

Birds on tangled wires

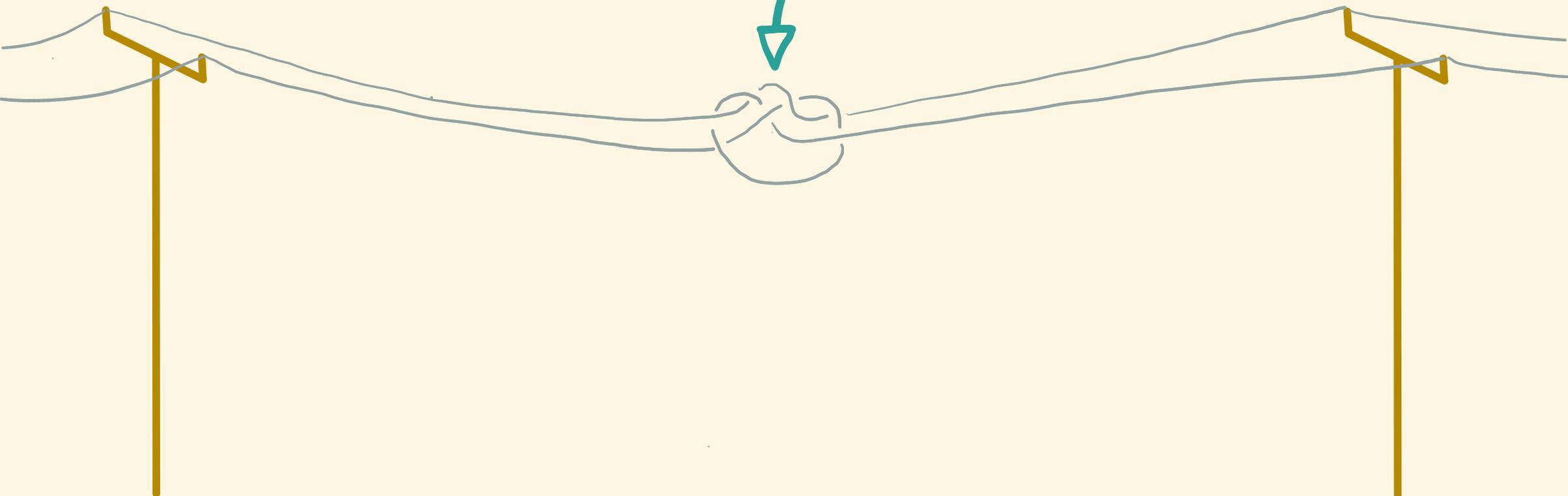
Jesse Fröhlich

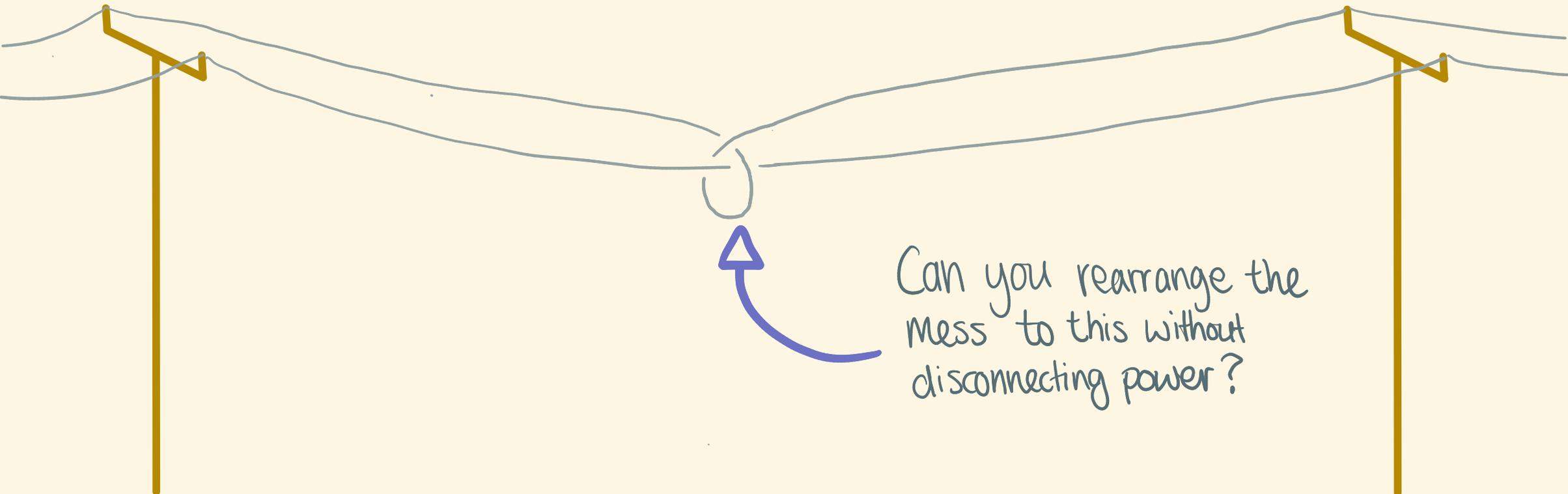
PhD advisor: Dror Bar-Natan



Bird's Eye Conference
University of Toronto
March 3, 2023

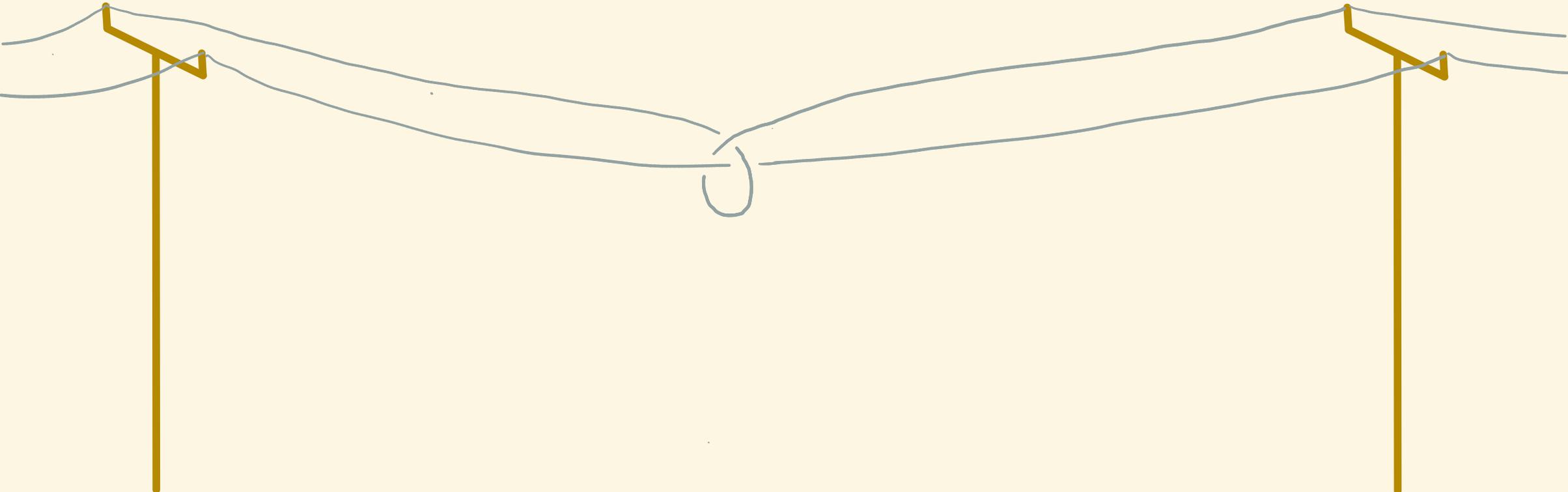
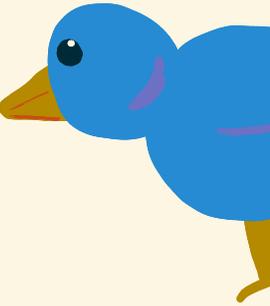
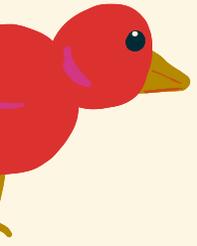
You have been tasked
with cleaning this up



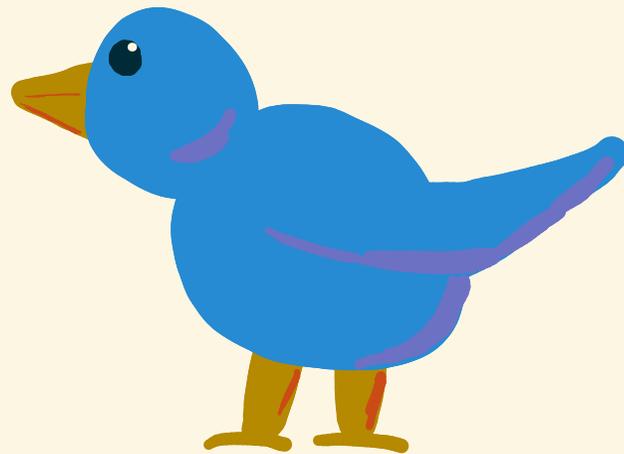
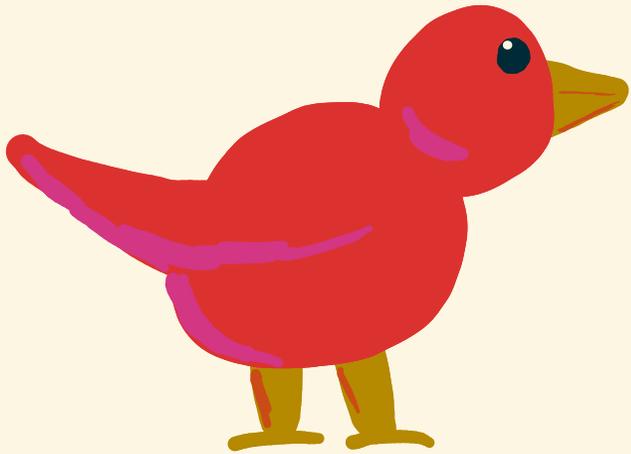


Can you rearrange the mess to this without disconnecting power?

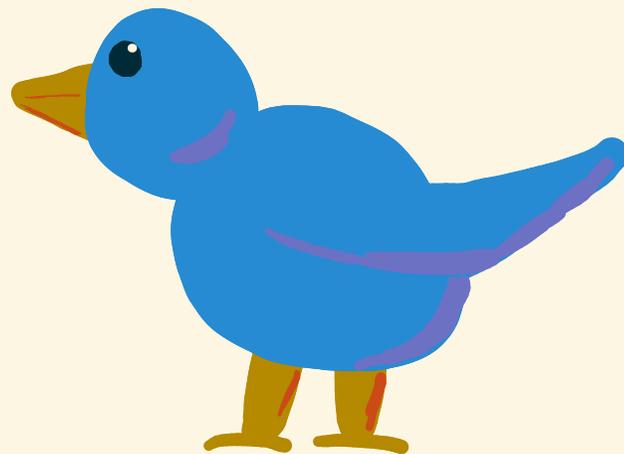
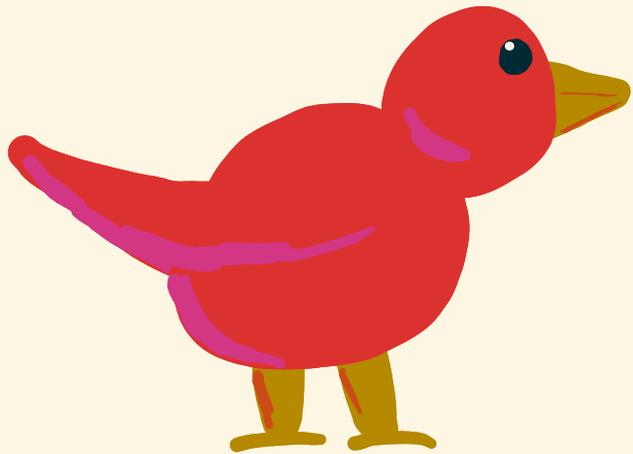
Let's enlist
the birds



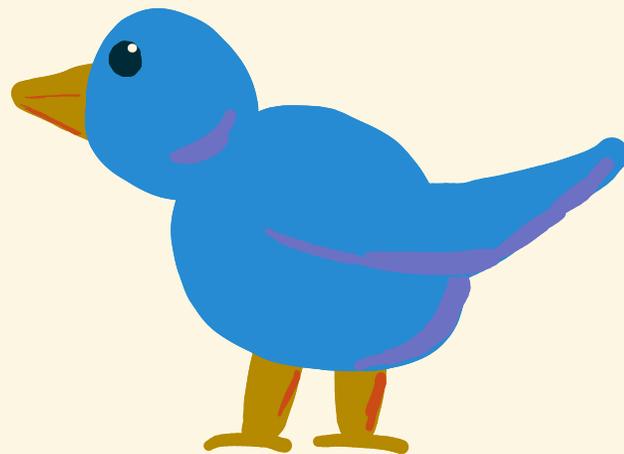
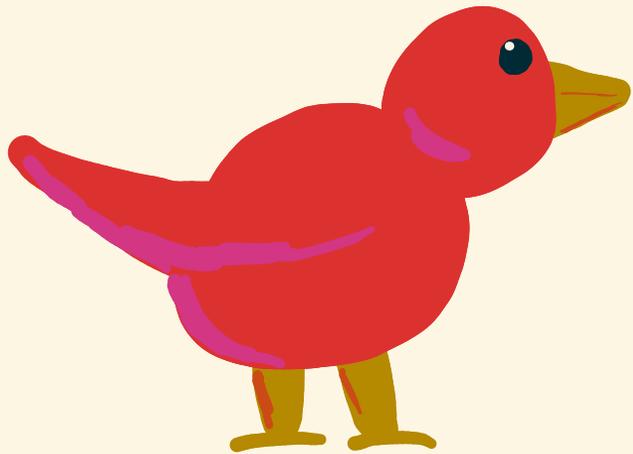
Meet the lovebirds.



Meet the lovebirds.
How strong is their bond?

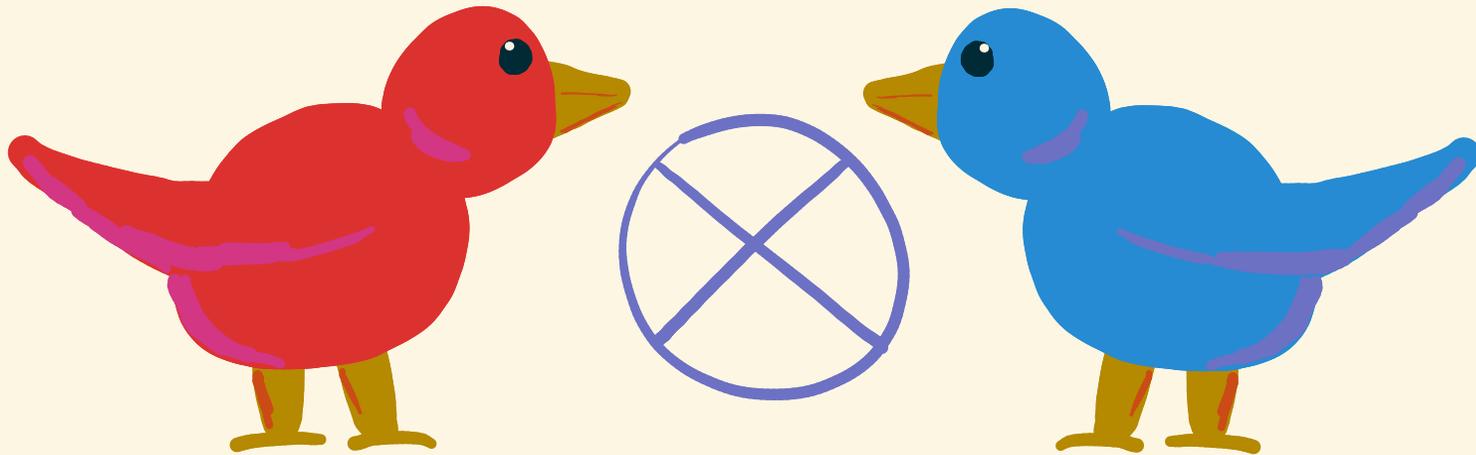


Meet the lovebirds.
How strong is their bond? Ten.

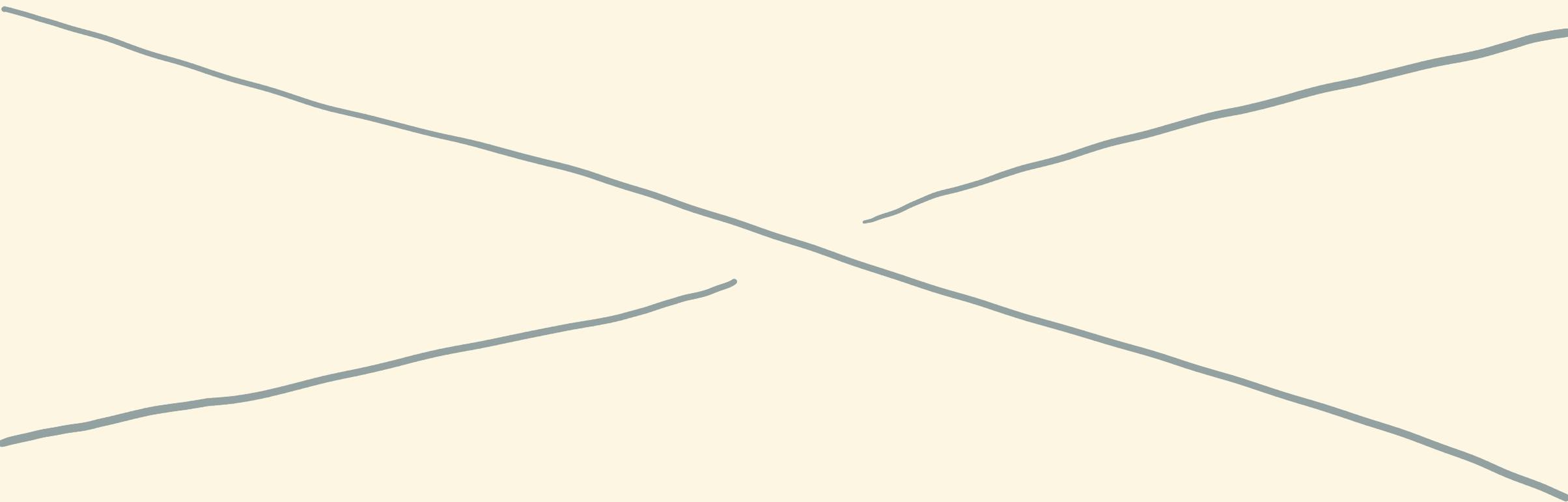


Meet the lovebirds.

How strong is their bond? **Tensor**

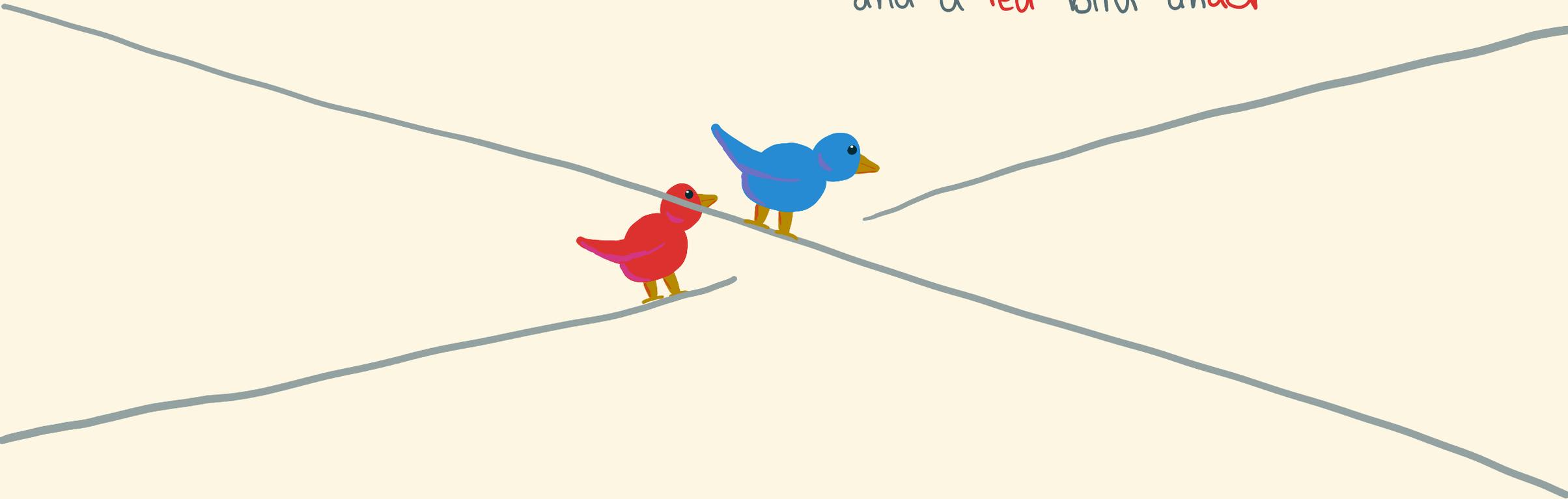


A bird's-eye view:



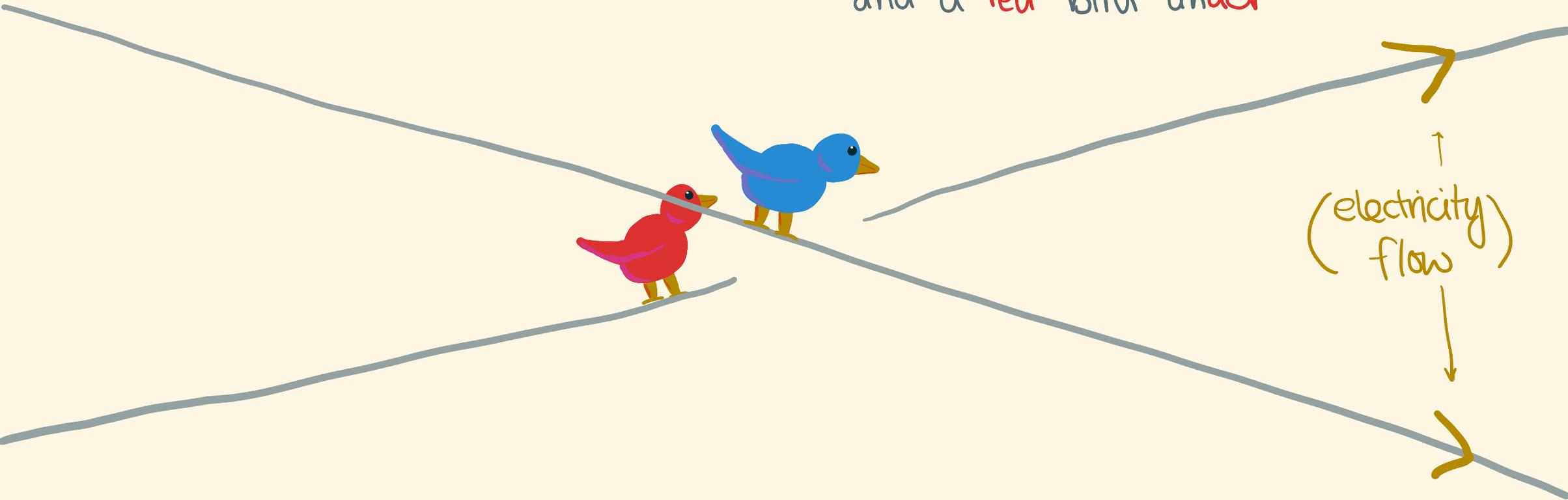
A bird's-eye view:

