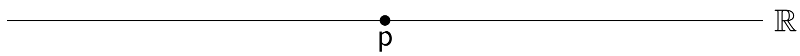
Localisation

In algebraic geometry, we study geometric spaces using methods and tools from algebra, such as ring theory. It turns out that of the most fruitful way to study a geometric space is by studying *functions* defined on this space. You have already seen examples of this very powerful shift of perspective: for example, every ideal in a ring is the kernel of some ring homomorphism; thus, given a ring A, studying the structure of all ideals in A is equivalent to studying all ring homomorphisms out of A. One of these very powerful algebraic techniques of studying functions on a space is called *localisation*, and it is used all the time in algebraic geometry. We begin with a discussion of an important motivating example, which will lead us to the right construction.

Suppose we are interested in studying the real line $\mathbb{R}$ at or *near* some specific point $\mathsf{p} \in \mathbb{R}$:



We consider all continuous real-valued functions defined near a point $\mathsf{p} \in \mathbb{R}$. (To be precise, a function f is **defined near** $\mathsf{p} \in \mathbb{R}$ if there exists some (no matter how small) open set $\mathsf{U} \subseteq \mathbb{R}$ containing p such that f is continuous on U.) Such functions form a unital[1] commutative ring:

$$\mathsf{A} := \Big\{ \text{ring of continuous } \mathbb{R}\text{-valued functions defined near } \mathsf{p} \in \mathbb{R} \Big\} .$$

Exercise 0.1: Check that A is a ring. What is the multiplicative identity in A?

Since we want to understand $\mathbb{R}$ in the vicinity of p, we only want to study the *local behaviour* of functions near $\mathsf{p} \in \mathbb{R}$. This means that if two functions behave differently far from p, we should just ignore this difference. Another way of saying this is that if two functions are the same near p, we want to make no distinction between them; i.e., we want to identify them. Here, "the same" means we identify two functions if they agree on some (possibly very small) open set around p. The mathematical way to say this is: if $f, g \in \mathsf{A}$, then

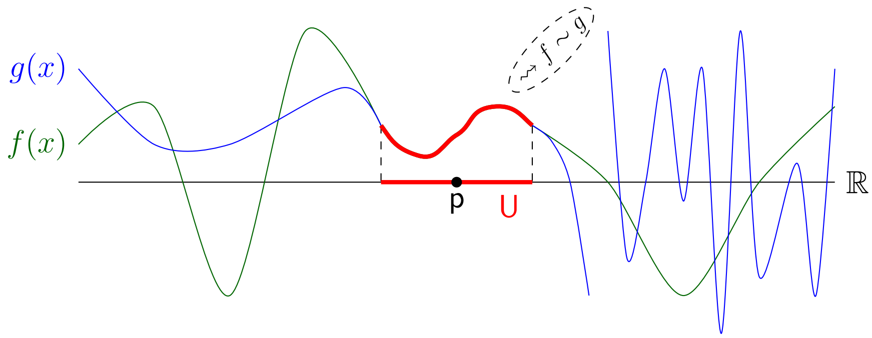
$$f \sim g \quad :\Leftrightarrow \quad \text{there is an open set } \mathsf{U} \text{ around } \mathsf{p} \text{ such that } f(x) = g(x) \text{ for all } x \in \mathsf{U}. \tag{1}$$

Then $\sim$ is an equivalence relation[2] on A.

Exercise 0.2: Check that $\sim$ is an equivalence relation.

[1] **unital** ring := ring with a multiplicative identity.
[2] A common notation for this equivalence relation $\sim$ is "$f \sim_{\mathsf{p}} g$"; we won't use it in this note.

Even if two functions f, g have completely different values outside of U, or even if either or both of f, g are wildly discontinuous outside of U – we don't care. The picture you should have in mind is this:



These equivalence classes are called **germs of functions**[3]. A common notation for germs of functions is "$[f]_\mathsf{p}$" — this the equivalence class represented by some function f defined near p. Another common notation, which doesn't reference a specific representative, is "φ_p". Any representative other g of $[f]_\mathsf{p}$ agrees with f on some open neighbourhood U of p, and this neighbourhood U depends on the choice[4] of g. Let's denote the set of equivalence classes by A_p:

$$\mathsf{A}_\mathsf{p} := \mathsf{A}/\!\sim\, = \{\text{germs of functions in } \mathsf{A}\} \quad .$$

Let me highlight a few general points regarding germs; they are meant to indicate to you that *germs are weird (but cool!)*.

