d_k})$. Hence

$$\prod_{i=1}^{d}(1-\alpha_i p^{-1})^{-1} = (1-p^{-1})(1-p^{-d_1})^{-1}\cdots(1-p^{-d_k})^{-1}.$$

Now we use Mertens' theorem:

$$\prod_{p\leqslant y}(1-p^{-1})^{-1} = e^\gamma(1+o(1))\log y.$$

Also $\prod_{p\leqslant y}(1-p^{-n})^{-1} = \zeta(n)(1+O(\frac{1}{y\log y}))$ if $n\geqslant 2$.

Hence the upper bound of $\prod_{i=1}^d(1-\alpha_i p^{-1})^{-1}$ is when $C=1$, and it is $(1-p^{-1})^{-d}$. The lower bound is when $C=(1,\cdots,d+1)$, and it is $(1-p^{-1})(1-p^{-d-1})^{-1}$. Moreover, $\prod_{i=1}^d(1-\alpha_i p^{-1})^{-1}$ takes only the values $(1-p^{-e_1})^{-a_1}\cdots(1-p^{-e_l})^{-a_l}(1-p^{-1})^{a_0}$, where $e_1,\ldots,e_l\geqslant 2$, and $-d\leqslant a_0\leqslant 1$. Here $a_0=1$ only when $a_1e_1+\cdots+a_le_l=d+1$. We summarise it as

$$(1-p^{-1})(1-p^{-d-1})^{-1} \leqslant \prod_{i=1}^{d}(1-\alpha_i p^{-1})^{-1} \leqslant (1-p^{-1})^{-d}. \tag{3·4}$$

We note that (3·4) is true even if p is ramified, i.e., when some of α_i's are zero. Hence by the above proposition, under GRH and the strong Artin conjecture for $L(s,\rho)$, for any $\epsilon>0$,

$$\frac{\zeta(d+1)}{(2+\epsilon)e^\gamma\log\log N}(1+o(1)) \leqslant L(1,\rho) \leqslant (e^\gamma(2+\epsilon)\log\log N)^d(1+o(1)).$$

Since ϵ is arbitrarily small, we have shown

$$\left(\frac{1}{2}+o(1)\right)\frac{\zeta(d+1)}{e^\gamma\log\log N} \leqslant L(1,\rho) \leqslant (2+o(1))^d(e^\gamma\log\log N)^d.$$

4. *Extreme residue values*

4·1. *Proof of Theorem* 1·1

Let $y = c_1 \log X$ with $c_1 > 0$. Recall that in Proposition 3·1, the conductor of $L(s, \rho)$ is $|D_K|$, and $X/2 < |D_K| < X$, and $x = (\log X)^\beta$ for some β.

In this section we show that except for $O(Xe^{-c' \frac{\log X}{\log\log X} \log\log\log X})$ in $L(X)$, the lower bound and upper bound on $L(1, \rho)$ are

$$(1 + o(1)) \frac{\zeta(d+1)}{e^\gamma (\log\log |D_K|)}, \quad (1 + o(1))(e^\gamma \log\log |D_K|)^d, \quad \text{resp.}$$

We apply Kowalski–Michel zero density theorem [**8**] to the family $L(X)$. Then except for $O\left((\log X)^{\beta B} X^{(\frac{5d}{2}+1)\frac{1-\alpha}{2\alpha-1}}\right)$ L-functions, every L-function $L(s, \rho)$ in $L(X)$ is zero-free on $[\alpha, 1] \times [-(\log X)^\beta, (\log X)^\beta]$ with $\beta(1-\alpha) > 2$. Here B is a constant depending on the family $L(X)$. We refer to [**1**] for the details.

Since except for $O\left((\log X)^{\beta B} X^{(\frac{5d}{2}+1)\frac{1-\alpha}{2\alpha-1}}\right)$ L-functions, the L-functions in $L(X)$ have the desired zero-free region, we apply Proposition 3·3 to the L-functions in $L(X)$ to obtain

$$L(1, \rho) = \prod_{p<x} \prod_{i=1}^{d} (1 - \alpha_i p^{-1})^{-1} \left(1 + O\left(\frac{1}{\log x}\right)\right).$$

Since

$$\sum_{y<p<x} \frac{1}{p^2} \leqslant \sum_{p>y} \frac{1}{p^2} \leqslant \frac{2}{y \log y},$$

we can show

$$\prod_{y<p<x} \prod_{i=1}^{d} (1 - \alpha_i p^{-1})^{-1} = \exp\left(\sum_{y<p<x} \frac{a_\rho(p)}{p}\right)\left(1 + O\left(\frac{1}{y \log y}\right)\right).$$