Where two wires cross,
put a blue bird above,
and a red bird under



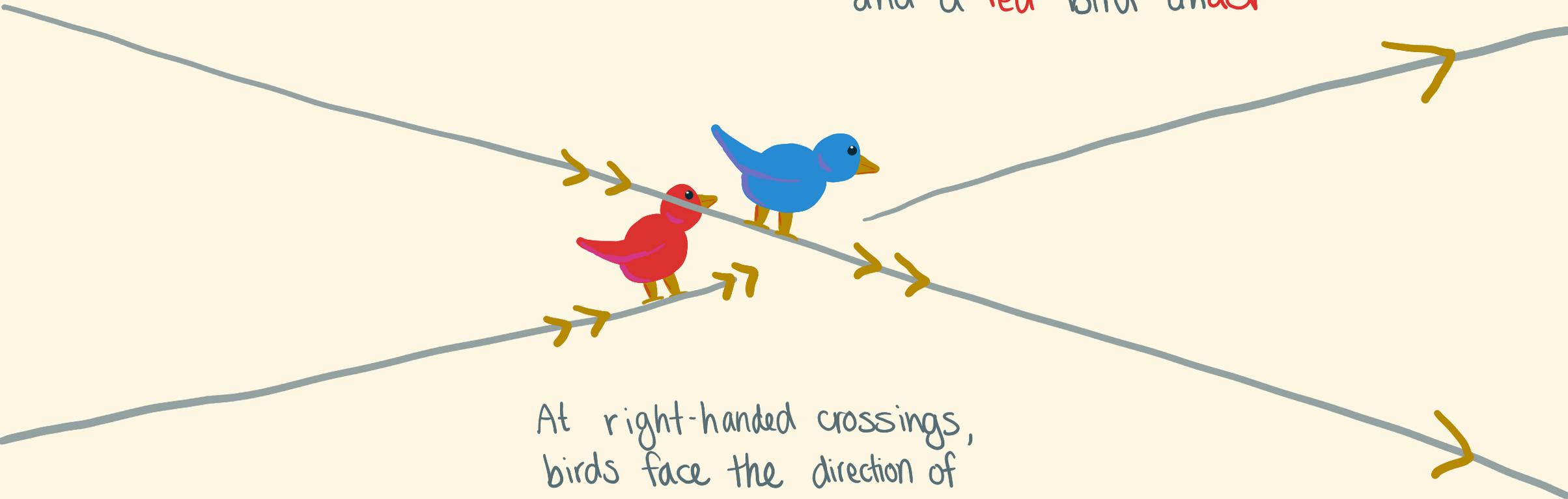
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A bird's-eye view:

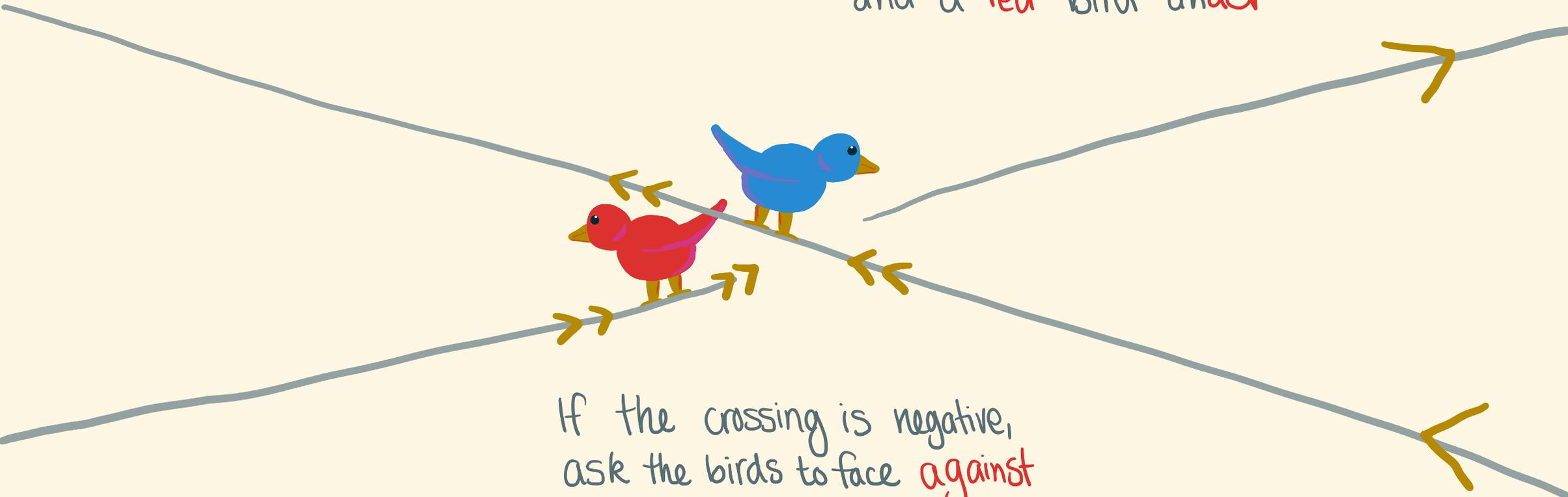
Where two wires cross,
put a **blue** bird above,
and a **red** bird under



At right-handed crossings,
birds face the direction of
electricity.

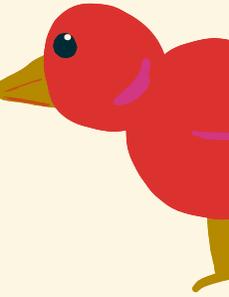
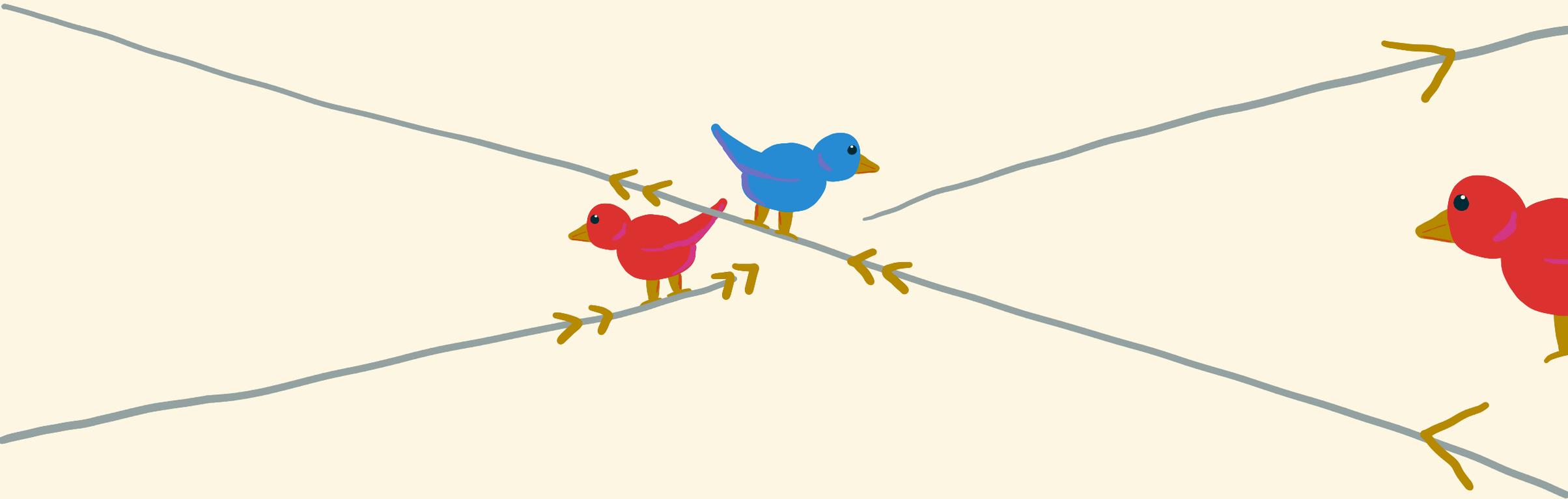
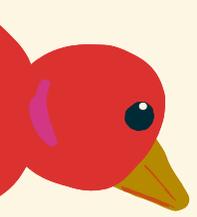
A bird's-eye view:

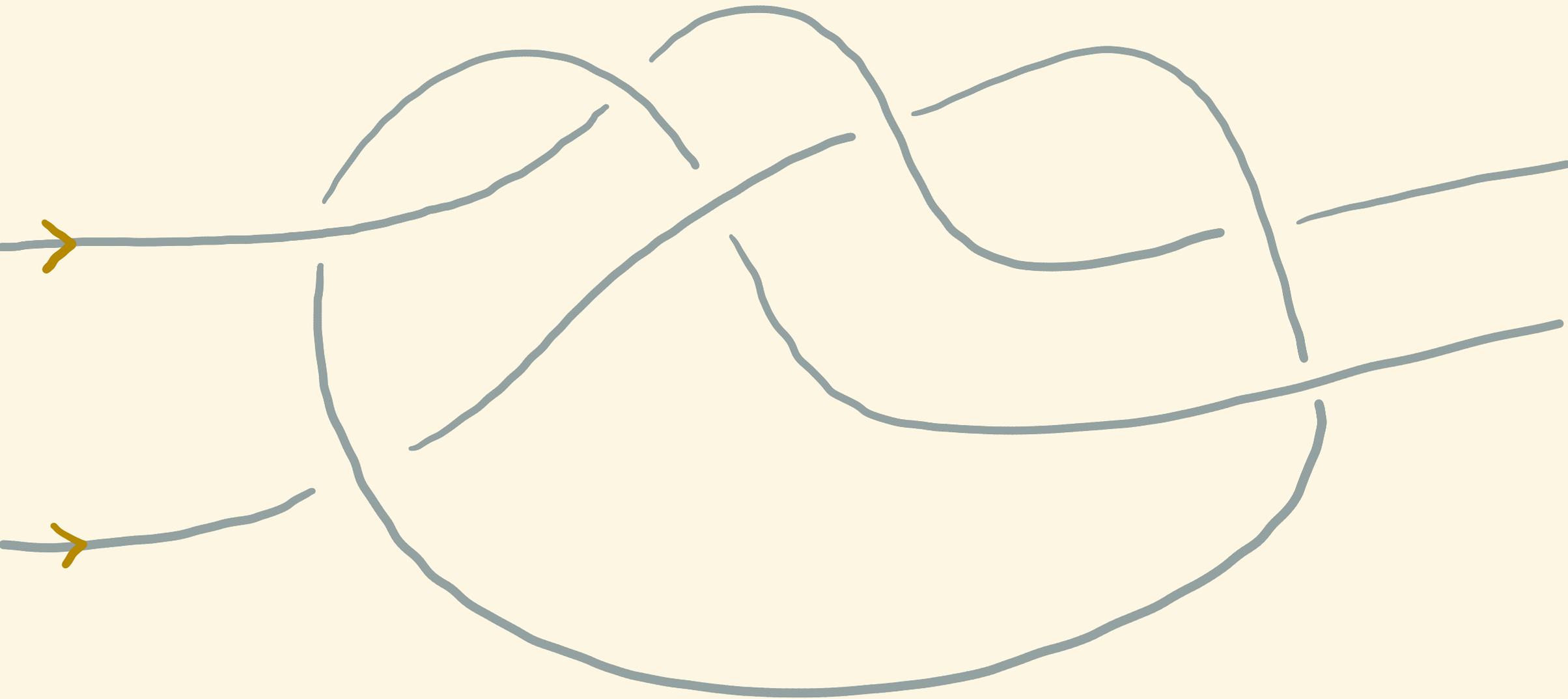
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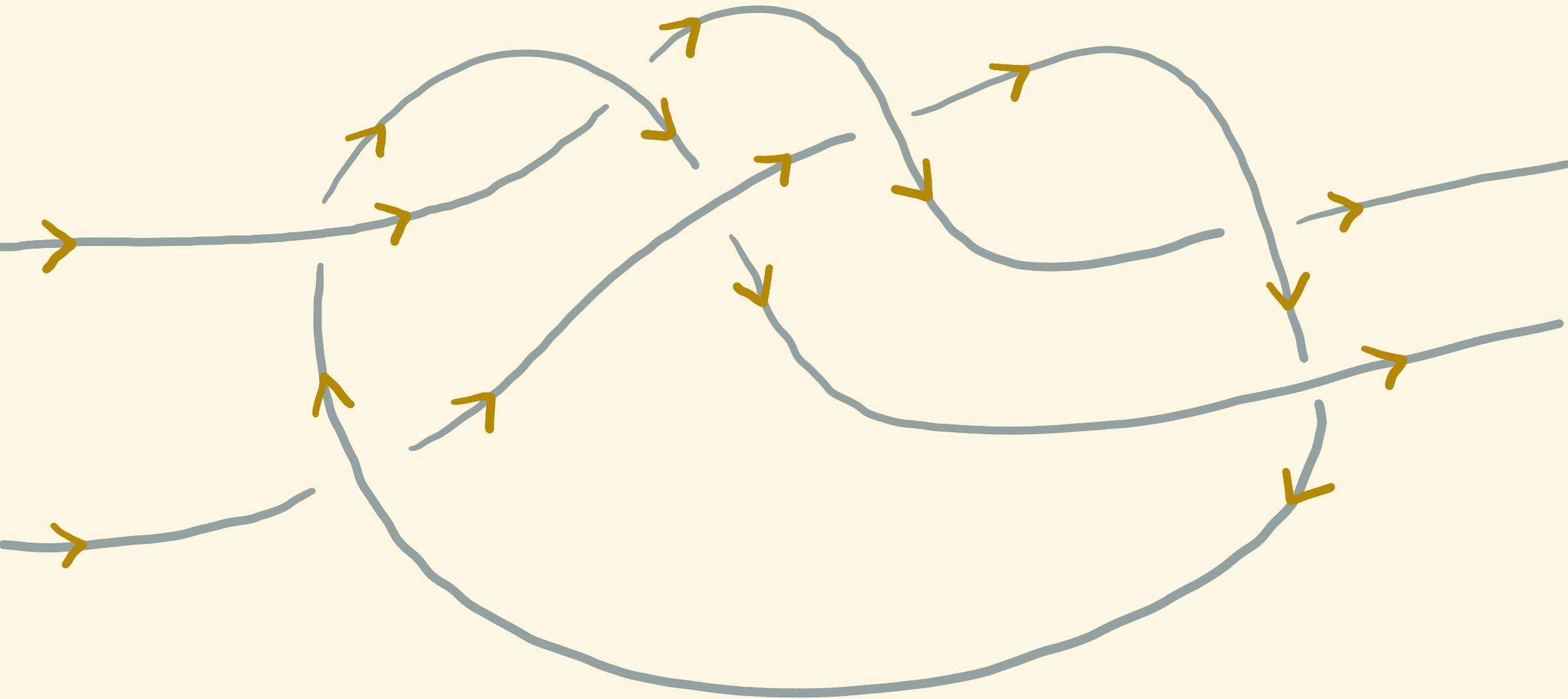


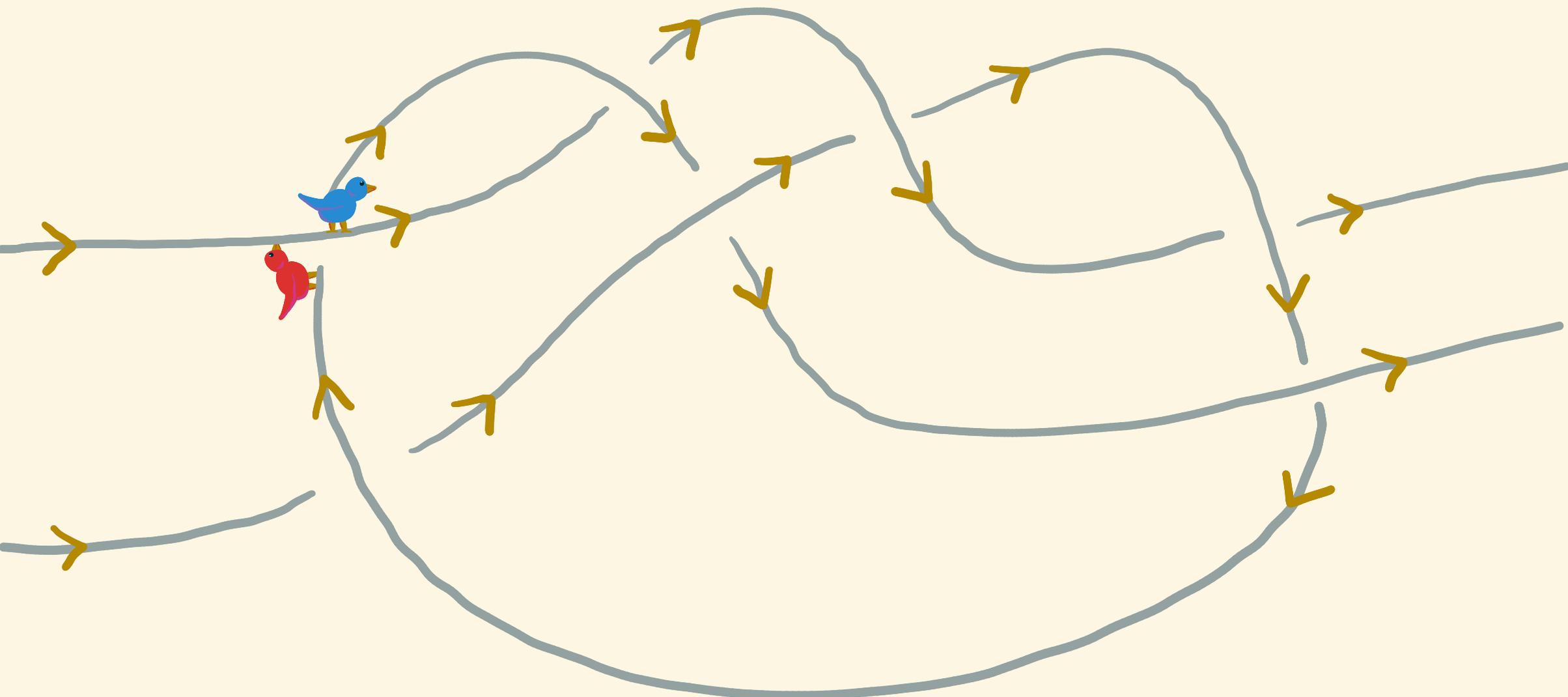
If the crossing is negative,
ask the birds to face **against**
electricity

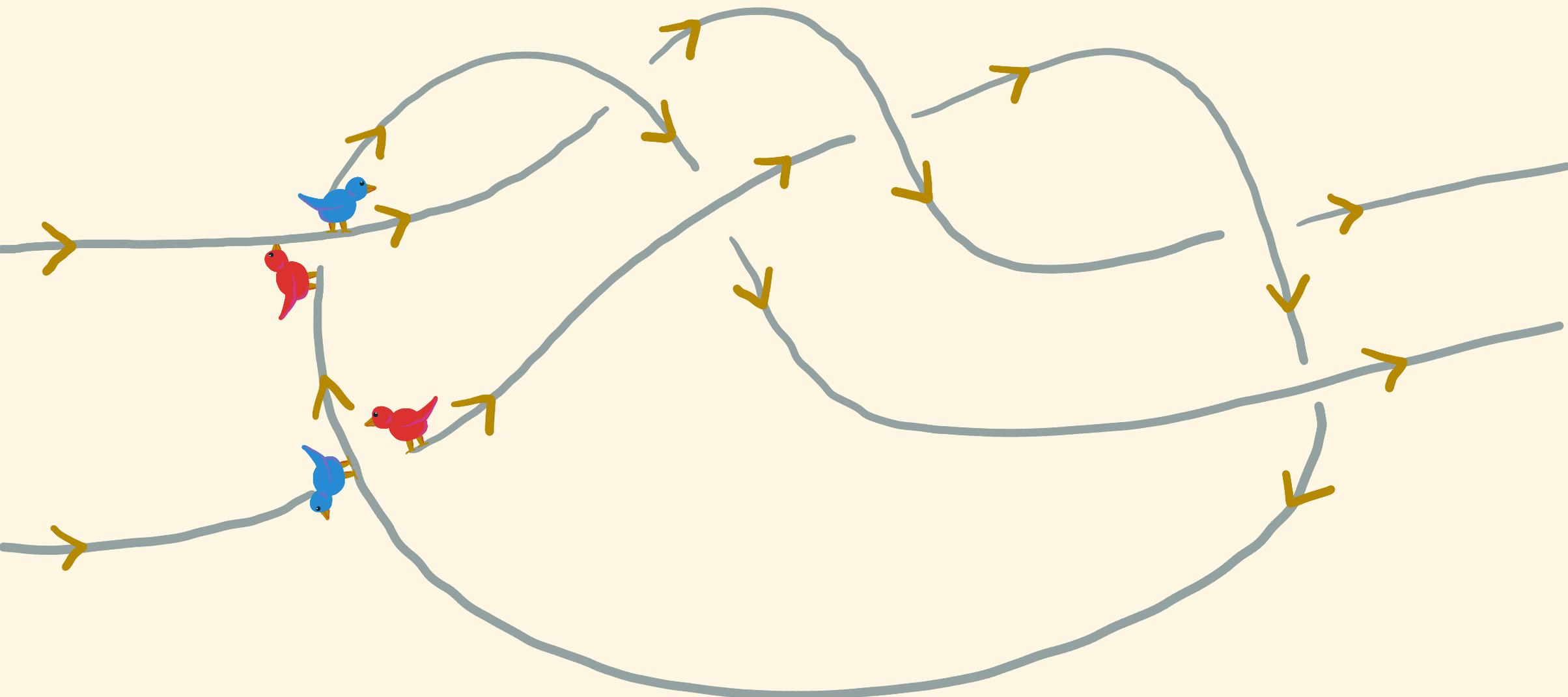
Let's put birds at every crossing...