- **A germ φ_p has a value at p.** Of course, any two representatives f, g of the same germ φ_p are defined near p and (crucially) agree on some open neighbourhood of p; in particular, they agree at p. That is, *all* representatives of φ_p agree at p:

$$\text{if } \varphi_\mathsf{p} \in \mathsf{A}_\mathsf{p}, \text{ then } \varphi_\mathsf{p}(\mathsf{p}) = f(\mathsf{p}) \text{ for } \textit{any} \text{ representative } f \text{ of } \varphi_\mathsf{p}.$$

- **But a germ φ_p has no "value" at any other point.** In general, it doesn't even make sense[5] to ask for a value of φ_p at some other point $\mathsf{q} \neq \mathsf{p}$. This is because A is the ring of functions that are required to be defined only *near* p, so we can always find a representative f of φ_p that is *undefined* at q; alternatively, we can always find two representatives f, g of φ_p which have completely different values at q.

- **Yet a germ $[f]_\mathsf{p}$ contains an enormous amount of information about f.** For example, let's assume for the moment that we are considering with differentiable functions. Since any two representatives f, g of a germ φ_p agree on some open neighbourhood U of p, they have the same derivative at p:

$$\text{if } f, g \in \varphi_\mathsf{p}, \text{ then } f'(\mathsf{p}) = g'(\mathsf{p}).$$

[3]The word *germ* is not in reference to the microorganisms; instead, it is derived from the French word *germe* (meaning *cereal germ*) which is the embryo of a seed out of which a plant grows. This metaphor is not at all ludicrous.

[4]In other words, given a germ φ_p, it is *not* true that there exists an open neighbourhood U of p such that *all* representatives of φ_p agree on U. Think about this until it makes total sense to you.

[5]In this argument, I'm using the fact that we chose to consider *continuous* functions. For more restricted kinds of functions — such as analytic functions — there is a sense in which one can ask the question "what is the value of the germ φ_p at q?" But, except in a special situation of a simply connected domain, this question is ambiguous and depends on exactly *which parth* one takes to go from p to q. For more information, look up analytic continuation.

Thus, *all* representatives of φ_p have the same derivative at p! In fact, the exact same argument holds for all higher derivatives; i.e., *all* representatives of φ_p have the same Taylor polynomial! This means that *the germ* $[f]_p$ *knows the Taylor polynomial of* f.

Germs A_p form a unital commutative ring under usual operations:

$$[f]_p + [g]_p := [f+g]_p \qquad \text{and} \qquad [f]_p \cdot [g]_p := [f \cdot g]_p \ .$$

Exercise 0.3: Check that A_p with these operations is a ring. What is the multiplicative identity in A_p? Notice that the fact that $\sim$ is an equivalence relation, these operations are well-defined; make sure you understand this point.

Before we continue, let me summarise the concept of a germ in a definition.

Definition 0.4 (Germ of Functions)
The equivalence classes φ_p on A under the equivalence relation (1),

$$f \sim g \quad :\Leftrightarrow \quad \text{there is an open set } U \text{ around } p \text{ such that } f(x) = g(x) \text{ for all } x \in U,$$

are called **germs of functions**. The germ of a function $f \in A$ is denoted by $[f]_p$. The set of equivalence classes is denoted by A_p and called the **ring of germs** on A.

What are the invertible elements[6] in the ring of germs A_p? Let $[f]_p \in A_p$ be a germ. Intuitively, the inverse of $[f]_p$ should be the germ of the reciprocal, $[1/f]_p$. But $1/f(x)$ only makes sense whenever $f(x) \neq 0$. If $f(p) \neq 0$, then by continuity f is non-zero in a neighbourhood U of p, so the reciprocal $1/f$ is well-defined on U, and therefore the germ $[1/f]_p$ is well-defined too. This argument fails if and only if $f(p) = 0$. Thus, the invertible elements in A_p are precisely the germs of functions from the subset

$$S := \left\{ f \in A \;\middle|\; f(p) \neq 0 \right\} \subseteq A \ .$$

Thus, in the ring A_p, we are allowed to divide only by germs of functions from the set S. Therefore, we can generally present elements of A_p as *fractions* whose denominators are germs of functions in S, like so:

$$A_p = \left\{ \frac{[f]_p}{[g]_p} \;\middle|\; f, g \in A \quad \text{and} \quad g \in S \right\} \ .$$

The set S is very important: it tells us precisely the germs of which functions can play the role of denominators. What properties does it have? The constant function $1 \in A$ is non-zero at p, so we have $1 \in S$. Moreover, if two functions f, g are non-zero at p, their product $f \cdot g$ is also non-zero at p; thus, S is closed under multiplication. However, S is not an ideal or subring of A: indeed, it fails closedness under addition (why?). Such sets bear a special name.