We prove

PROPOSITION 4·1. *Let $y = c_1 \log X$ with $c_1 > 0$. Then except for $O(Xe^{-c' \frac{\log X}{\log\log X} \log\log\log X})$ L-functions in $L(X)$ for some constant $c' > 0$, L-functions in $L(X)$ satisfy*

$$\left|\sum_{y<p<x} \frac{a_\rho(p)}{p}\right| \leqslant \frac{1}{(\log\log X)^{1/2}}. \tag{4·1}$$

Hence, for L-functions which have the desired zero-free region and satisfy (4·1),

$$L(1, \rho) = \prod_{p \leqslant y} \prod_{i=1}^{d} \left(1 - \alpha_i p^{-1}\right)^{-1} \left(1 + \frac{1}{(\log\log |D_K|)^{1/2}}\right).$$

This and (3·4) imply immediately Theorem 1·1.

In order to prove Proposition 4·1, we follow the idea in [**9**]. Namely we prove

PROPOSITION 4·2. *Let* $y = c_1 \log X$ *and* $r \leqslant c_2 \frac{\log X}{\log\log X}$ *for some positive constants* c_1 *and* c_2*. Then,*

$$\sum_{L(s,\rho)\in L(X)} \left(\sum_{y<p<x} \frac{a_\rho(p)}{p} \right)^{2r} \ll 2^{2r-1} (dN_d)^{2r} \frac{(2r)!}{r!} \frac{2^{2r}}{(y\log y)^r} X,$$

with an absolute implied constant, where N_d *is the number of splitting types in* S_{d+1}*-fields.*

By Stirling's formula,

$$2^{2r-1}(dN_d)^{2r} \frac{(2r)!}{r!} \frac{2^{2r}}{(y\log y)^r} \ll \left(\frac{cd^2N_d^2 r}{y\log y} \right)^r \text{ for some constant } c.$$

Here $N_1 = 3$, $N_2 = 5$, $N_3 = 11$, $N_4 = 17$ (cf. [**2**]).

Proof. By the multinomial formula, the left hand side is

$$\sum_{L(s,\rho)\in L(X)} \sum_{u=1}^{2r} \frac{1}{u!} \sum\nolimits^{(1)}_{r_1,\ldots,r_u} \frac{(2r)!}{r_1!\cdots r_u!} \sum\nolimits^{(2)}_{p_1,\ldots,p_u} \frac{a_\rho(p_1)^{r_1}\cdots a_\rho(p_u)^{r_u}}{p_1^{r_1}\cdots p_u^{r_u}}, \tag{4·2}$$

where $\sum^{(1)}_{r_1,\ldots,r_u}$ means that the sum is over the ordered u-tuples $(r_1, \ldots, r_u)$ of positive integers such that $r_1 + \cdots + r_u = 2r$, and $\sum^{(2)}_{p_1,\ldots,p_u}$ means the sum over the u-tuples $(p_1, \ldots, p_u)$ of distinct primes such that $y < p_i < x$ for each i. Each ordered u-tuple $(r_1, \ldots, r_u)$ gives a composition of $2r$. Here a composition means an ordered partition.

Write

$$(4·2) = \sum_{u=1}^{2r} \sum\nolimits^{(1)}_{r_1,\ldots,r_u} \frac{(2r)!}{r_1!\cdots r_u!} \frac{1}{u!} \sum\nolimits^{(2)}_{p_1,\ldots,p_u} \frac{1}{p_1^{r_1}\cdots p_u^{r_u}} \left(\sum_{L(s,\rho)\in L(X)} a_\rho(p_1)^{r_1}\cdots a_\rho(p_u)^{r_u} \right).$$

We will show that for any composition $r_1 + r_2 + \cdots + r_u = 2r$,

$$\begin{aligned} &\frac{(2r)!}{r_1!\cdots r_u!} \frac{1}{u!} \sum\nolimits^{(2)}_{p_1,\ldots,p_u} \frac{1}{p_1^{r_1}\cdots p_u^{r_u}} \left(\sum_{L(s,\rho)\in L(X)} a_\rho(p_1)^{r_1}\cdots a_\rho(p_u)^{r_u} \right) \\ &\ll (dN_d)^{2r} X \frac{(2r)!}{r!} \frac{2^{2r}}{(y\log y)^r}. \end{aligned} \tag{4·3}$$