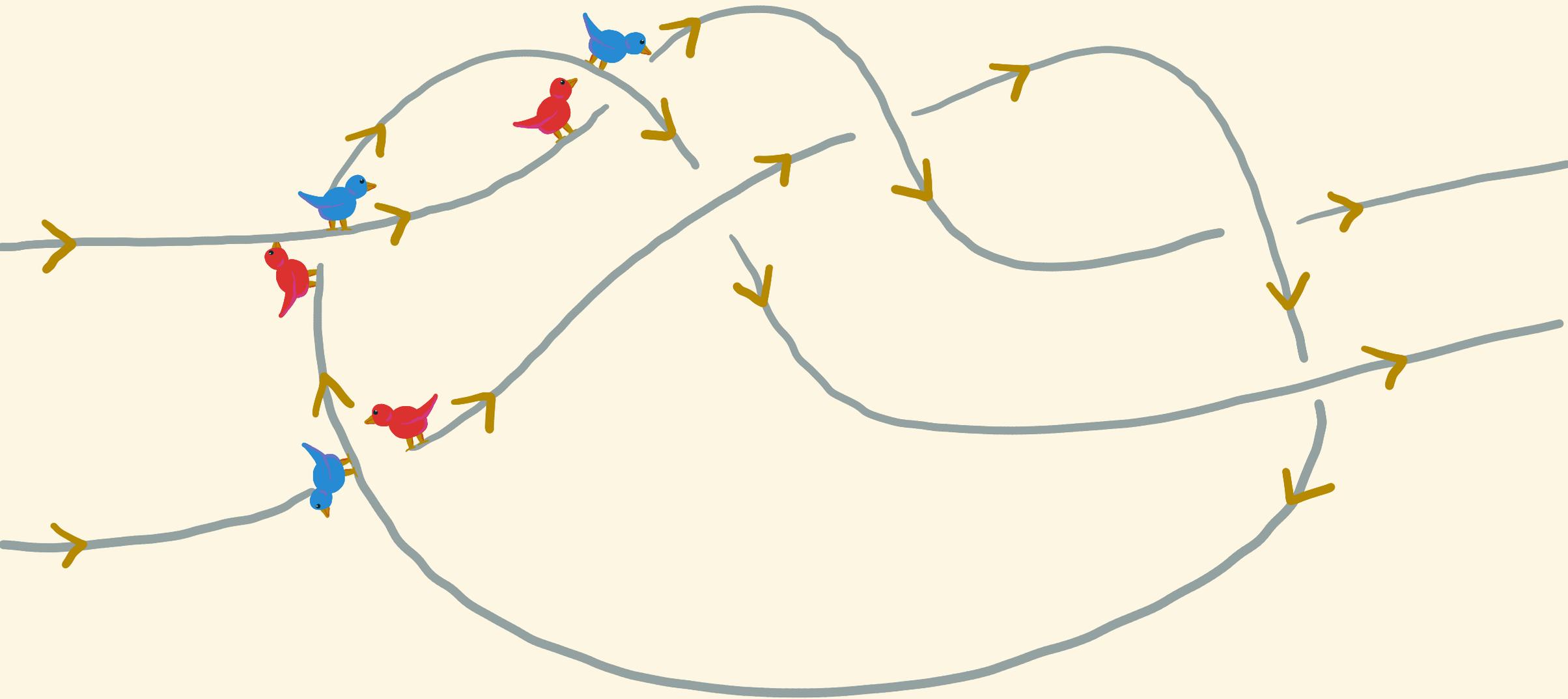


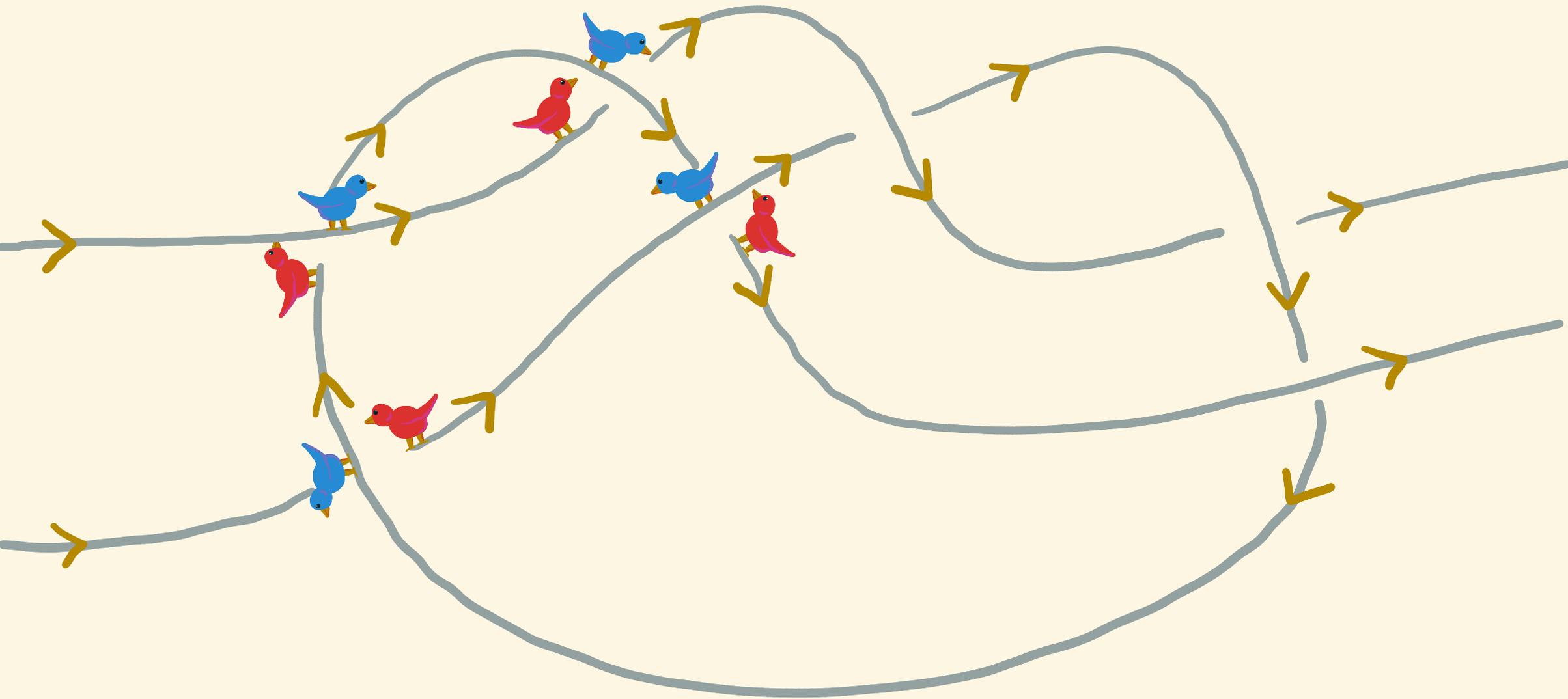


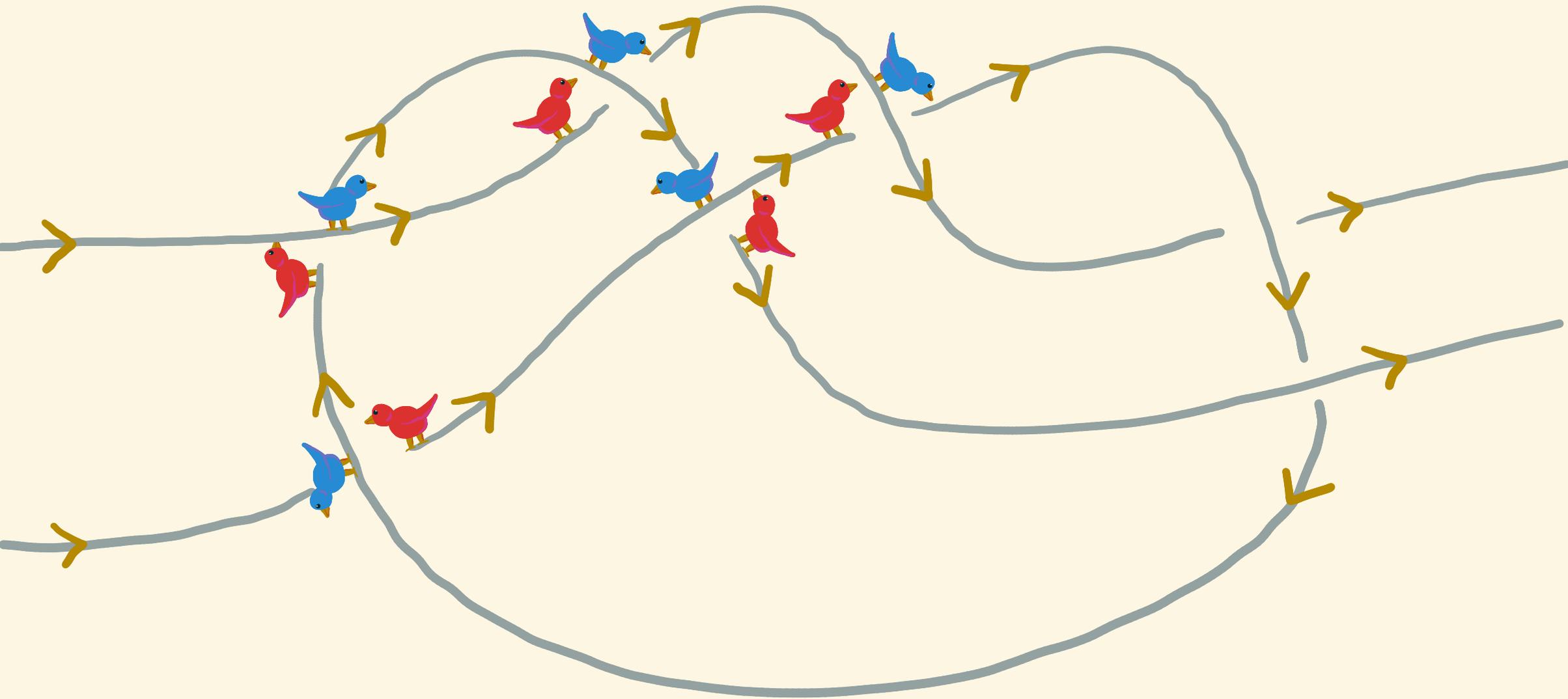


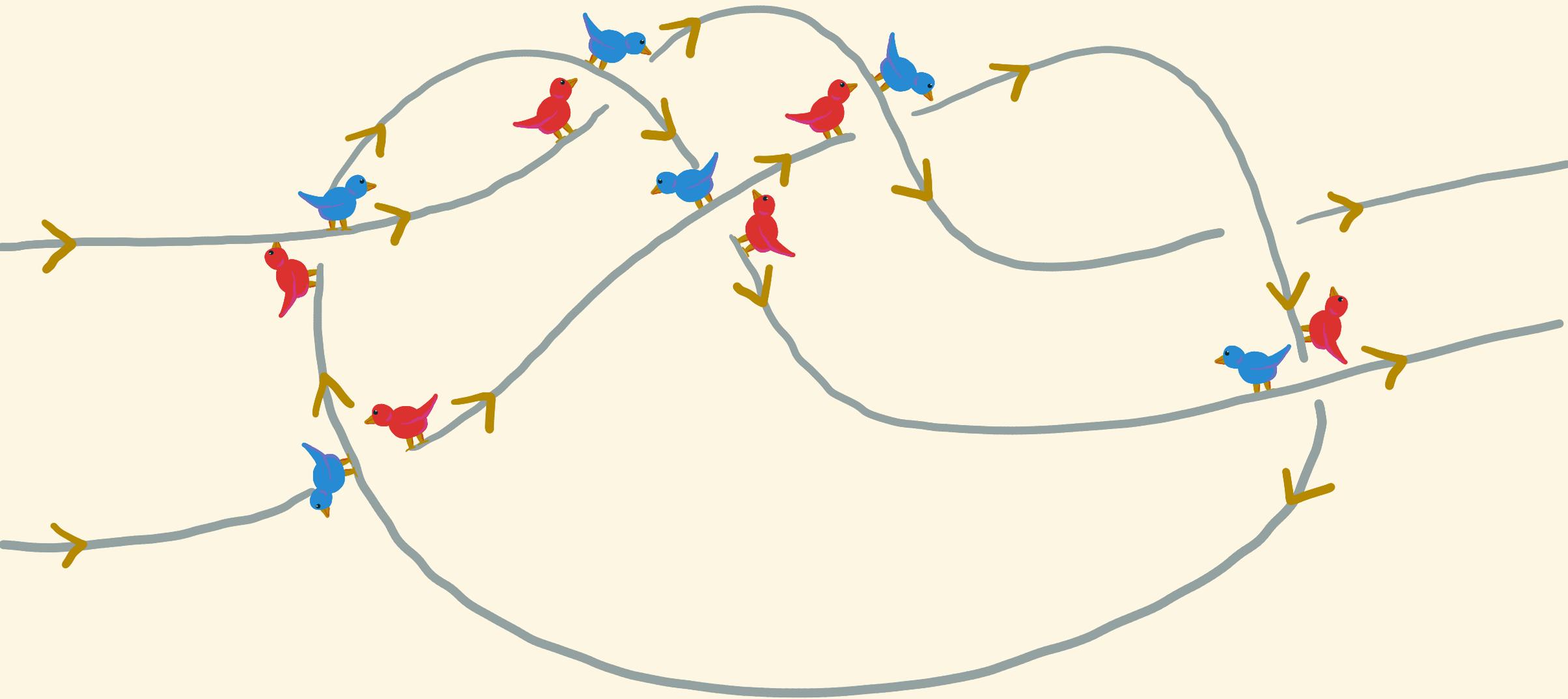


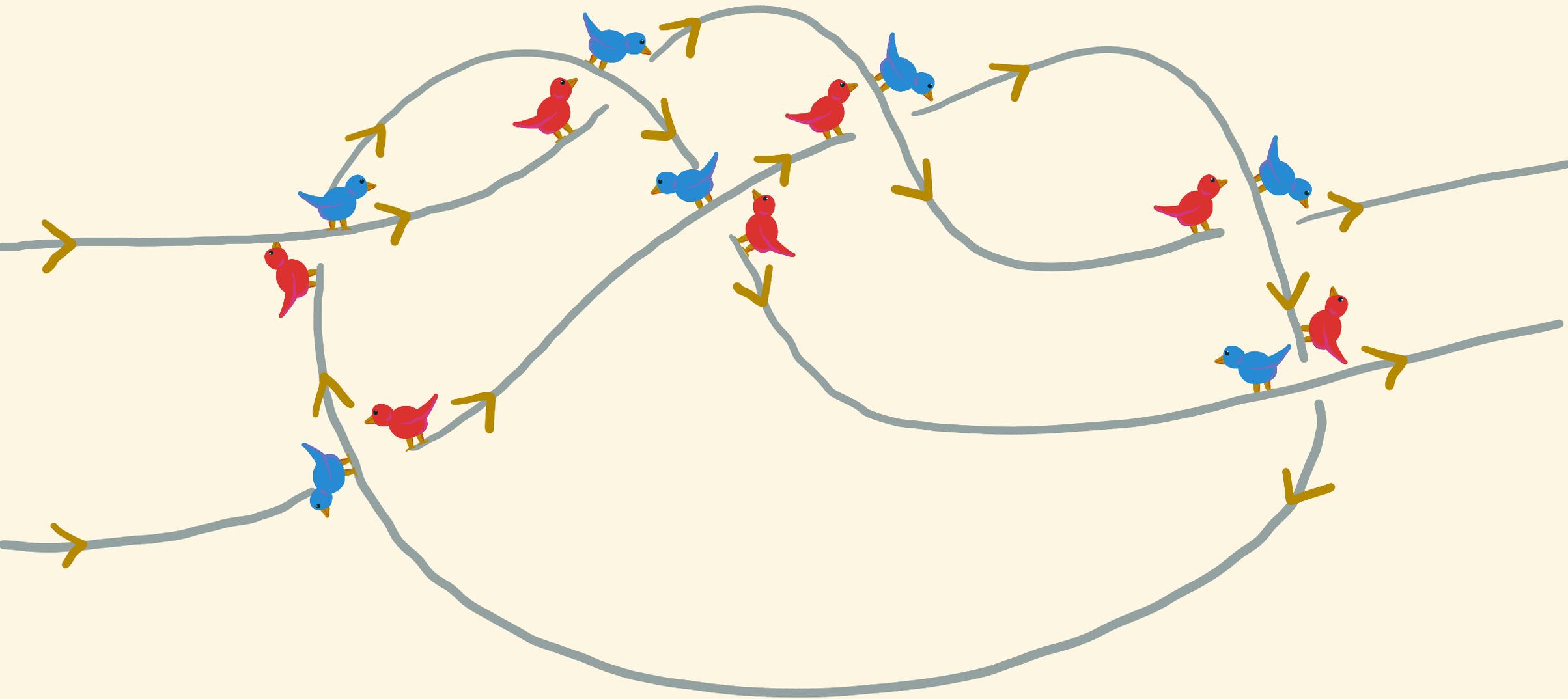














Now slide the
birds to the beginning...