Definition 0.5 (Multiplicatively Closed Subset)
A subset S of a ring A is a **multiplicatively closed subset** if

- $1 \in S$,
- $f, g \in S \implies f \cdot g \in S$.

We are now ready to give the main definition of this note.

[6] **invertible element** := unit.

Definition 0.6 (Localisation)
Let A be an integral domain, and $\mathsf{S} \subseteq \mathsf{A}$ be a multiplicatively closed subset.
The **localisation of A at S** is the ring

$$\mathsf{S}^{-1}\mathsf{A} := \left\{ \frac{a}{s} \;\middle|\; a \in \mathsf{A} \quad \text{and} \quad s \in \mathsf{S} \right\} \Big/ \sim \quad ,$$

where the equivalence relation $\sim$ is given by

$$\frac{a_1}{s_1} \sim \frac{a_2}{s_2} \qquad :\Leftrightarrow \qquad a_1 s_2 - a_2 s_1 = 0 \quad ,$$

with operations given by usual fraction addition and multiplication:

$$\frac{a_1}{s_1} + \frac{a_2}{s_2} := \frac{a_1 s_2 + a_2 s_1}{s_1 s_2} \qquad \text{and} \qquad \frac{a_1}{s_1} \cdot \frac{a_2}{s_2} := \frac{a_1 a_2}{s_1 s_2} \quad .$$

The element $1/1$ is the multiplicative identity in $\mathsf{S}^{-1}\mathsf{A}$.

A few remarks regarding the definition. Notice that in the rules for addition and multiplication, the denominators are always multiplied; this is the reason we always localise at a *multiplicatively closed subset*. A note on notation: "$\mathsf{S}^{-1}\mathsf{A}$" should be read as a single item; there is no such thing as the ring S^{-1}. Lastly, the assumption that A be an integral domain can be dropped, and localisation still makes sense, except the equivalence relation needs a slight modification.

Example 0.7 (germs are localisation at non-zero elements)
The ring A_p of germs we described above is the localisation of A at the set S of functions non-vanishing at p:

$$\mathsf{A}_\mathsf{p} = \mathsf{S}^{-1}\mathsf{A} \quad .$$

Actually, I claim that you have already seen localisation of rings. It's not a coincidence that the definition of the *field of fractions* of a ring looks very similar to the definition of localisation: the construction of a field of fractions is a special case of localisation.

Example 0.8 (rationals are the localisation of integers)
Let $\mathsf{A} := \mathbb{Z}$, and $\mathsf{S} := \mathbb{Z}^\times = \mathbb{Z} \setminus \{0\}$. Then $\mathsf{S}^{-1}\mathsf{A} = \mathbb{Q}$, the field of rational numbers.

Example 0.9
Let $\mathsf{A} := \mathbb{Z}$, and choose any nonzero integer $m \in \mathbb{Z}$. Let $\mathsf{S} := \{1, m, m^2, m^3, \ldots\} = \left\{ m^k \;\middle|\; k \in \mathbb{Z}_{\geqslant 0} \right\}$. Then the localisation of A at S is

$$\mathsf{S}^{-1}\mathsf{A} = \left\{ \frac{n}{m^k} \;\middle|\; n \in \mathbb{Z} \right\} \quad ;$$

i.e., it is the subset of rationals whose denominators are powers of m.

Example 0.10
Let $\mathsf{A} := \mathbb{R}[x]$. Choose the element $x \in \mathsf{A}$, and let $\mathsf{S} := 1, x, x^2, x^3, \ldots = \left\{ x^n \;\middle|\; n \in \mathbb{Z}_{n \geqslant 0} \right\}$. Then, intuitively, in the ring $\mathsf{S}^{-1}\mathsf{A}$, we are allowed to divide by elements of S; i.e., by powers of x. Thus, $\mathsf{S}^{-1}\mathsf{A}$ contains elements like $\frac{1}{x}, \frac{1}{x^2}, \ldots$, and their $\mathbb{R}$-linear combinations. In other words, $\mathsf{S}^{-1}\mathsf{A}$ is the ring of Laurent polynomials:

$$\mathsf{S}^{-1}\mathsf{A} = \mathbb{R}[x, x^{-1}] \quad .$$