Since the number of compositions of $2r$ is 2^{2r-1}, it implies that

$$(4·2) \ll 2^{2r-1}(dN_d)^{2r} \frac{(2r)!}{r!} \frac{2^{2r}}{(y\log y)^r} X.$$

First, we consider compositions with $r_i \geqslant 2$ for all i. Then by using the trivial bound,

$$\begin{aligned} &\sum\nolimits^{(2)}_{p_1,\ldots,p_u} \frac{1}{p_1^{r_1}\cdots p_u^{r_u}} \left(\sum_{L(s,\rho)\in L(X)} a_\rho(p_1)^{r_1}\cdots a_\rho(p_u)^{r_u} \right) \\ &\ll d^{2r} X \left(\sum_{y<p_1<x} \frac{1}{p_1^{r_1}} \right) \cdots \left(\sum_{y<p_u<x} \frac{1}{p_u^{r_u}} \right) \ll d^{2r} X \frac{2^{2r}}{(y\log y)^r} \left(\frac{\log y}{y} \right)^{r-u}. \end{aligned}$$

Hence (4·3) is proved once we show that for any $r_1, \ldots, r_u$ such that $r_1 + \cdots + r_u = 2r$, and $r_i \geqslant 2$ for all i

$$\frac{1}{u!r_1!\cdots r_u!}\left(\frac{\log y}{y}\right)^{r-u} \leqslant \frac{1}{r!},$$

or equivalently

$$\frac{r!}{u!r_1!\cdots r_u!} \leqslant \left(\frac{y}{\log y}\right)^{r-u}. \tag{4·4}$$

Since $r_i \geqslant 2$ for all $i = 1, 2, \ldots, u$, we have $u \leqslant r$. Since $y = c_1 \log X$ and $r \leqslant c_2 \frac{\log X}{\log\log X}$, $r \leqslant y/\log y$ for sufficiently small c_2. Then

$$\frac{r!}{u!r_1!\cdots r_u!} \leqslant \frac{r!}{u!} = r(r-1)\ldots(r-u+1) \leqslant r^{r-u} \leqslant \left(\frac{y}{\log y}\right)^{r-u}.$$

Next, suppose $r_i = 1$ for some i. We may assume that $r_1+\cdots+r_m+r_{m+1}+\cdots+r_u = 2r$, $r_1 = \cdots = r_m = 1$, and $r_{m+1} > 1, \ldots, r_u > 1$. First, we need a technical combinatorial lemma.

LEMMA 4·3. *Let r_i's be as above. Then*

$$\frac{1}{u!}\cdot\frac{1}{r_1!r_2!\ldots r_m!r_{m+1}!\ldots r_u!}\cdot\frac{y^u}{y^{m+r}}\cdot\frac{(\log y)^r}{(\log y)^u} \leqslant \frac{1}{r!}. \tag{4·5}$$

Proof. First, we assume that m is even. Then since $r_{m+1}, \ldots, r_u \geqslant 2$, and $r_{m+1}+\cdots+r_u = 2r - m$, by (4·4),

$$\frac{\left(\frac{2r-m}{2}\right)!}{(u-m)!r_{m+1}!\ldots r_u!} \leqslant \left(\frac{y}{\log y}\right)^{(r-m/2)-(u-m)} \leqslant \left(\frac{y}{\log y}\right)^{r+m/2-u}.$$

Hence

$$\frac{1}{r_{m+1}!\ldots r_u!} \leqslant \frac{(u-m)!}{(r-m/2)!}\left(\frac{y}{\log y}\right)^{r+m/2-u}.$$

So

$$\begin{aligned}
&\frac{1}{u!}\cdot\frac{1}{r_1!r_2!\ldots r_m!r_{m+1}!\ldots r_u!}\frac{y^u}{y^{m+r}}\frac{(\log y)^r}{(\log y)^u}\\
&\leqslant \frac{(u-m)!}{u!}\frac{1}{(r-m/2)!}\left(\frac{y}{\log y}\right)^{r+m/2-u}\frac{y^u}{y^{m+r}}\frac{(\log y)^r}{(\log y)^u}\\
&\leqslant \frac{(u-m)!}{u!}\frac{1}{(r-m/2)!}\frac{1}{(y\log y)^{m/2}}
\end{aligned}$$

Since $r < y$ and $\frac{(u-m)!}{u!} < 1$,

$$\frac{r!}{(r-\frac{m}{2})!}\frac{(u-m)!}{u!} \leqslant (y\log y)^{m/2}.$$

This implies

$$\frac{(u-m)!}{u!}\frac{1}{(r-m/2)!}\frac{1}{(y\log y)^{m/2}} \leqslant \frac{1}{r!}.$$

Hence we have (4·5).