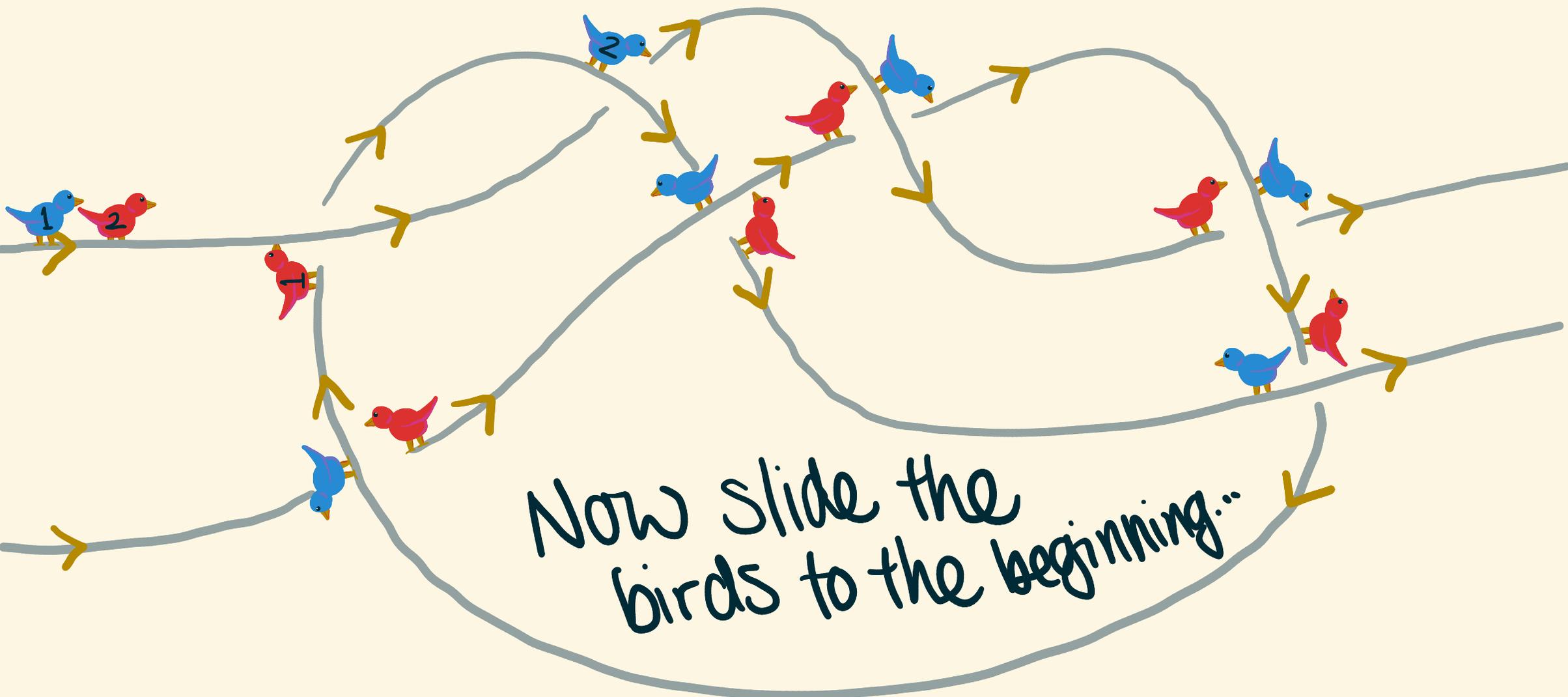
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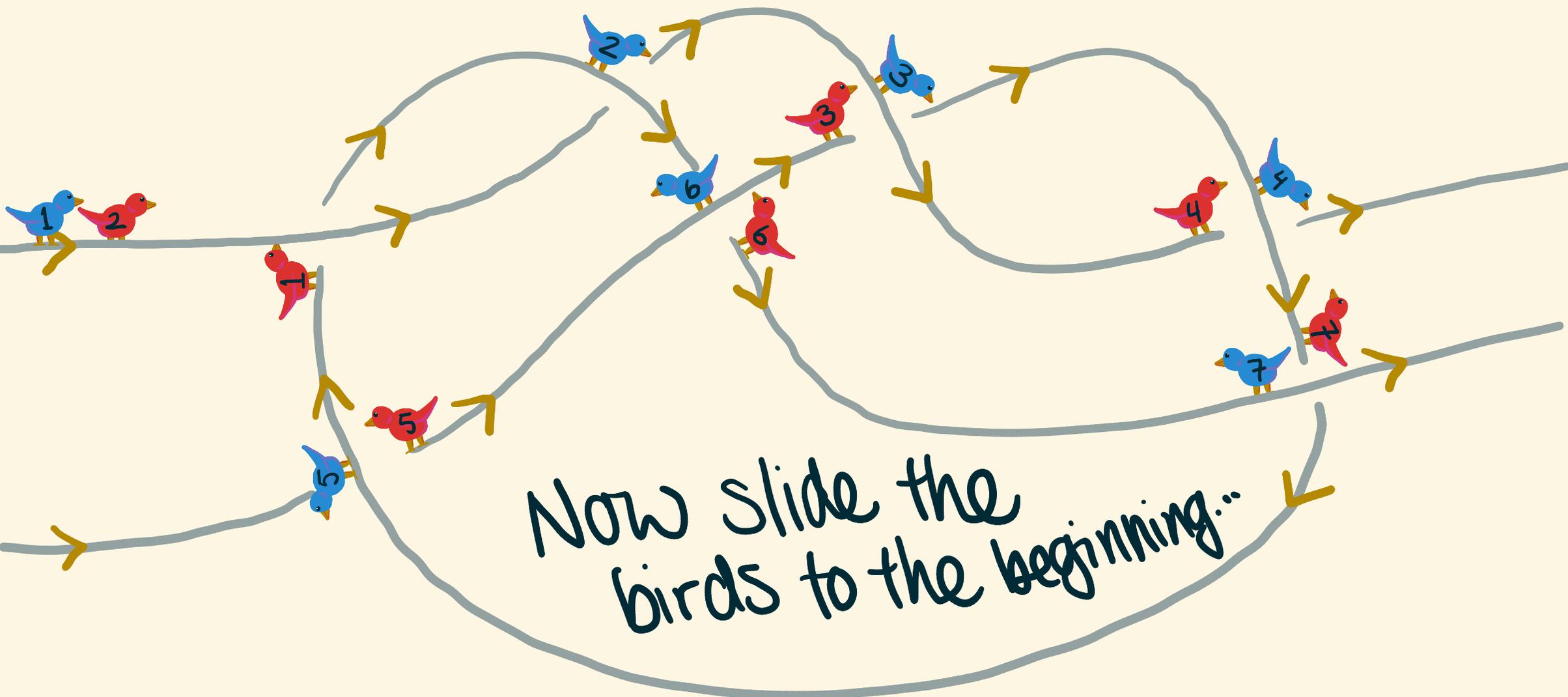
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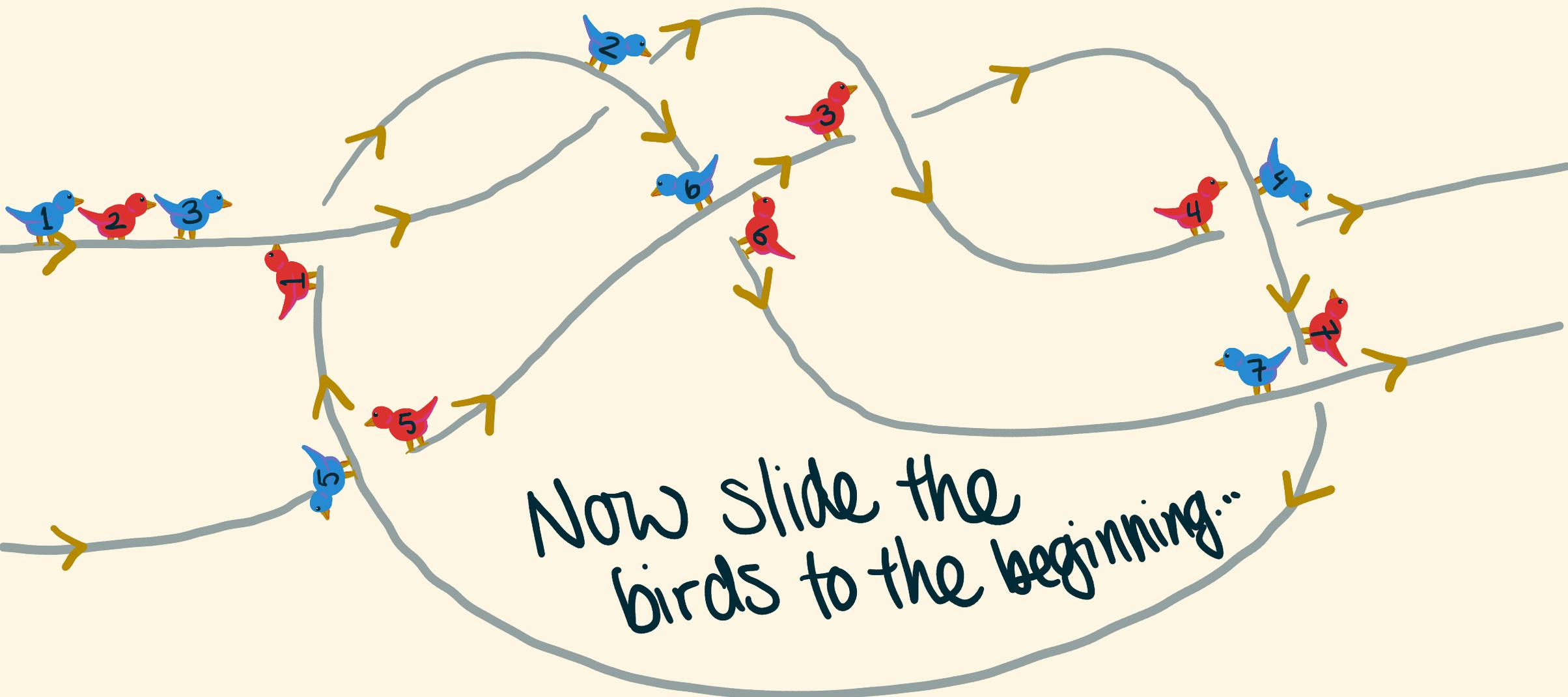
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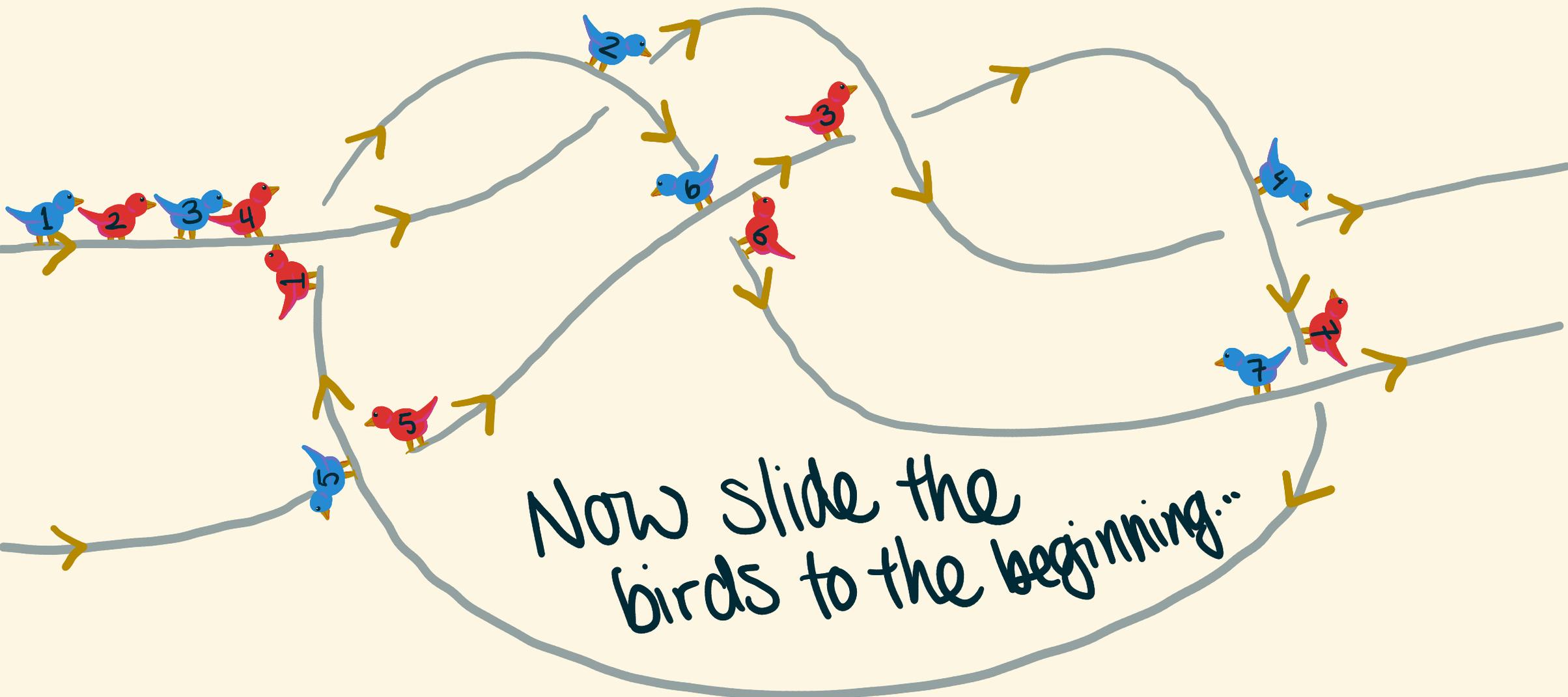
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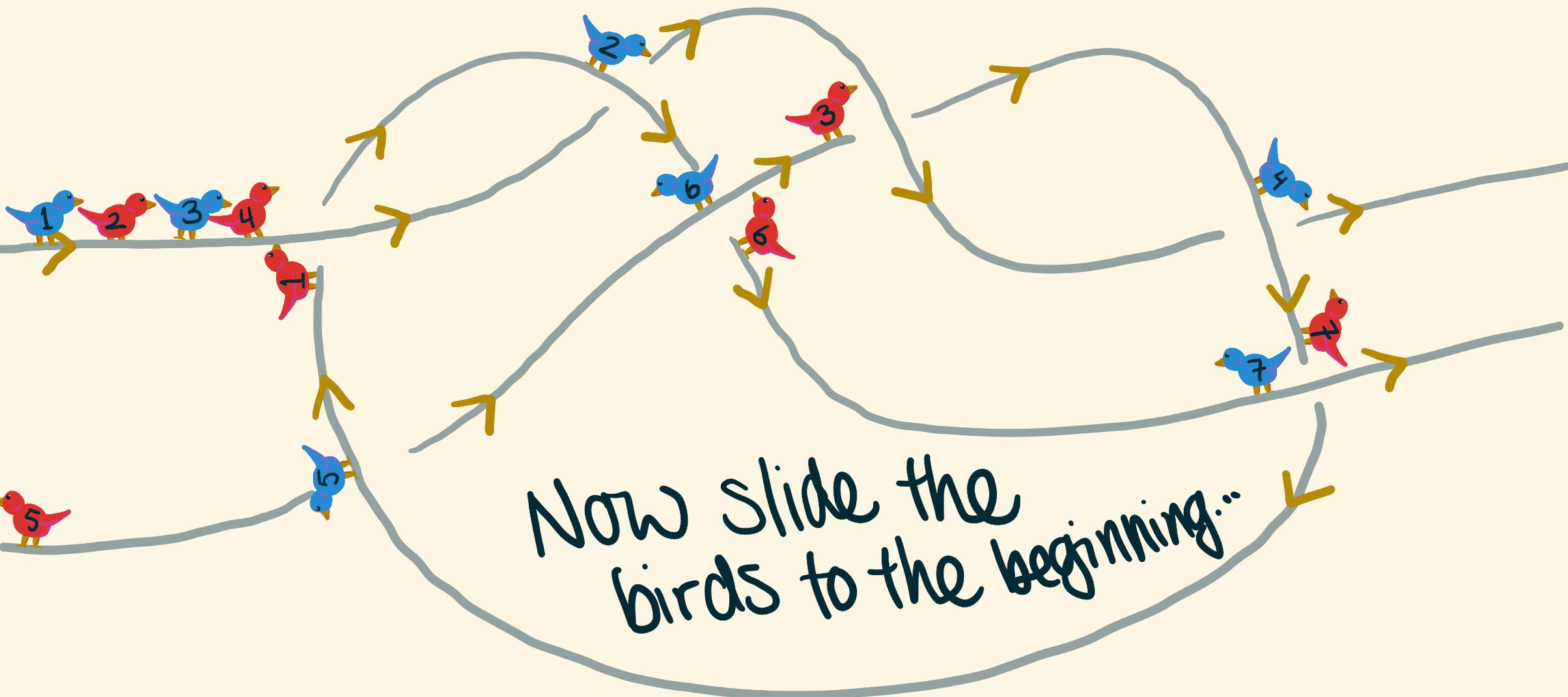
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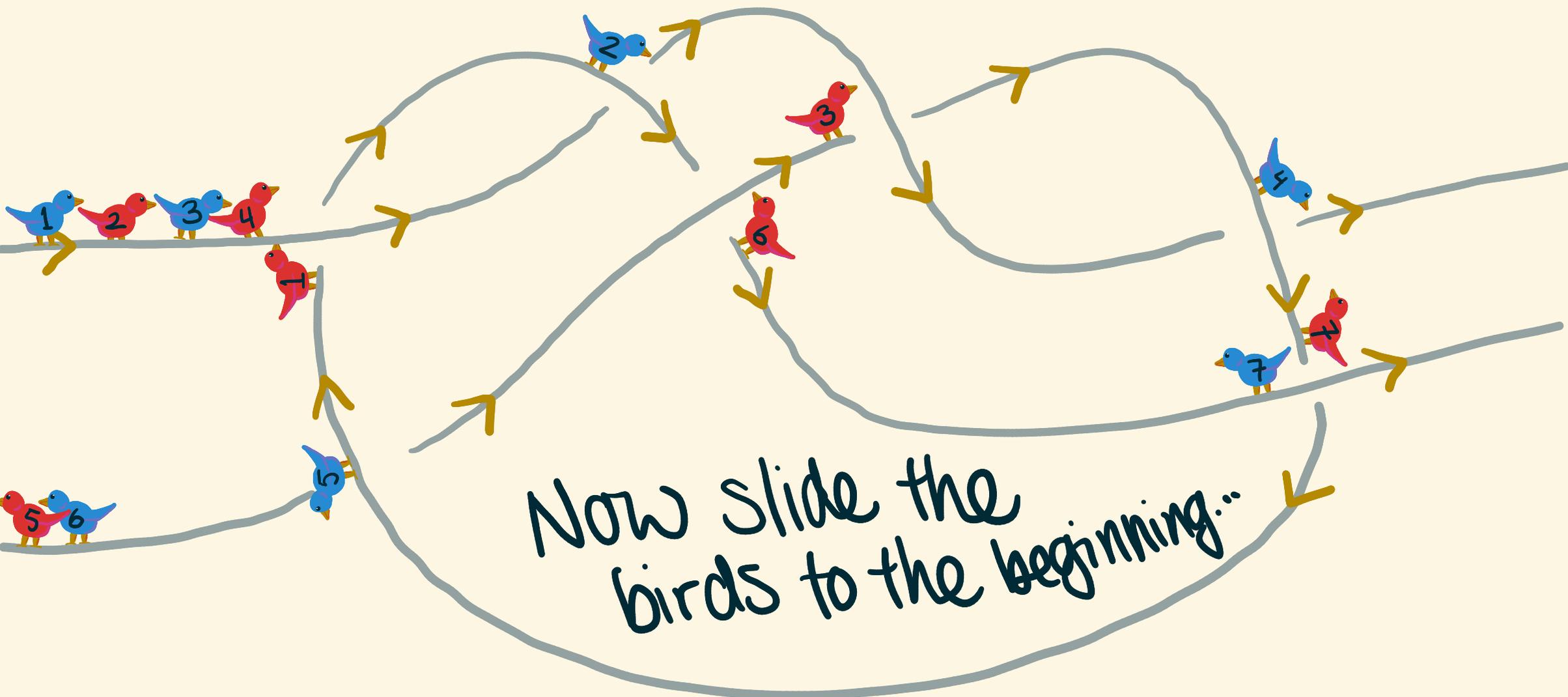
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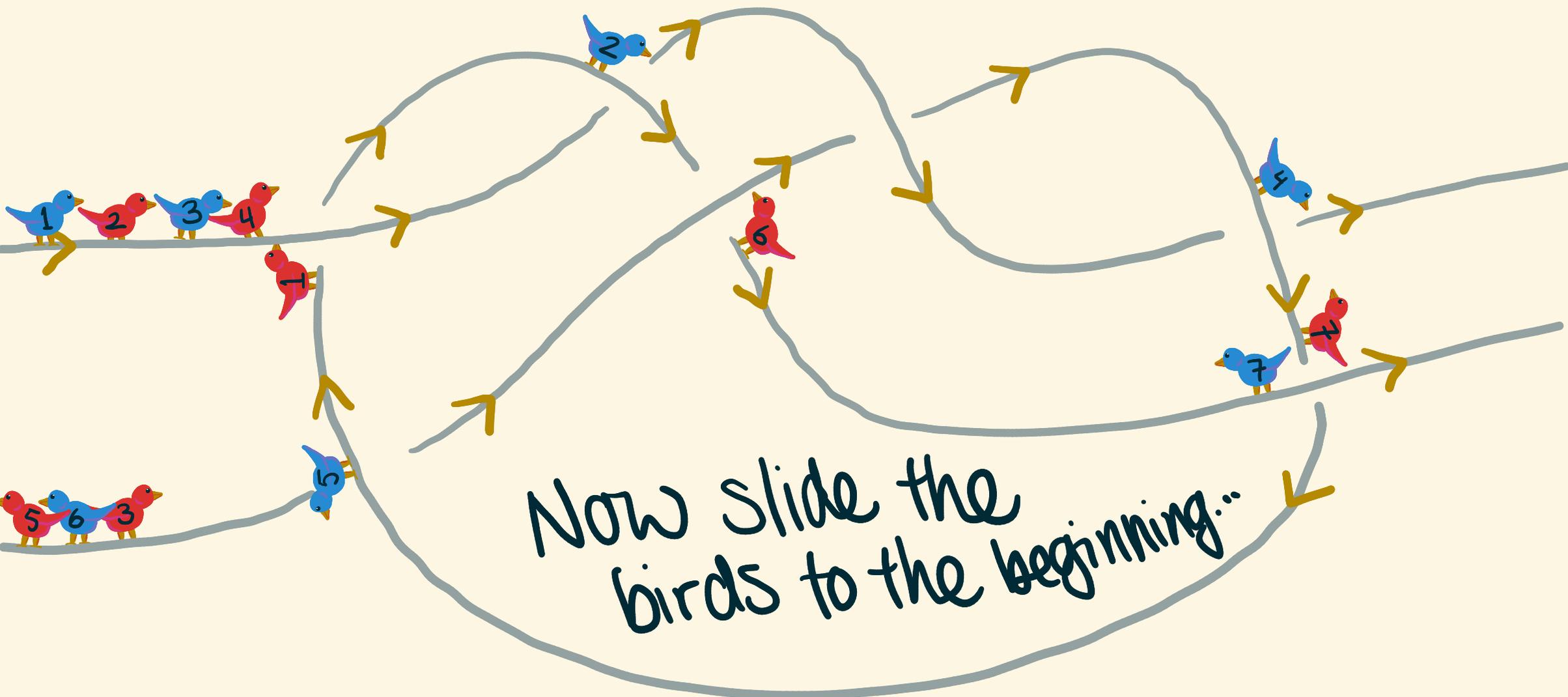
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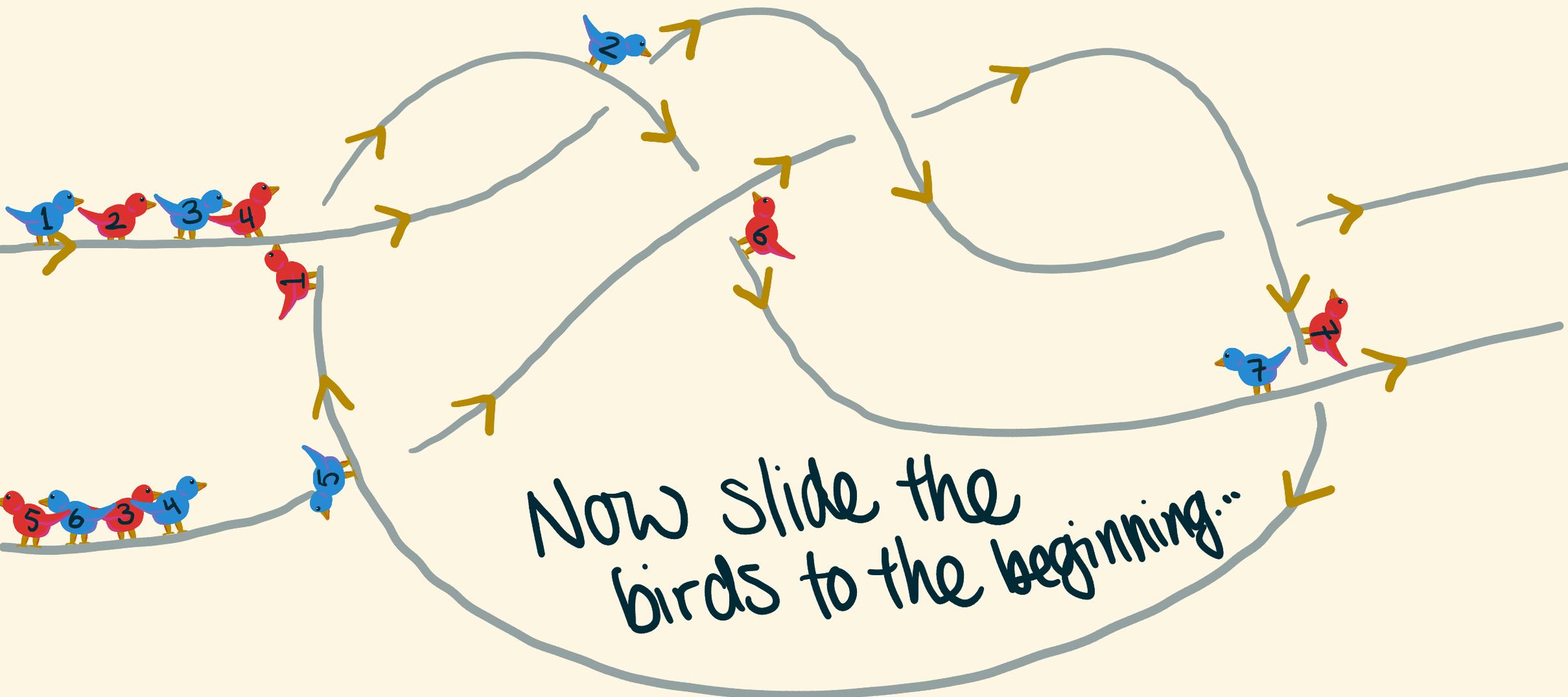
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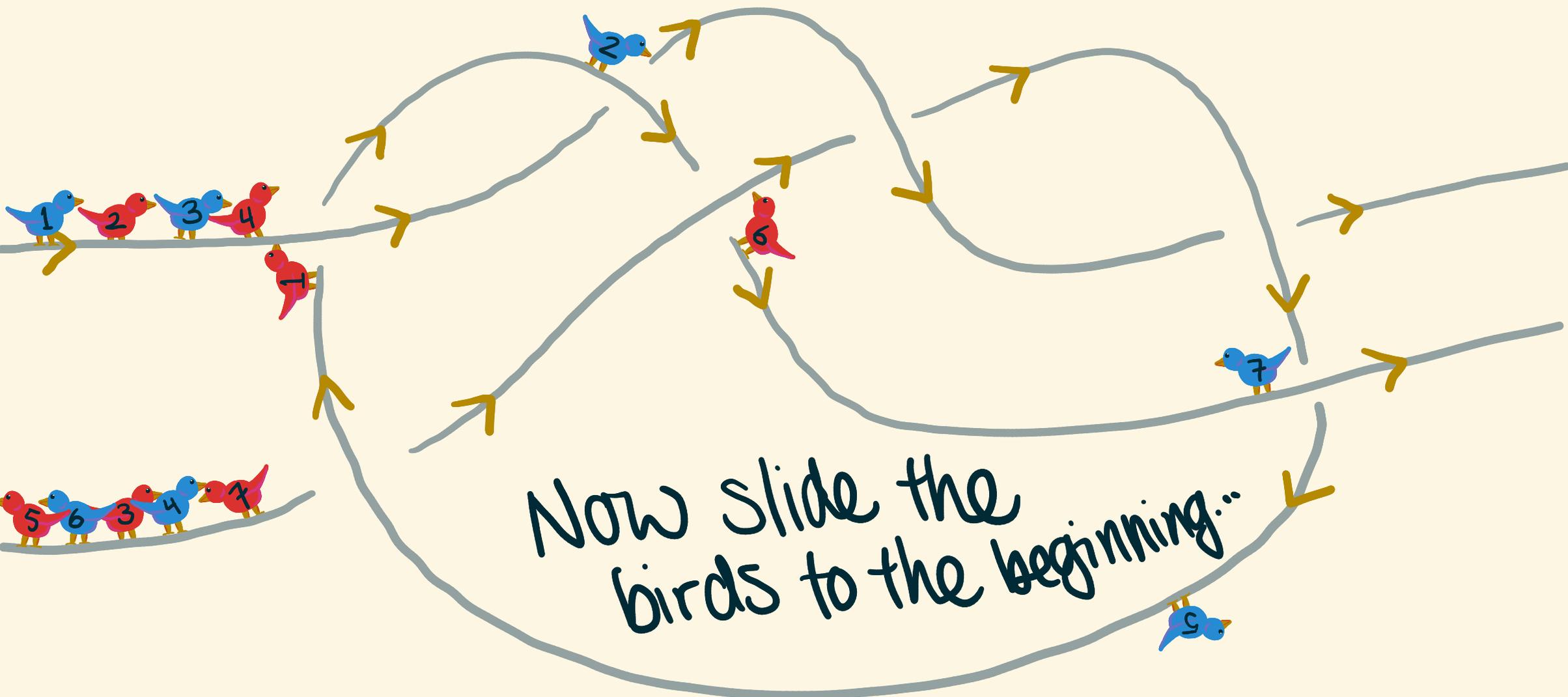
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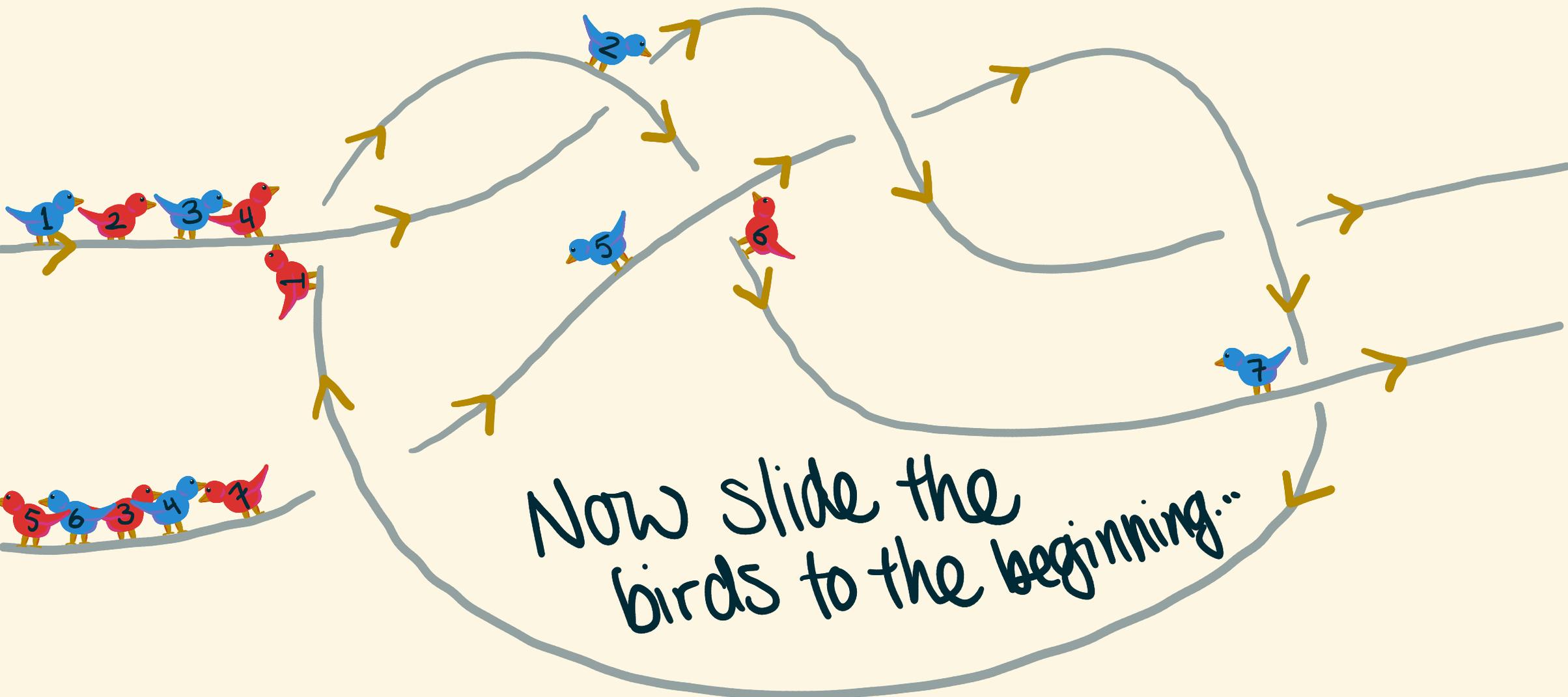
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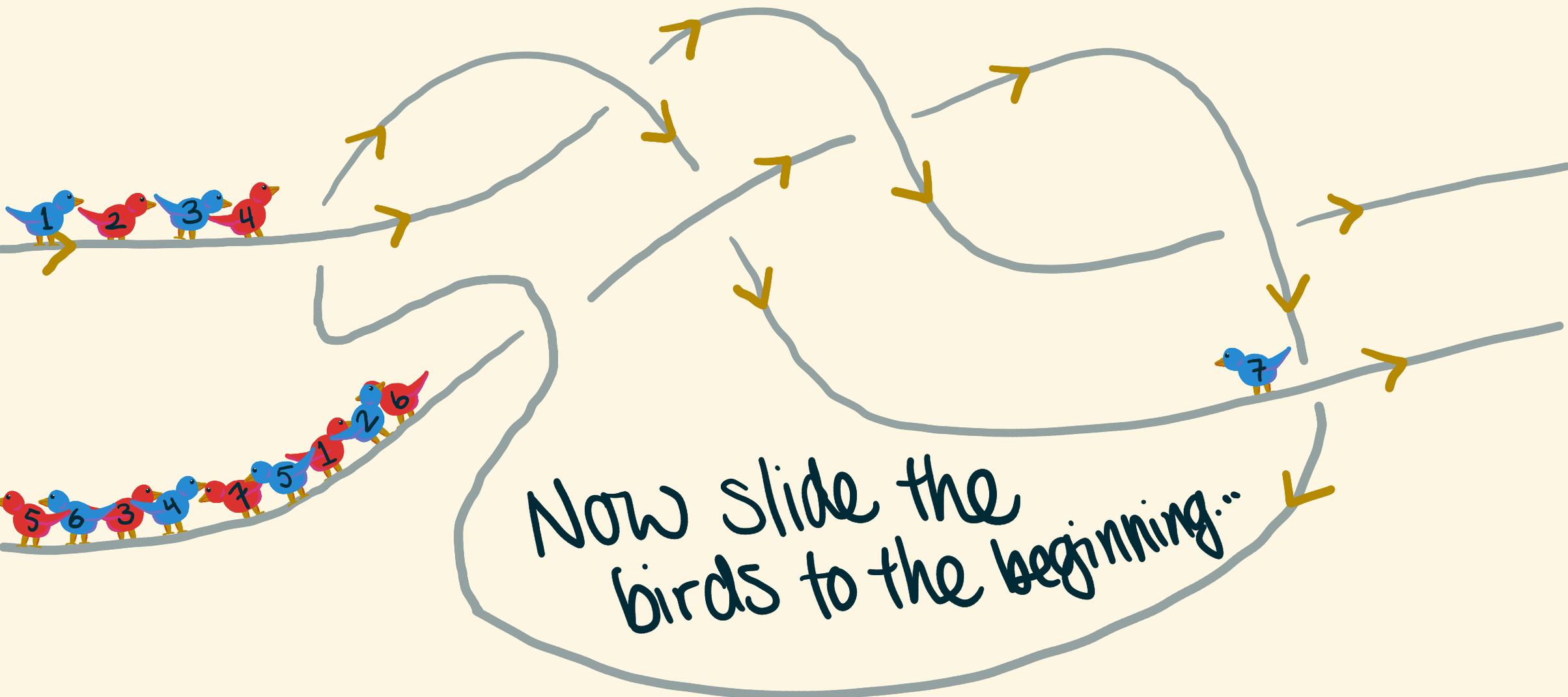
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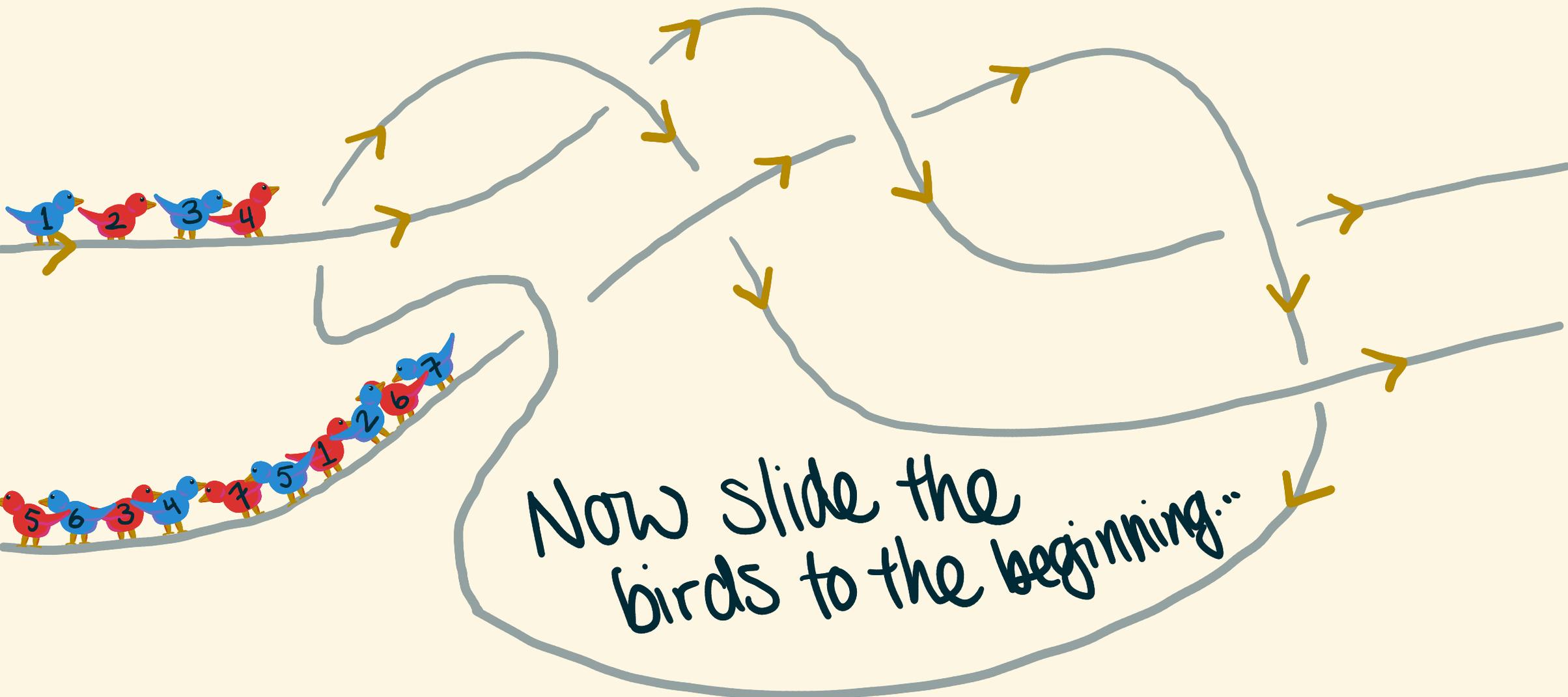
Now slide the birds to the beginning...