When m is odd, we consider a composition of $2r - m + 3$ of the form:

$$r'_{m+1} = r_{m+1}, r'_{m+2} = r_{m+2}, \ldots, r'_u = r_u, \text{ and } r'_{u+1} = 3.$$

With this composition, by (4·4),

$$\frac{\left(\frac{2r-m+3}{2}\right)!}{(u-m+1)! r_{m+1}! \ldots r_u! 3!} = \frac{\left(\frac{2r-m+3}{2}\right)!}{(u-m+1)! r'_{m+1}! \ldots r'_u! r'_{u+1}!} \leqslant \left(\frac{y}{\log y}\right)^{r+m/2+1/2-u}.$$

As we did for the case of even m, since $r < y$ and $\frac{(u-m+1)!}{u!} \leqslant 1$, we have

$$\frac{r!}{(r - \frac{m-3}{2})!} \frac{(u-m+1)!}{u!} \leqslant \frac{1}{6}(y \log y)^{\frac{m-1}{2}} \log y.$$

This implies (4·5).

Recall that we are treating a composition $r_1 + r_2 + \cdots + r_u = 2r$ with $r_1 = r_2 = \cdots = r_m = 1$. Consider

$$\sum_{L(s,\rho) \in L(X)} a_\rho(p_1)^{r_1} \cdots a_\rho(p_u)^{r_u}. \tag{4·6}$$

Let N be the number of conjugacy classes of G. Recall that there are w ramified splitting types so that $N_d = N + w$. Partition the sum $\sum_{\rho \in L(X)}$ into N_d^u sums, namely, given $(\mathcal{S}_1, \ldots, \mathcal{S}_u)$, where $\mathcal{S}_i$ is either $\mathcal{S}_{p_i,C}$ or $\mathcal{S}_{p_i,r_j}$, we consider the set of $\rho \in L(X)$ with the local conditions $\mathcal{S}_i$ for each i. Note that in each such partition, $a_\rho(p_1)^{r_1} \cdots a_\rho(p_u)^{r_u}$ remains a constant.

Suppose one of $p_1, \ldots, p_m$ is unramified, say p_1. Consider $N(N+w)^{u-1}$ such partitions in (4·6). Fix the splitting types of $p_2, \ldots, p_u$ and let Frob_{p_1} runs through the conjugacy classes of G. Let $L(X; p_2, \ldots, p_u)$ be the set of $\rho \in L(X)$ with the fixed splitting types. Then the sum of such N partitions is

$$\sum_C a_\rho(p_1) a_\rho(p_2) \cdots a_\rho(p_u)^{r_u} \sum_{\rho \in L(X, p_2, \ldots, p_u)} 1.$$

By (2·1),

$$\sum_{\rho \in L(X, p_2, \ldots, p_u)} 1 = \frac{|C|}{|G|(1 + f(p_1))} A(p_2, \ldots, p_u) X + O((p_1 \cdots p_u)^\kappa X^\delta),$$

for some constant $A(p_2, \ldots, p_u)$. Let χ_ρ be the character of ρ. Then $a_\rho(p) = \chi_\rho(g)$, where $g = \mathrm{Frob}_p$. By orthogonality of characters, $\sum_C |C| a_\rho(p_1) = \sum_{g \in G} \chi_\rho(g) = 0$. Hence the above sum is $O(N(p_1 \cdots p_u)^\kappa X^\delta)$. By the trivial bound, $|a_\rho(p_2) \cdots a_\rho(p_u)^{r_u}| \leqslant d^{2r}$. Hence the contribution from these partitions to (4·3) is

$$\begin{aligned}
&\ll N d^{2r} X^\delta \frac{(2r)!}{r_1! \cdots r_u!} \frac{1}{u!} \sum\nolimits^{(2)}_{p_1, \ldots, p_u} p_1^{\kappa-1} \cdots p_m^{\kappa-1} p_{m+1}^{\kappa - r_{m+1}} \cdots p_u^{\kappa - r_u} \\
&\ll N d^{2r} X^\delta \frac{(2r)!}{r_1! \cdots r_u!} \frac{1}{u!} \prod_{i=1}^m \left(\sum_{y < p_i < x} p_i^{\kappa-1}\right) \prod_{i=m+1}^u \left(\sum_{y < p_i < x} p_i^{\kappa - r_i}\right) \\
&\ll N d^{2r} X^\delta \frac{(2r)!}{r_1! \cdots r_u!} \frac{1}{u!} \frac{x^{u\kappa}}{(\log x)^u} \ll N d^{2r} X^\delta \frac{(2r)!}{r!} \frac{x^{u\kappa}}{(\log x)^u} y^{m+r-u} (\log y)^{u-r} \\
&\ll N d^{2r} X^\delta \frac{(2r)!}{r!} (\log X)^{u\kappa\beta + r}.
\end{aligned}$$

Here we used Lemma 4·3 for the second last inequality. Hence the contribution from the cases when one of $p_1, \ldots, p_m$ is unramified, is

$$\ll (N+w)^u d^{2r} X^{\delta} \frac{(2r)!}{r!} (\log X)^{u\kappa\beta+r} \ll N_d^{2r} d^{2r} X^{\delta} \frac{(2r)!}{r!} (\log X)^{2r(\kappa\beta+1)}.$$

If we choose c_2 sufficiently small, for example, taking $c_2 = \frac{1-\delta}{20(\kappa\beta+1)}$, $X^{\delta}(\log X)^{2r\kappa\beta+1} \ll X\frac{2^{2r}}{(y\log y)^r}$. This verifies (4·3).

Now suppose that $p_1, \ldots, p_m$ are all ramified. Then by (2·1), the number of elements in the set of $\rho \in L(X)$ with the local condition $\mathcal{S}_{p_i,r}$ for $i = 1, \ldots, m$, is

$$\prod_{i=1}^{m} \frac{f(p_i)}{1+f(p_i)} A(r_2)X + O((p_1\cdots p_m)^{\kappa} X^{\delta}),$$

Note that $\frac{f(p)}{1+f(p)} \ll 1/p$. By the trivial bound, $a_\rho(p_1)^{r_1}\cdots a_\rho(p_u)^{r_u} \leqslant d^u \leqslant d^{2r}$. Hence the main term contributes to (4·3)

$$Xd^{2r}\sum\nolimits^{(2)}_{p_1,\ldots,p_u} \frac{1}{p_1^2\cdots p_m^2 p_{m+1}^{r_{m+1}}\cdots p_u^{r_u}} \ll Xd^{2r}\prod_{i=1}^{m}\left(\sum_{y<p_i<x} p_i^{-2}\right)\prod_{i=m+1}^{u}\left(\sum_{y<p_i<x} p_i^{-r_i}\right)$$
$$\ll Xd^{2r}2^{2r}(y\log y)^{-r}\frac{y^u}{y^{m+r}}\cdot\frac{(\log y)^r}{(\log y)^u}.$$

By Lemma 4·3, (4·3) is verified in this case.

The contribution of the error term $O((p_1\cdots p_m)^{\kappa}X^{\delta})$ to (4·3) is similar to the case when p_1 is unramified.

Now take $y = c_1\log X$, and $r = c_2\frac{\log X}{\log\log X}$. Then from Proposition 4·2, the number of $\rho \in L(X)$ such that $\left|\sum_{y<p<x}\frac{a_\rho(p)}{p}\right| > \frac{1}{(\log\log X)^{1/2}}$, is

$$\ll Xe^{-c'\frac{\log X}{\log\log X}\log\log\log X}, \tag{4·7}$$

for some $c' > 0$. This proves Proposition 4·1.

4·2. *Infinite family of number fields with extreme residues*

Let C be a conjugacy class of S_{d+1}, and $\mathcal{S} = (S_{p,C})_{p\leqslant y}$ be the set of local conditions such that for every prime $p \leqslant y$, $\mathrm{Frob}_p \in C$. We denote $L(X,\mathcal{S})^{r_2}$ by $L(X,\mathcal{S})$. Conjecture 2·1 says that

$$|L(X,\mathcal{S})| = A(r_2)X\prod_{p\leqslant y}\frac{\frac{|C|}{|S_{d+1}|}}{1+f(p)} + O\left(\left(\prod_{p\leqslant y}p\right)^{\gamma}X^{\delta}\right).$$

The main term is

$$A(r_2)\frac{X}{\log y}\exp\left(-\log\frac{|S_{d+1}|}{|C|}\cdot\frac{\log X}{\log\log X}\right). \tag{4·8}$$