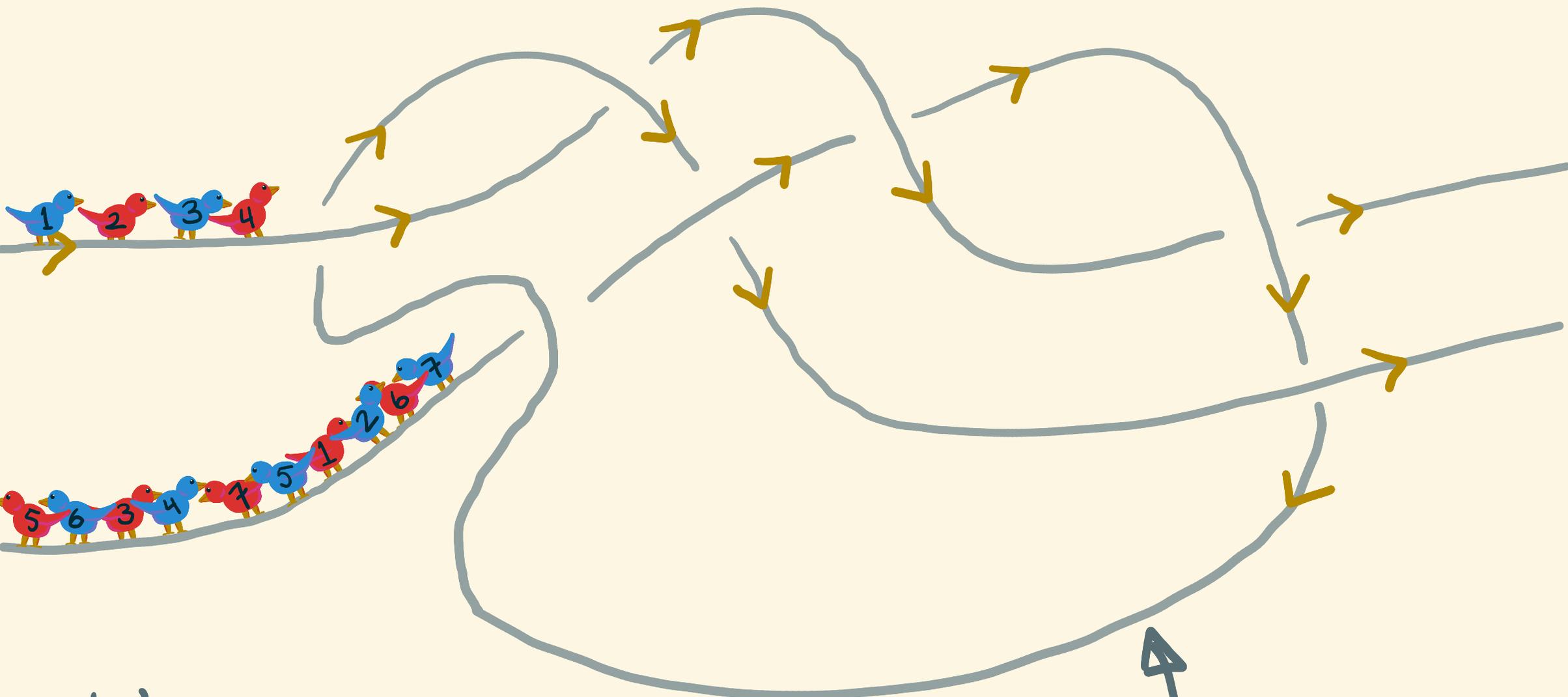
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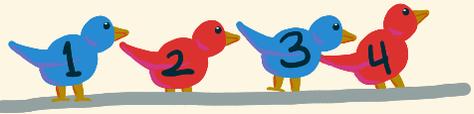


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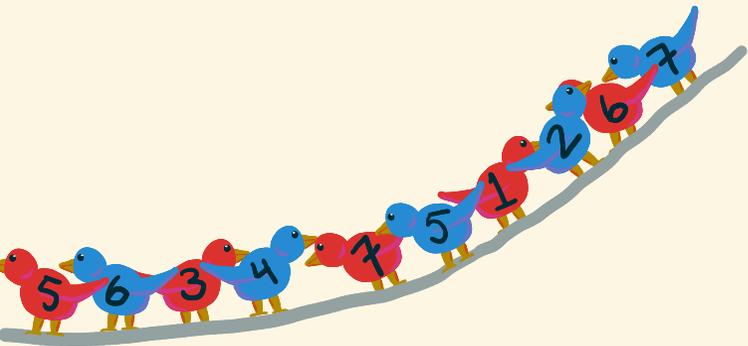
We converted this tangle...

... into two sequences of
birds!

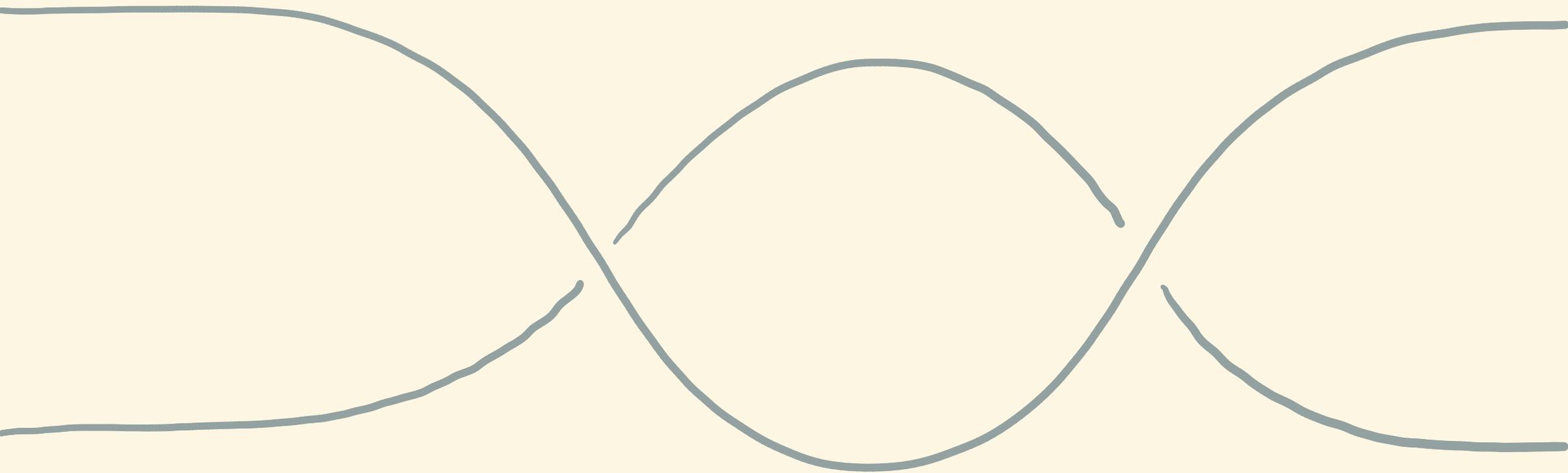


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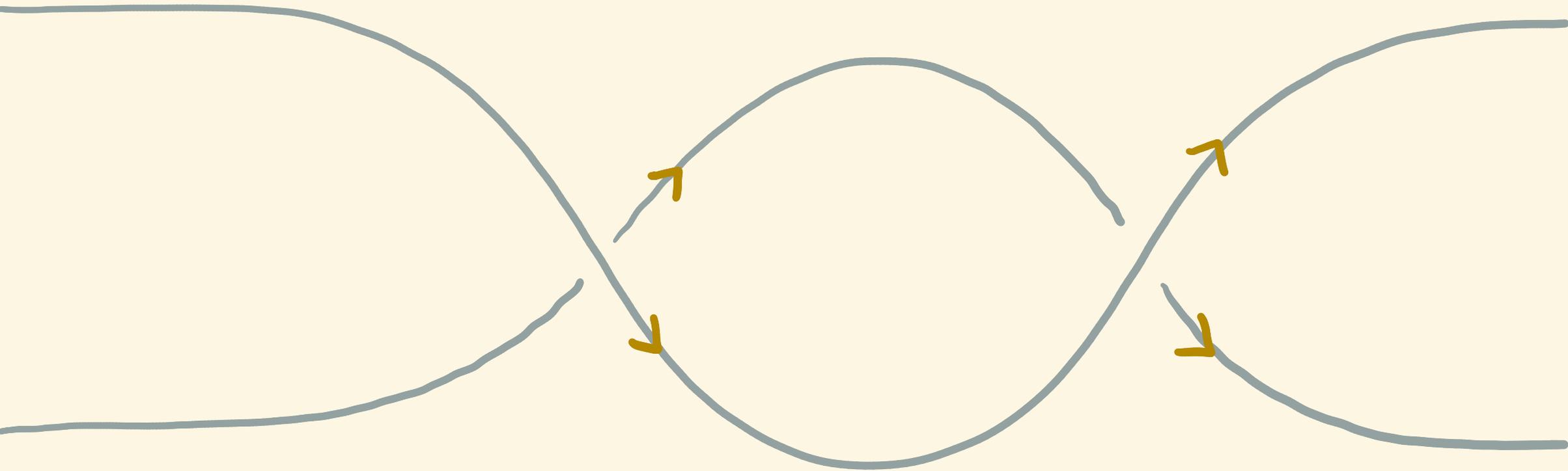
How do we compare
bird-sequences?



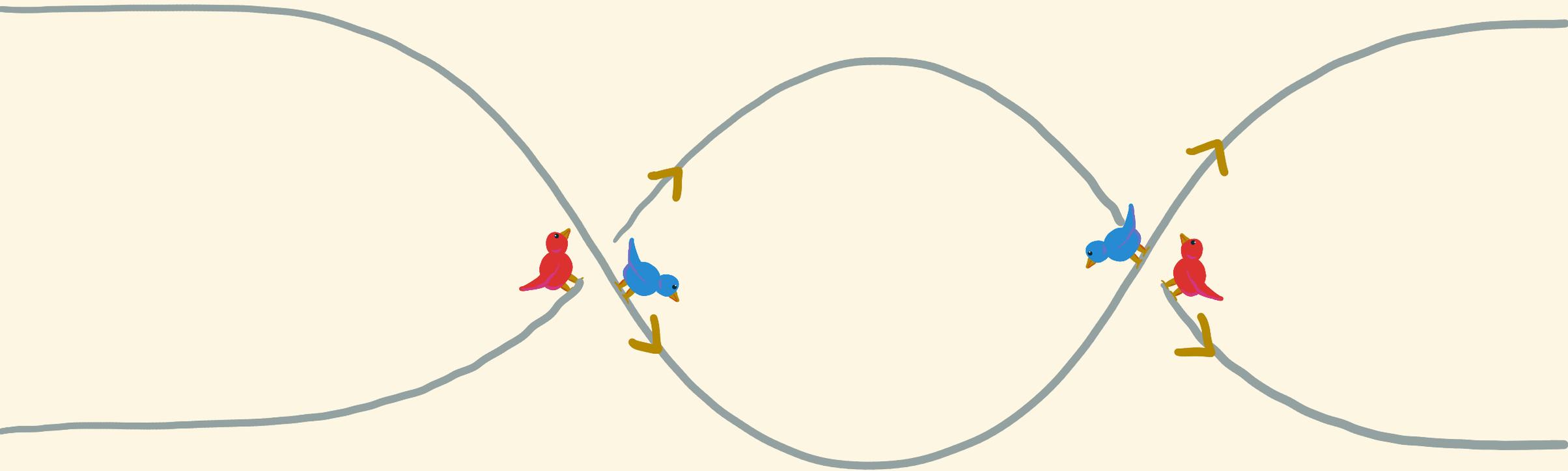
Here's a simple tangle:



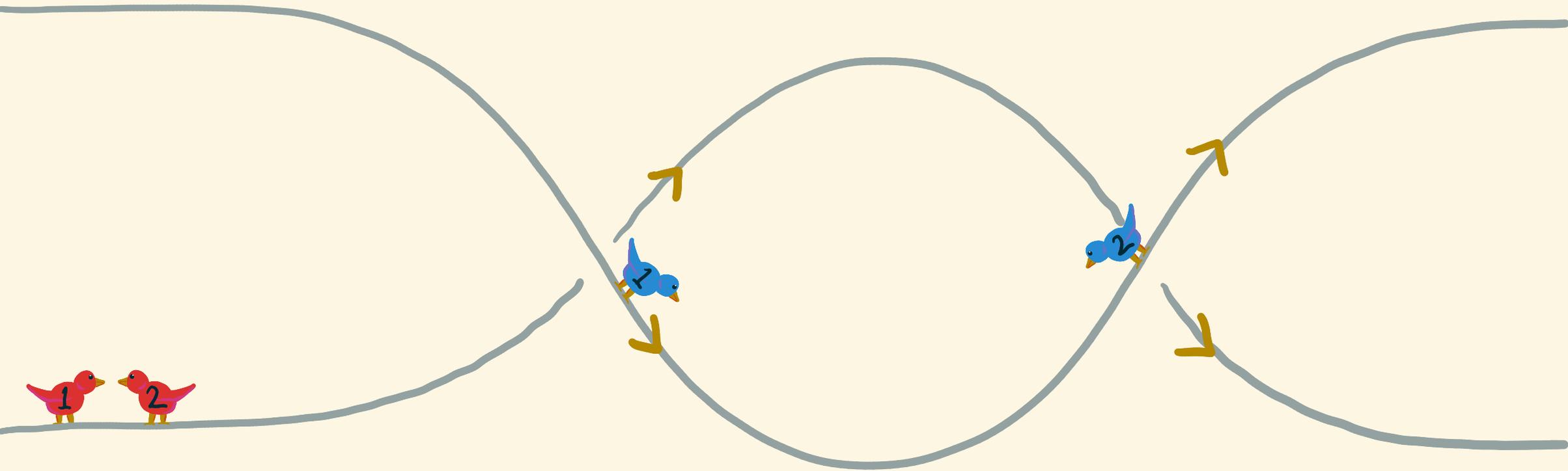
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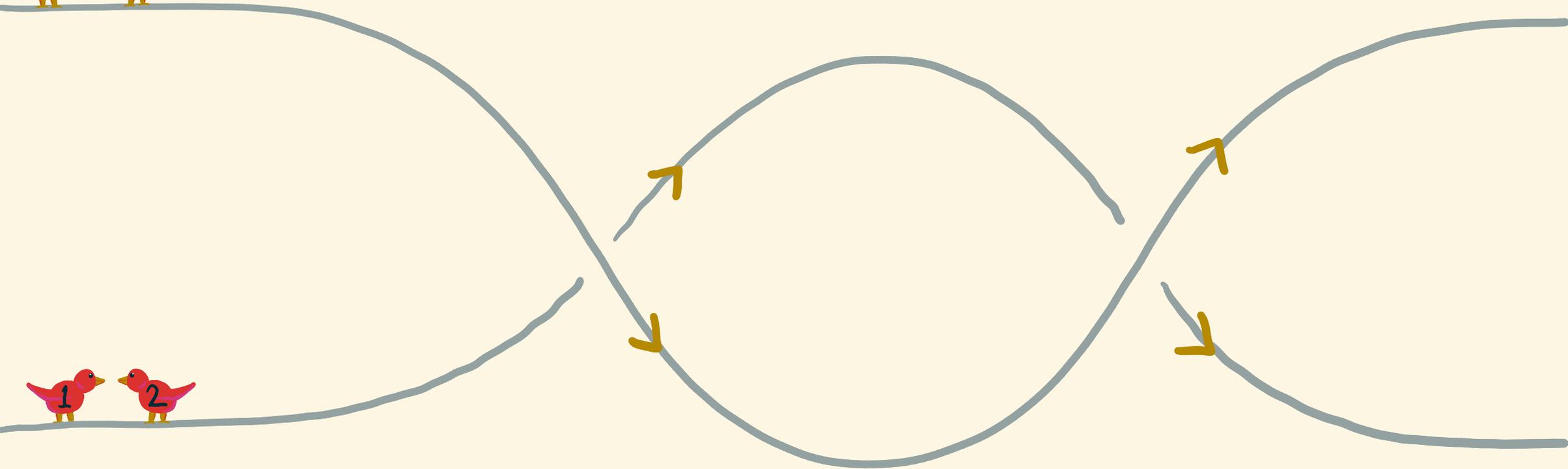
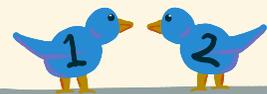
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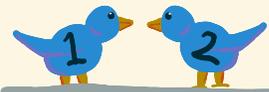
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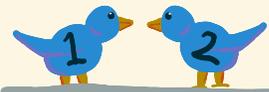
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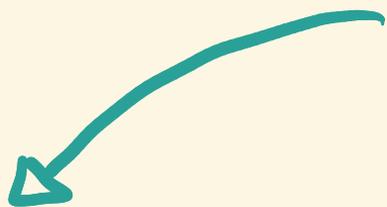
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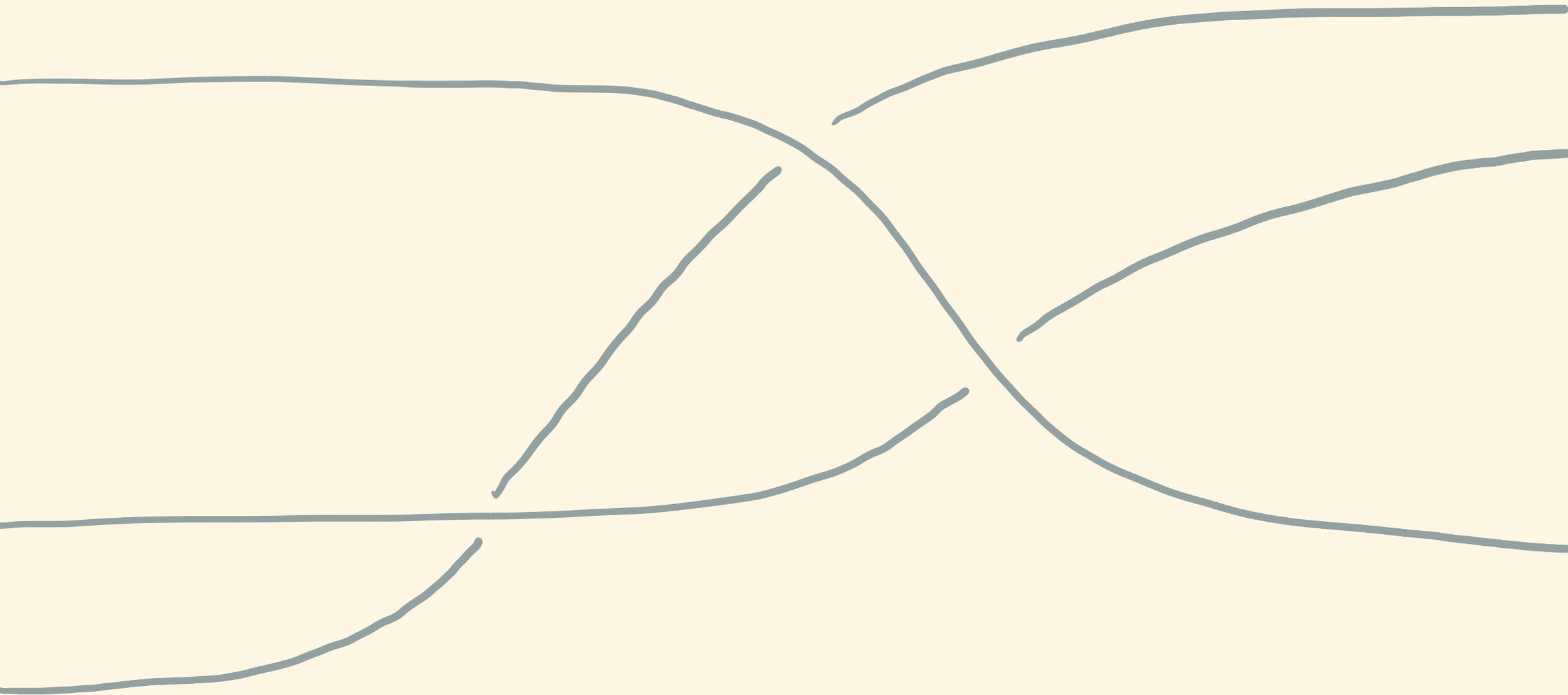
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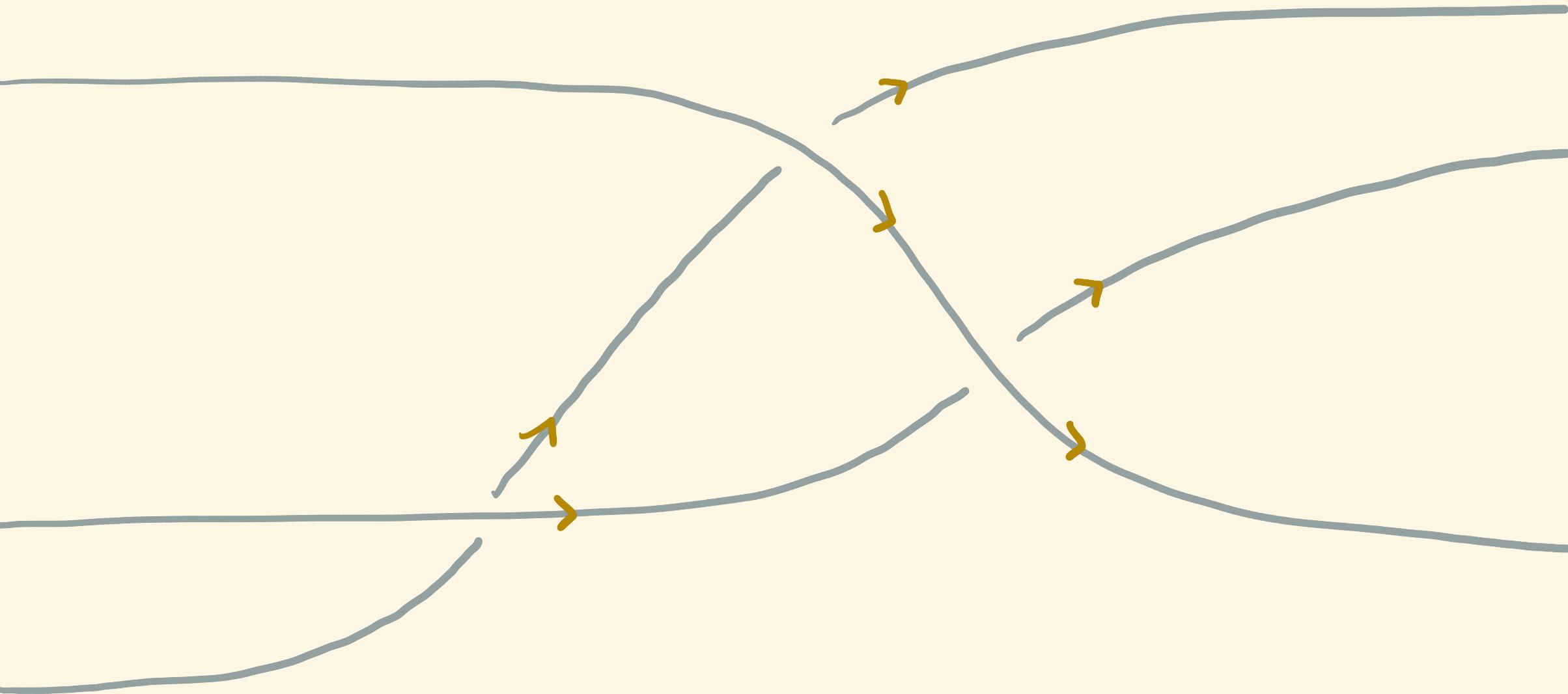
These lovebirds should tell us
the same thing as no birds at all!



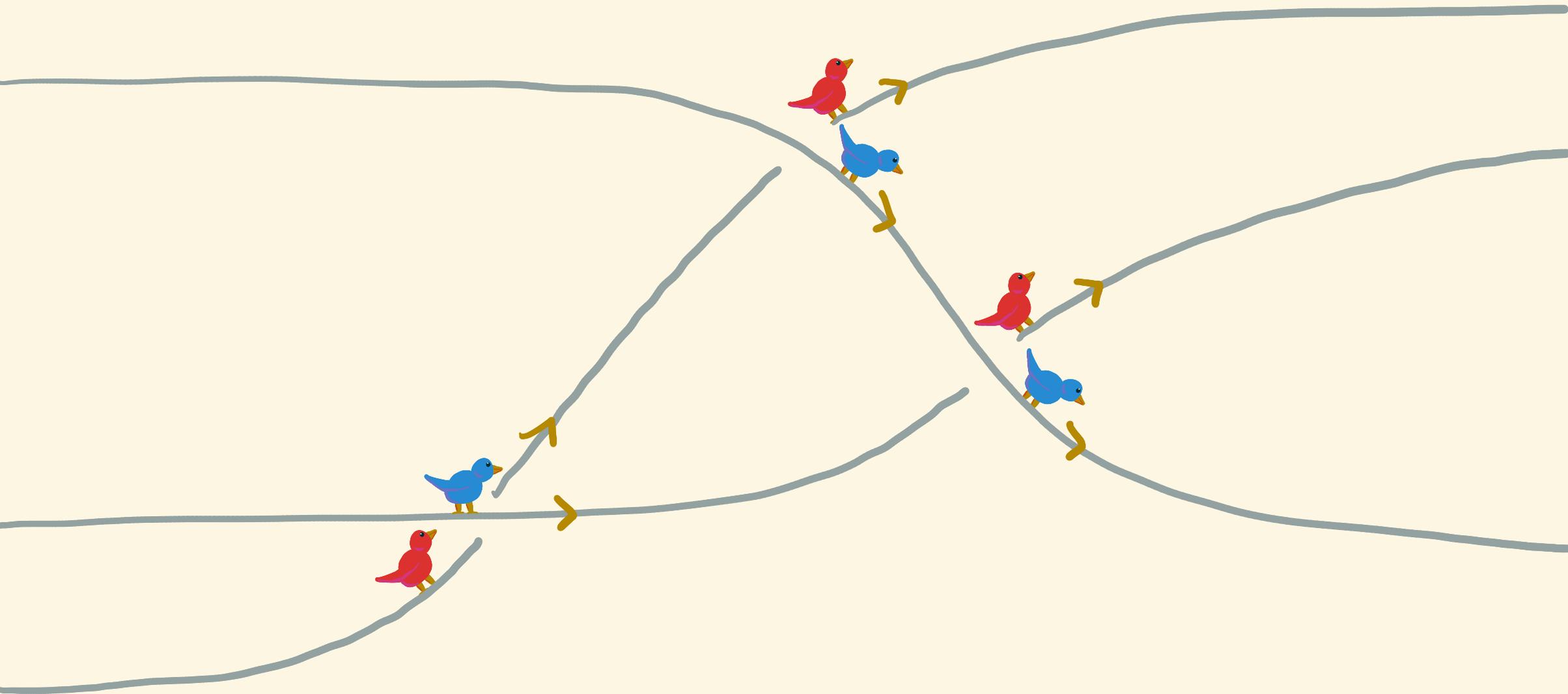
One more example:



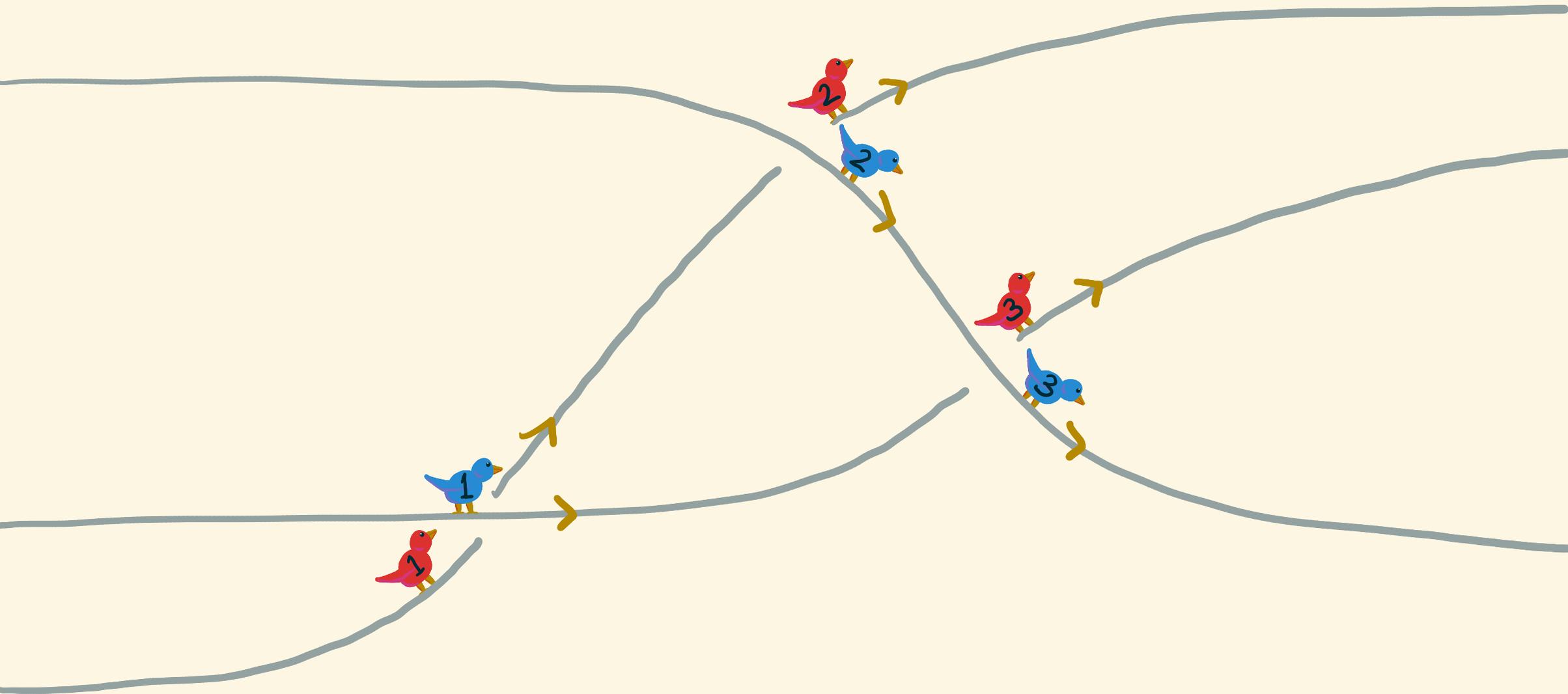
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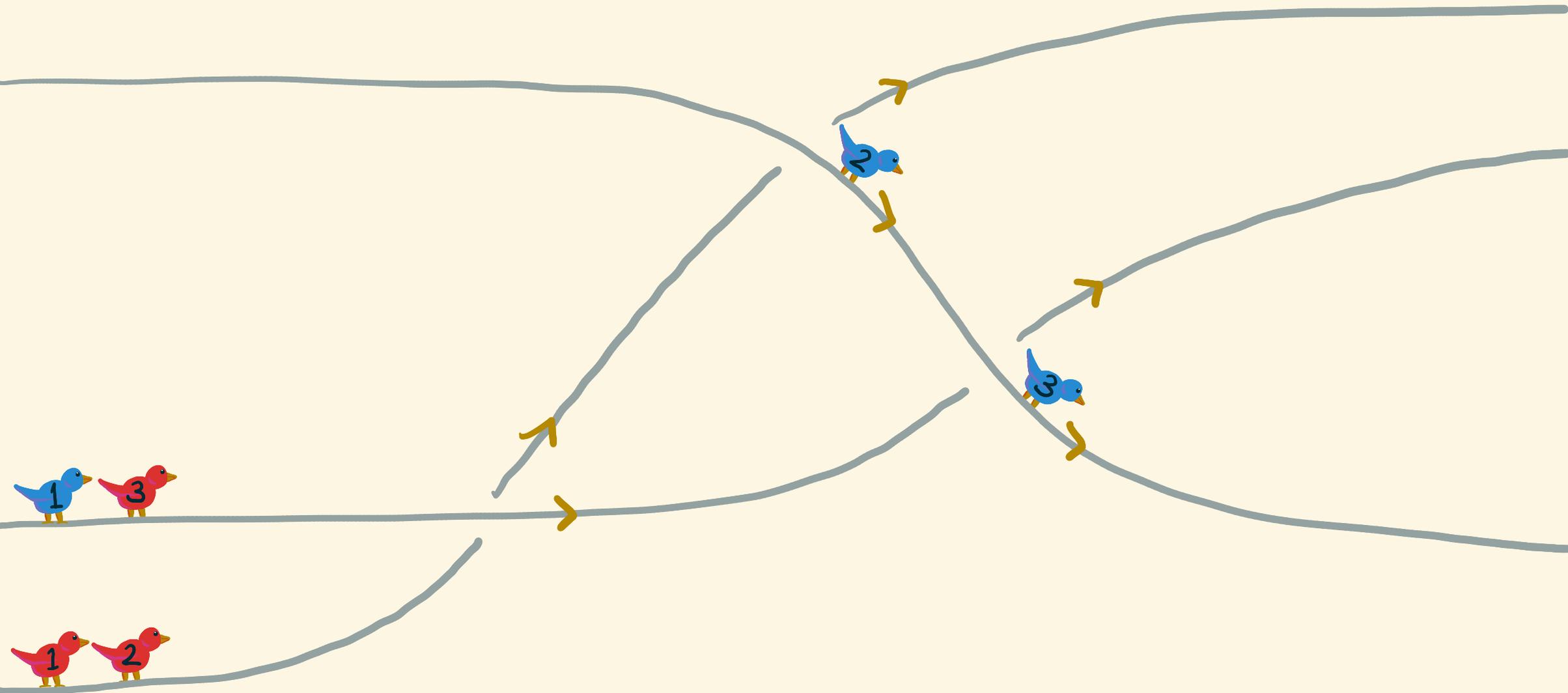
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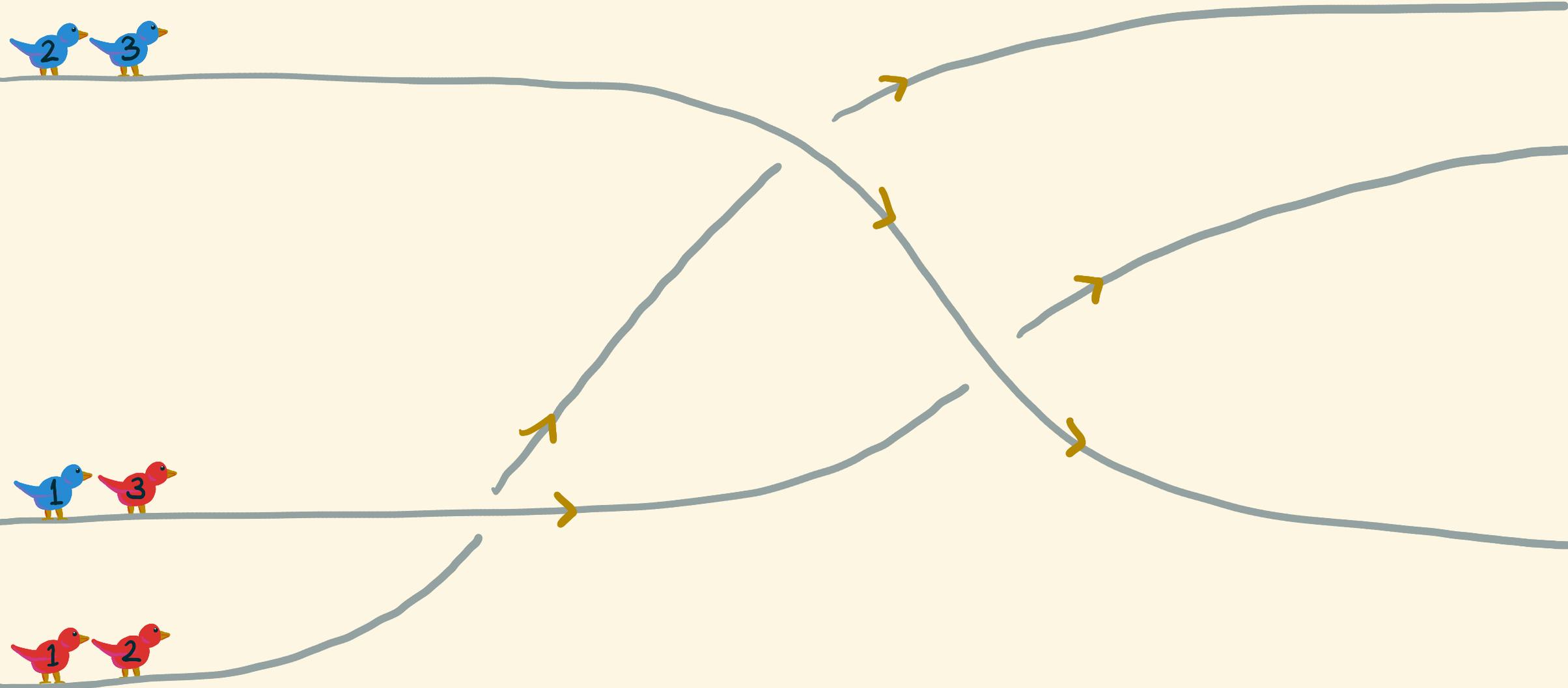
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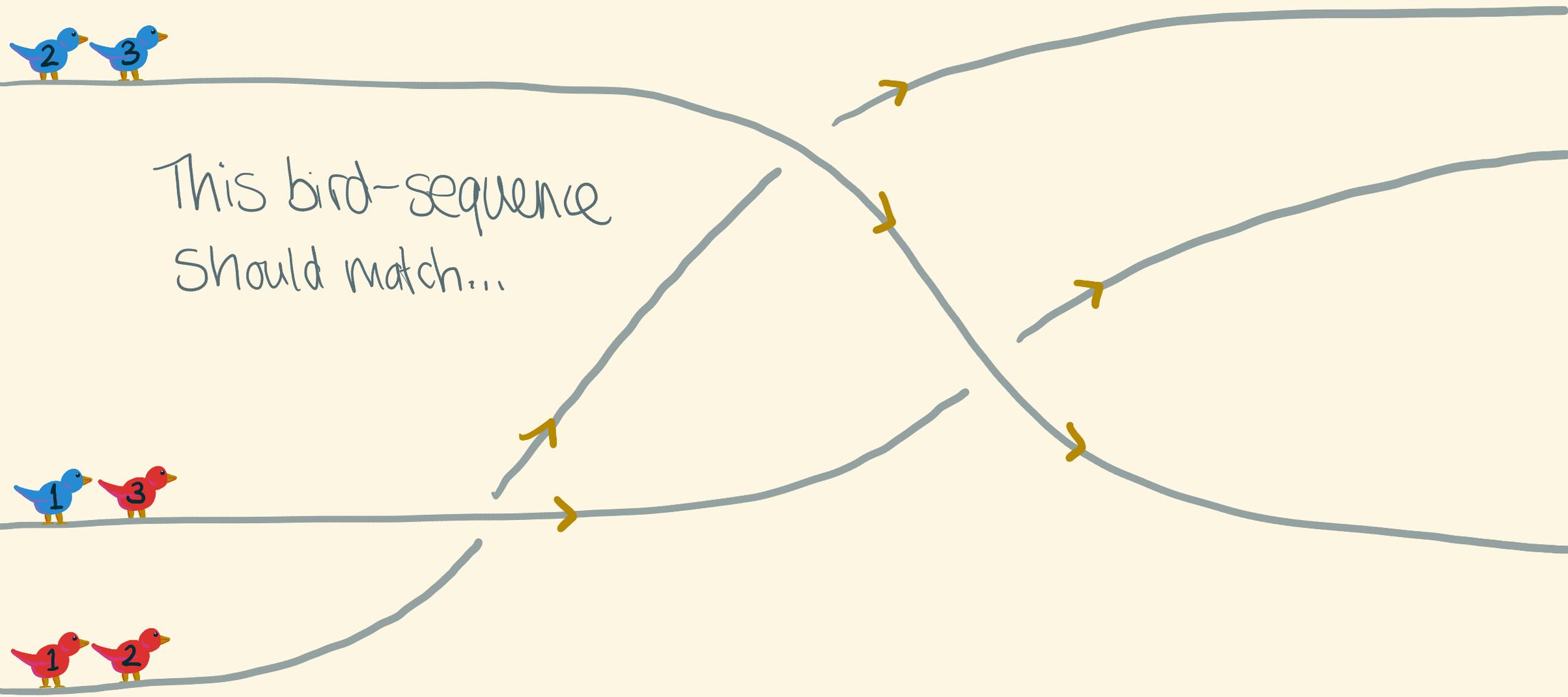
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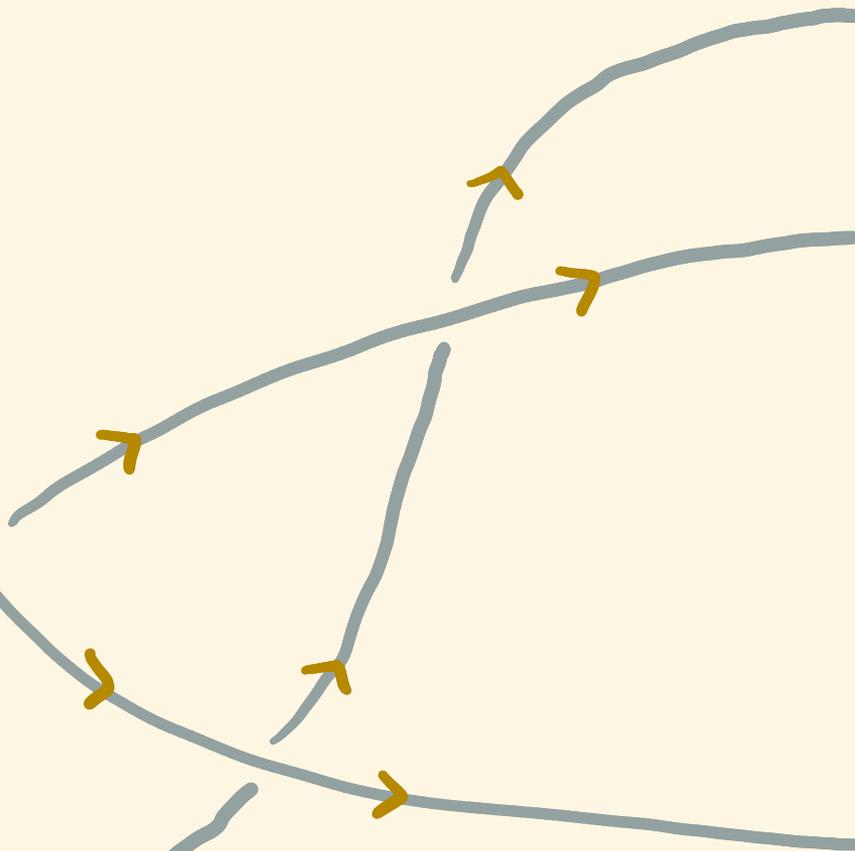
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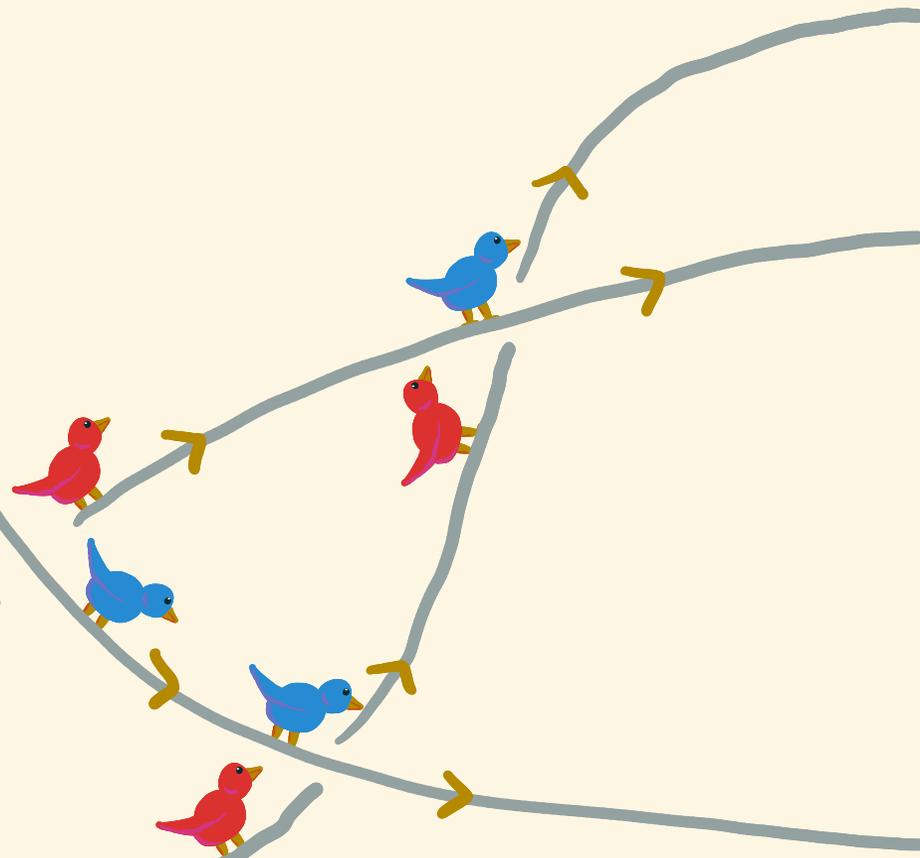
This bird-sequence
should match...