This is larger than (4·7). Also we may assume that almost all L-functions in $L(X,\mathcal{S})$ have the desired zero-free region of the form in Proposition 3·3. Hence, by Proposition 4·1, except $O(Xe^{-c'\frac{\log X}{\log\log X}\log\log\log X})$ fields,

$$L(1,\rho) = \prod_{\substack{p\leqslant y\\ \mathrm{Frob}_p\in C}}\prod_{i=1}^{d}(1-\alpha_i p^{-1})^{-1}\left(1+O\left(\frac{1}{(\log\log|D_K|^{\frac{1}{2}}}\right)\right).$$

By taking $C = 1$, we obtain an infinite family of number fields with the upper bound. On the other hand, by taking $C = (1, \cdots, d+1)$, we obtain an infinite family of number fields with the lower bound. This proves Theorem 1·2.

In a similar way, for each $0 \leqslant i \leqslant d$, $d - i$ even, we can construct an infinite family of number fields with the residue

$$\zeta(2)^{\frac{d-i}{2}} e^{\gamma i} (\log\log |D_K|)^i (1 + o(1)).$$

In particular we obtain an infinite family of number fields with bounded residues by taking

$$C = \begin{cases} (1,2)(3,4)\cdots(d-1,d), & \text{if } d \text{ is even}, \\ (1,2)(3,4)\cdots(d-4,d-3)(d-2,d-1,d), & \text{if } d \text{ is odd}, \end{cases}$$

for which

$$\mathrm{Res}_{s=1}\zeta_K(s) = L(1,\rho) = \begin{cases} \zeta(2)^{\frac{d}{2}}(1+o(1)), & \text{if } d \text{ is even}, \\ \zeta(2)^{\frac{d-3}{2}}\zeta(3)(1+o(1)), & \text{if } d \geqslant 3 \text{ is odd}, \end{cases}$$

and it proves Theorem 1·3.

Acknowledgments. We would like to thank the referee for their careful reading of the paper and for many comments.

REFERENCES

[**1**] P.J. CHO and H.H. KIM. Probabilistic properties of number fields. *J. Number Theory* **133** (2013), 4175–4187.

[**2**] P.J. CHO and H.H. KIM. Central limit theorem for Artin L-functions. *Int. J. Number Theory* **13** (2017), no. 1, 1–14.

[**3**] S. CHOWLA. Improvement of a theorem of Linnik and Walfisz. *Proc. London Math. Soc.* **50** (1949), 423–429.

[**4**] R.C. DAILEDA. Non-abelian number fields with very large class numbers. *Acta Arith.* **125** (2006), 215–255.

[**5**] A. GRANVILLE and K. SOUNDARARAJAN. The Distribution of Values of $L(1, \chi_d)$. *Geom. Funct. Anal.* **13** (2003), no. 5, 992–1028.

[**6**] A. GRANVILLE and K. SOUNDARARAJAN. Large character sums. *J. Amer. Math. Soc.* **14** (2000), no. 2, 365–397.

[**7**] N. KLINGEN. *Arithmetical Similarities.* Oxford Math. Monogr., Oxford Sci. Publ. (The Clarendon Press, Oxford University Press, New York 1998).

[**8**] E. KOWALSKI and P. MICHEL. Zeros of families of automorphic L-functions close to 1. *Pacific J. Math.* **207** (2002), No. 2, 411–431.

[**9**] Y. LAMZOURI. Extreme values of class numbers of real quadratic fields. *Int. Math. Res. Not.* 2015, no. 2, 632–653.

[**10**] Y. LAMZOURI. Large values of $L(1, \chi)$ for k-th order characters χ and applications to character sums. *Mathematika* **63** (2016), 53–71

[**11**] J.E. LITTLEWOOD. On the class number of corpus $P(\sqrt{-k})$. *Proc. London Math. Soc.* **27**, no.1 (1928): 358–372.

[**12**] H.L. MONTGOMERY and R.C. VAUGHAN. Extreme values of Dirichlet L-functions at 1. *Number Theory in Progress*, Vol. 2 (Zakopane-Kościelisko, 1997) (de Gruyter, Berlin, 1999), 1039–1052.

[**13**] M.R. MURTY. *Problems in analytic number theory*, Second edition. Graduate Texts in Math. *206* (Springer, New York, 2008).

[**14**] T. TANIGUCHI and F. THORNE,. Secondary terms in counting functions for cubic fields. *Duke Math. J.* **162** (2013), 2451–2508.