One more example:



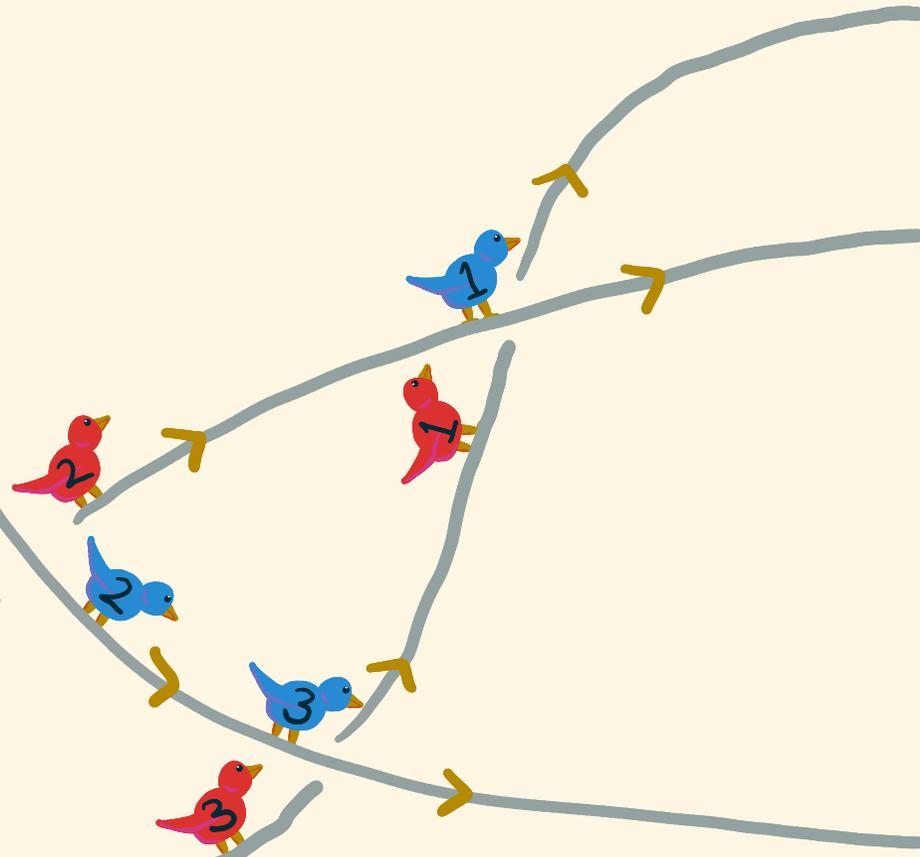
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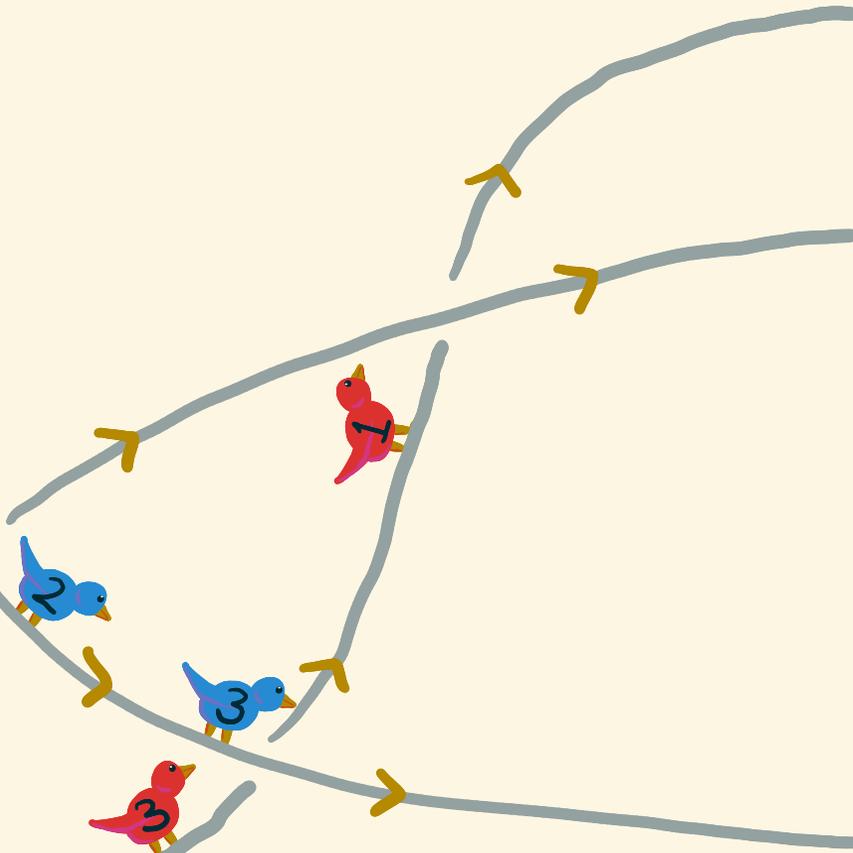
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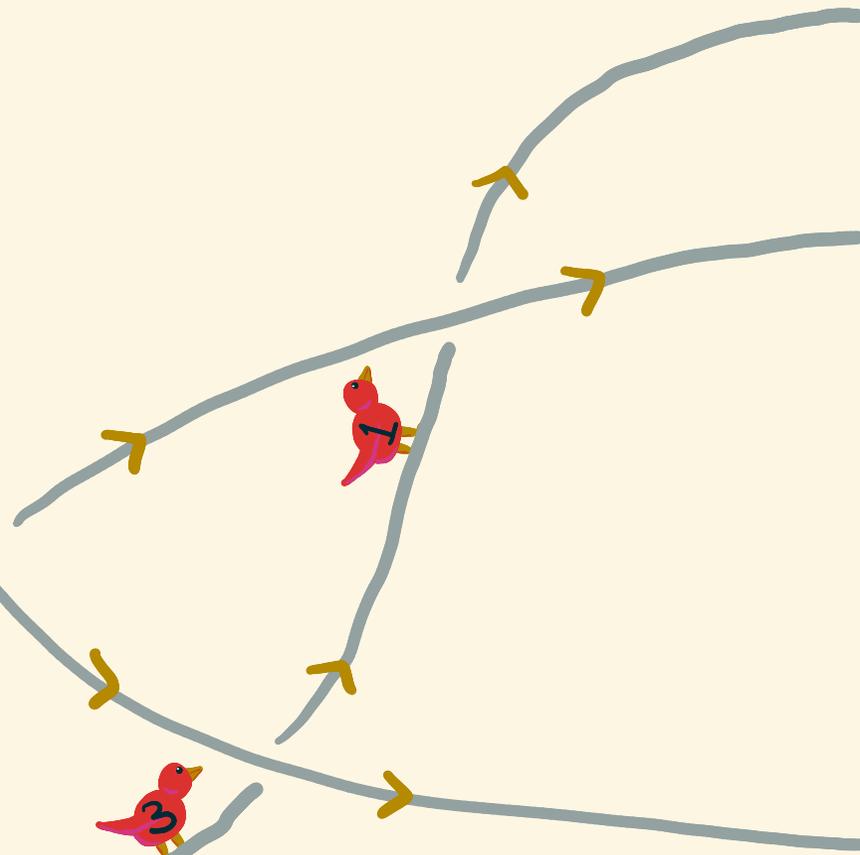
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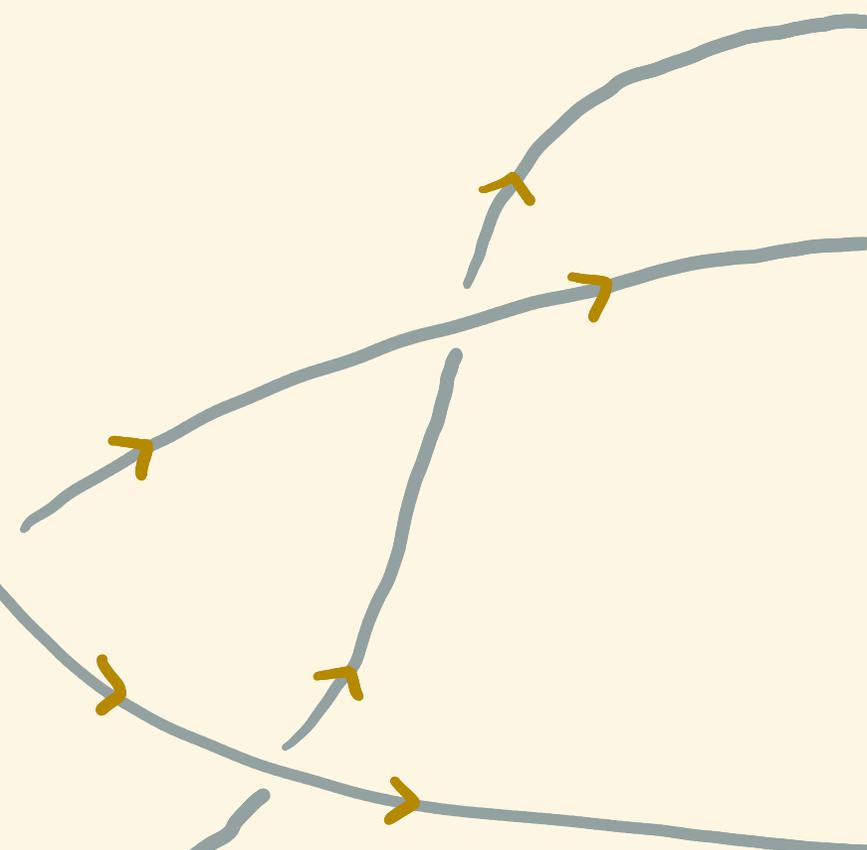
This bird-sequence
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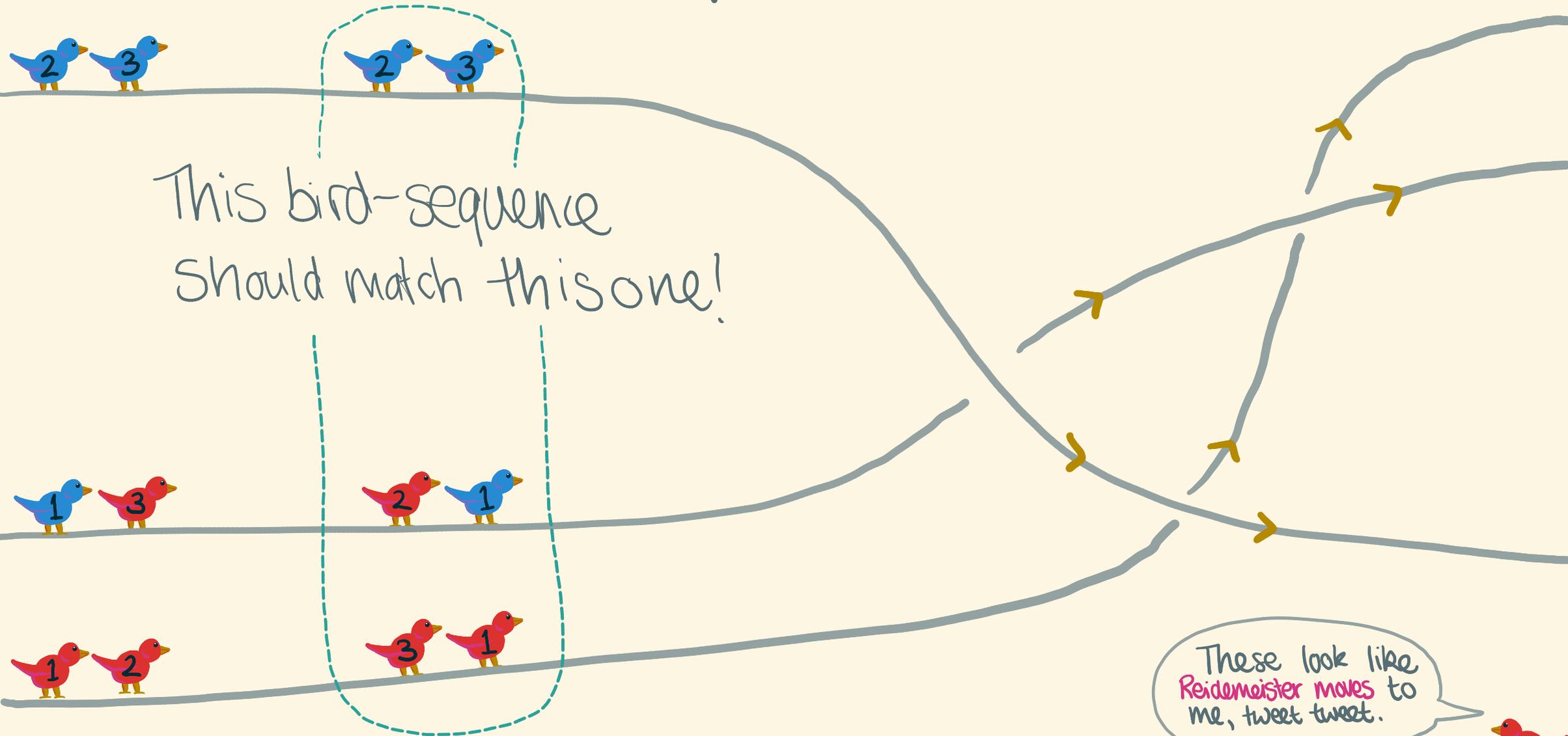
One more example:



This bird-sequence
should match...



One more example:



Thm (Reidemeister)

Using the above bird-sequence relations, you can tell whether two tangles can be rearranged into each other.

Thm (Coward & Lackenby)

Using the above bird-sequence relations, you can tell whether two tangles can be rearranged into each other.

Thm (Conrad & Lackenby)

Using the above bird-sequence relations, you can tell whether two tangles can be rearranged into each other, but not in your lifetime.



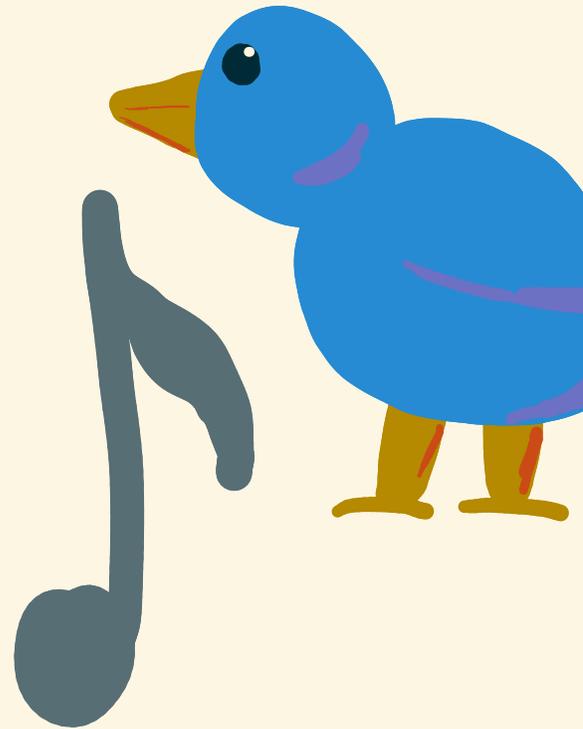
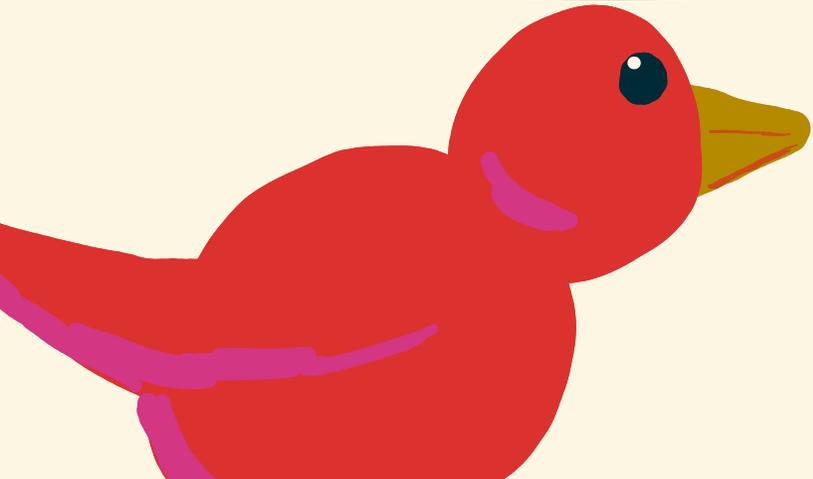
Juan Antonio Segal from Madrid, Spain, CC BY 2.0 <<https://creativecommons.org/licenses/by/2.0/>>, via Wikimedia Commons

Instead of comparing birds directly,

Let's compare

Instead of comparing birds directly,
let's compare their

songs



Idea: If each bird sings a note,
and each note is strung together,
then maybe equivalent tangles will
result in similar-sounding songs!

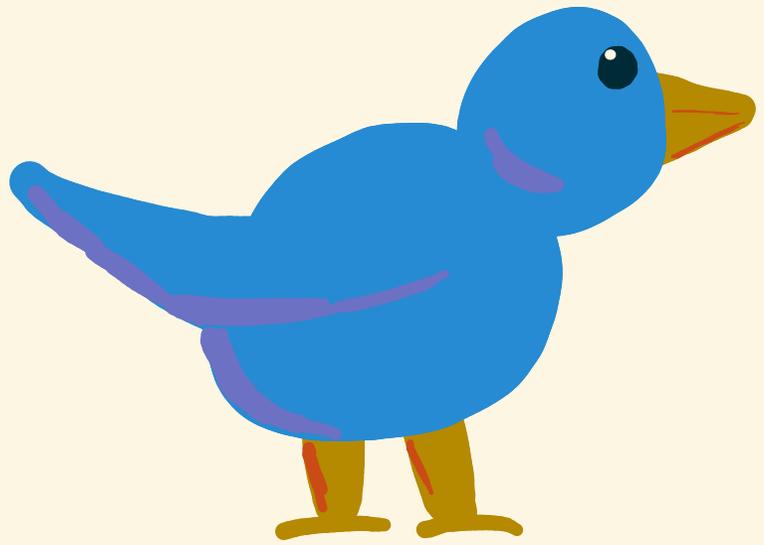
Idea: If each bird sings an algebra element,
and each note is strung together,
then maybe equivalent tangles will
result in similar-sounding songs!

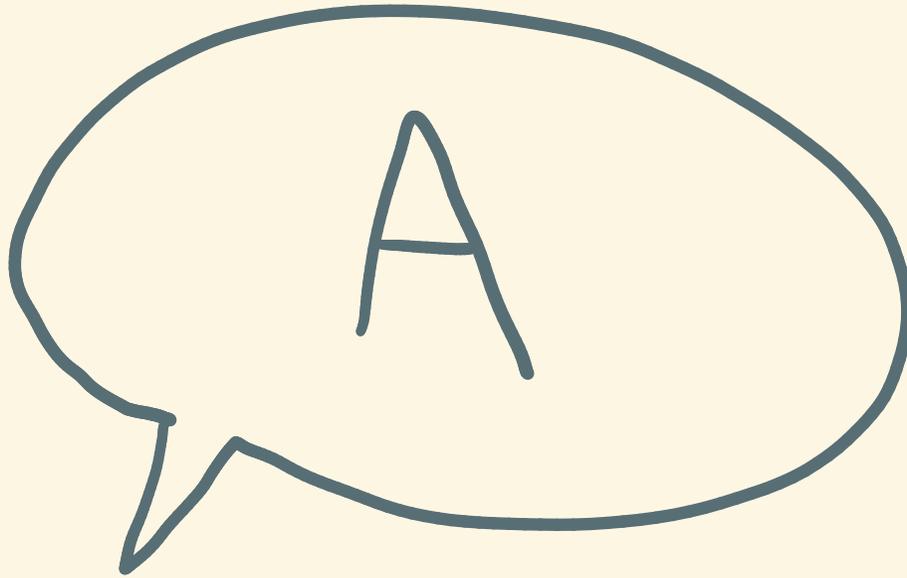
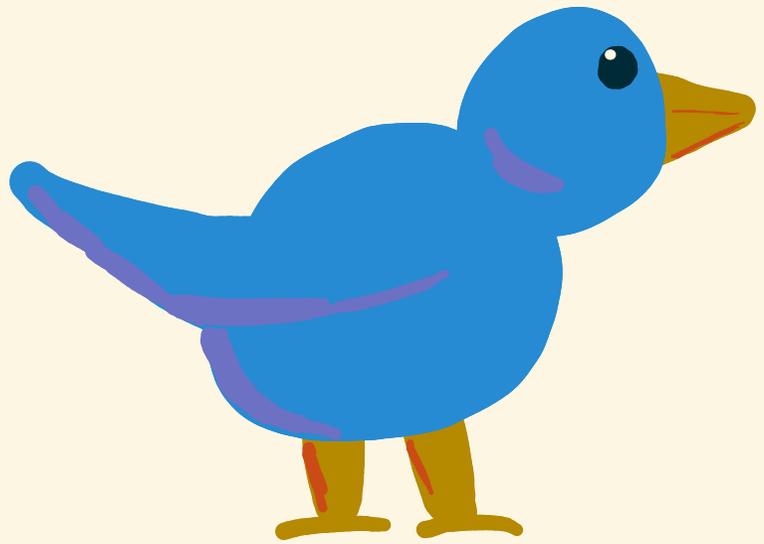
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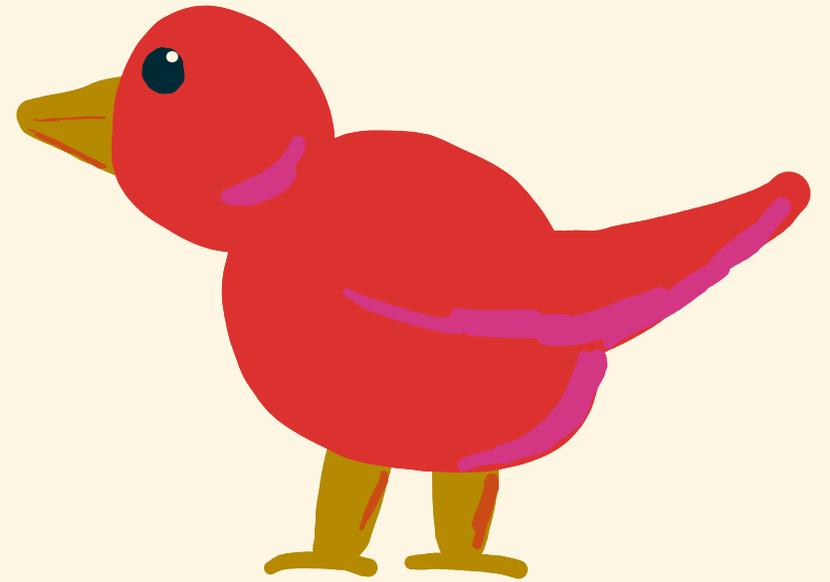
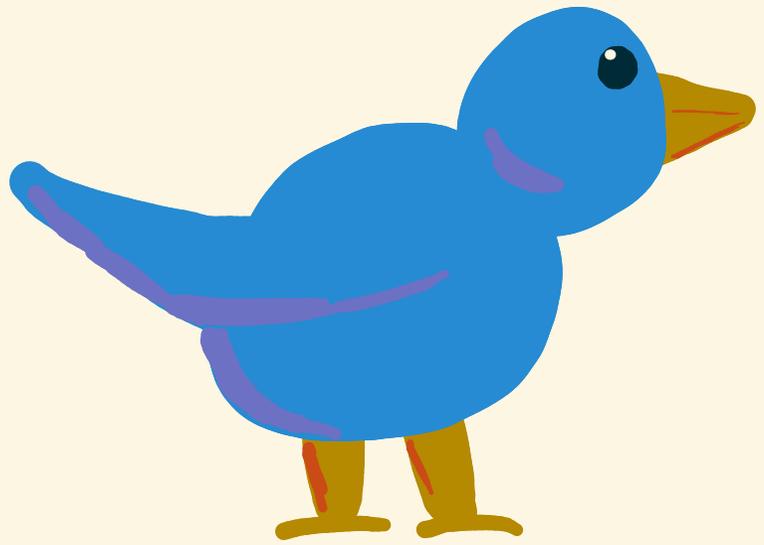
Idea: If each bird sings an algebra element,
and each element is multiplied together,
then maybe equivalent tangles will
result in similar-sounding songs!

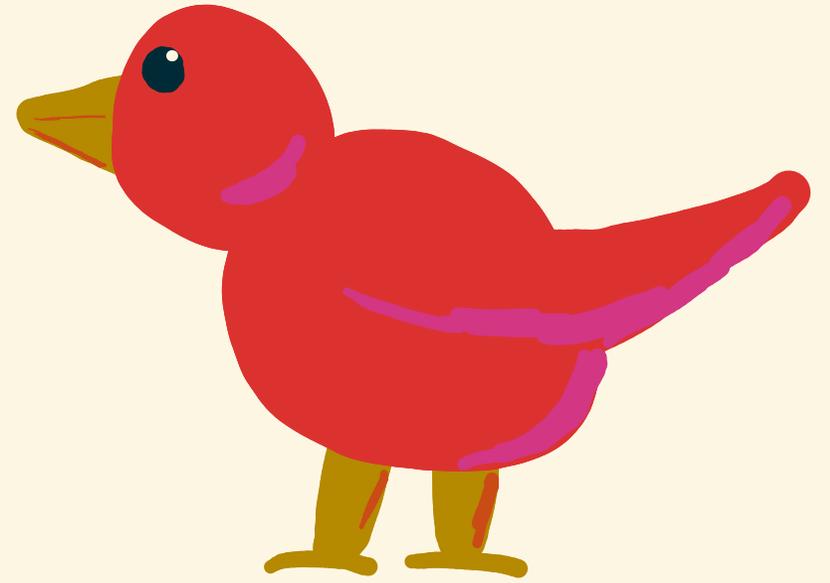
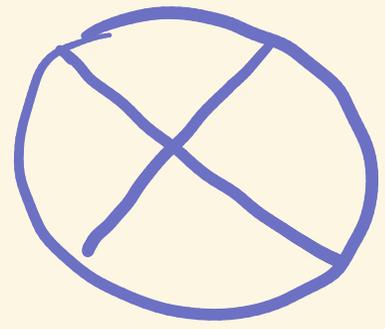
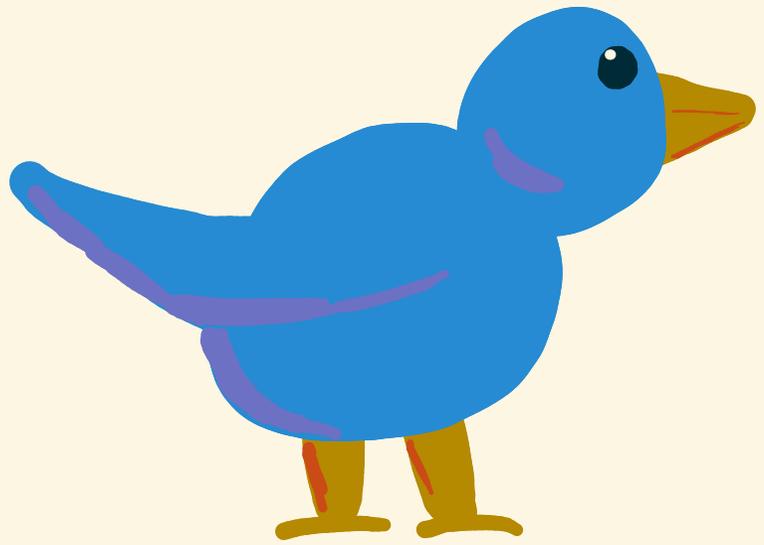
Idea: If each bird sings an algebra element,
and each element is multiplied together,
then maybe equivalent tangles will
result in equal songs!

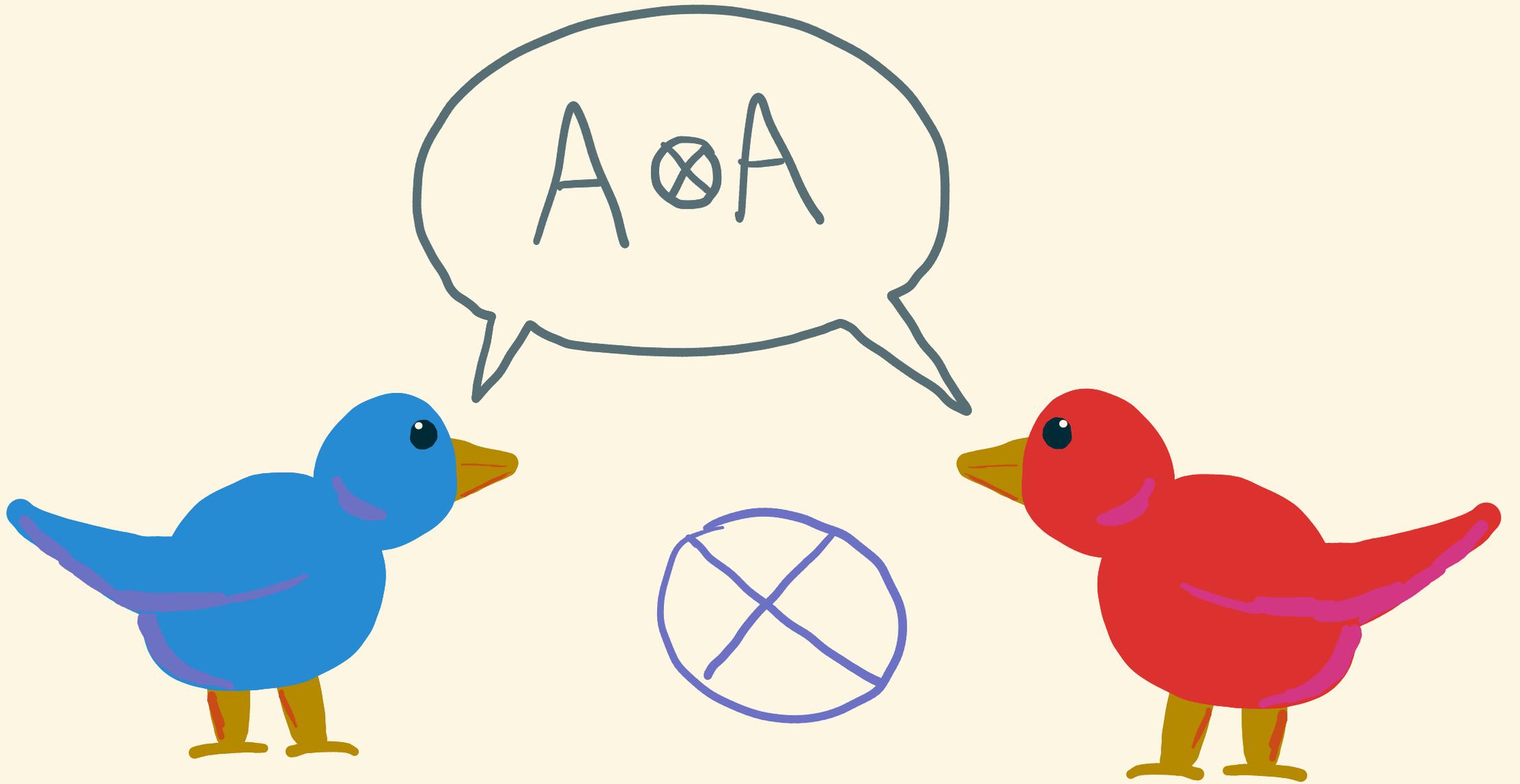
Idea: If each bird sings an algebra element,
and each element is multiplied together,
then maybe equivalent tangles will
result in equal products in the algebra!











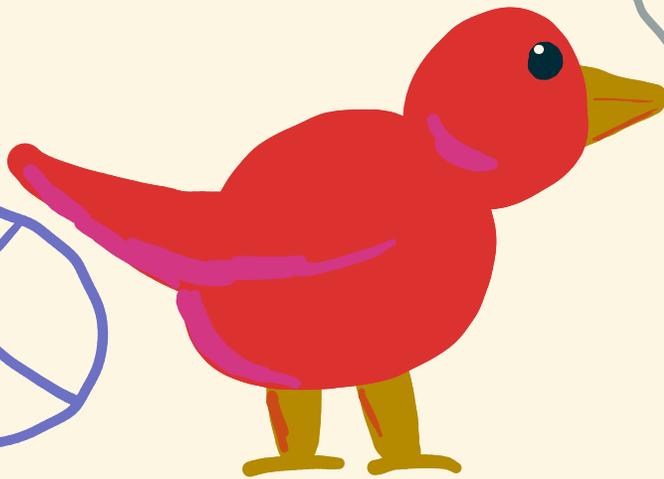
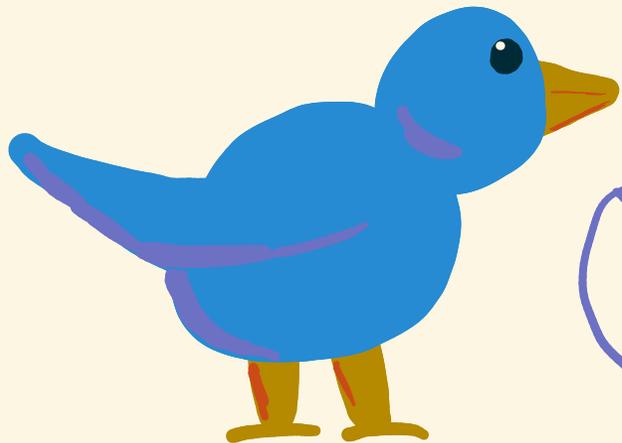
When $A = \left\langle \begin{pmatrix} 0 & 0 \\ q^{1/2} & 0 \end{pmatrix}, \begin{pmatrix} 0 & q^{1/2} \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} q^{-1} & 0 \\ 0 & q \end{pmatrix}, \begin{pmatrix} q & 0 \\ 0 & q^{-1} \end{pmatrix} \right\rangle$

Tweet tweet, that looks like
the 2-dimensional representation
of $U_q(\mathfrak{sl}_2)$!



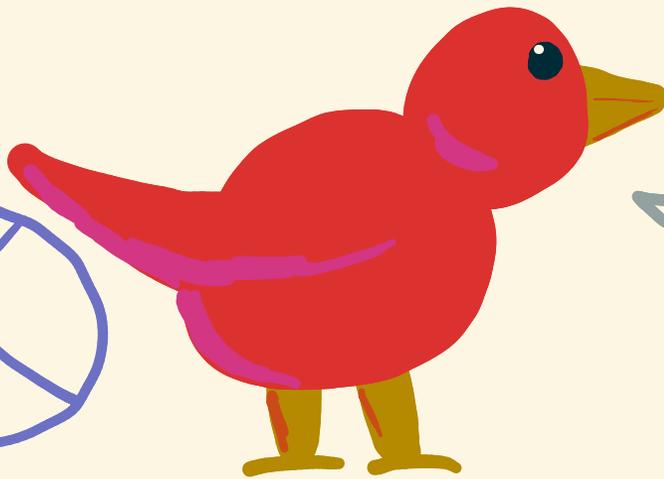
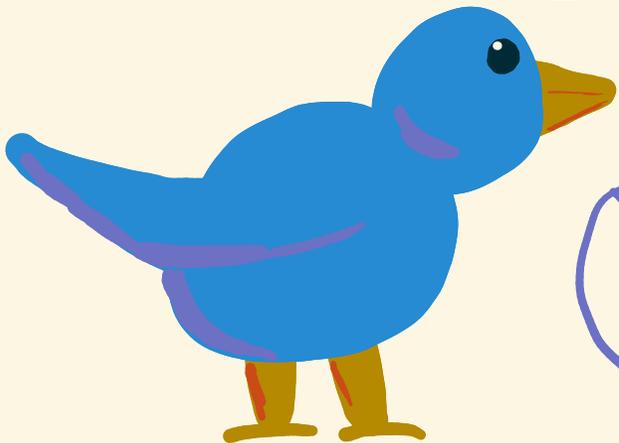
When $A = \left\langle \begin{pmatrix} 0 & 0 \\ q^{1/2} & 0 \end{pmatrix}, \begin{pmatrix} 0 & q^{1/2} \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} q^{-1} & 0 \\ 0 & q \end{pmatrix}, \begin{pmatrix} q & 0 \\ 0 & q^{-1} \end{pmatrix} \right\rangle$

$$q^{-1/2} \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & 1 & q - q^{-1} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & q \end{pmatrix}$$



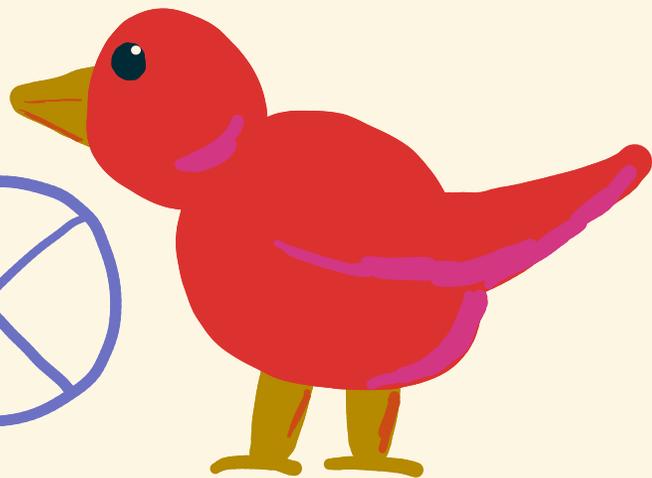
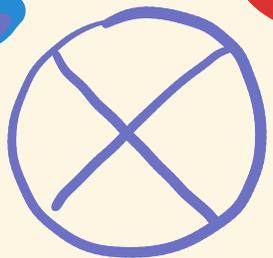
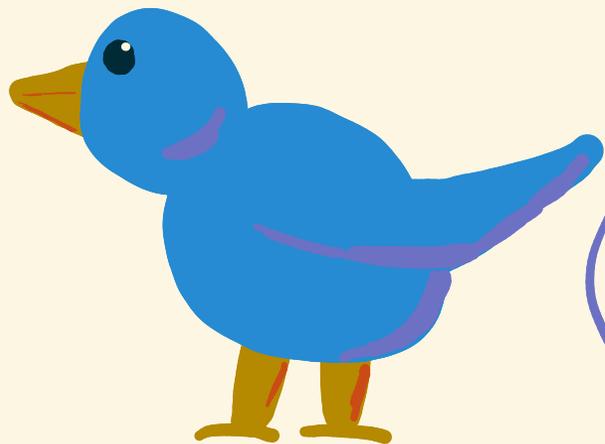
When $A = \left\langle \begin{pmatrix} 0 & 0 \\ q^{1/2} & 0 \end{pmatrix}, \begin{pmatrix} 0 & q^{1/2} \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} q^{-1} & 0 \\ 0 & q \end{pmatrix}, \begin{pmatrix} q & 0 \\ 0 & q^{-1} \end{pmatrix} \right\rangle$

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} q^{1/2} & 0 \\ 0 & q^{-1/2} \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} q^{-1/2} & 0 \\ 0 & q^{1/2} \end{pmatrix} \\ + q^{-1/2}(q - q^{-1}) \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$



When $A = \left\langle \begin{pmatrix} 0 & 0 \\ q^{1/2} & 0 \end{pmatrix}, \begin{pmatrix} 0 & q^{1/2} \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} q^{-1} & 0 \\ 0 & q \end{pmatrix}, \begin{pmatrix} q & 0 \\ 0 & q^{-1} \end{pmatrix} \right\rangle$

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} q^{1/2} & 0 \\ 0 & q^{1/2} \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} q^{1/2} & 0 \\ 0 & q^{-1/2} \end{pmatrix} \\ - q^{1/2}(q - q^{-1}) \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$



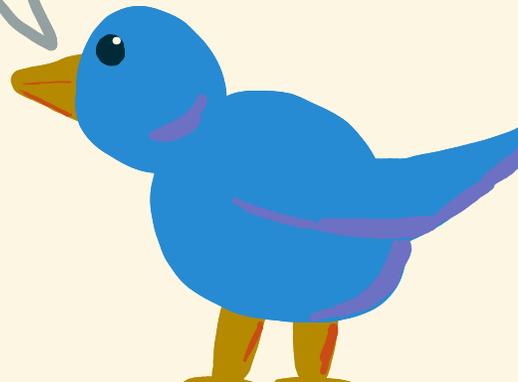
This birdsong gives
us the Jones polynomial.

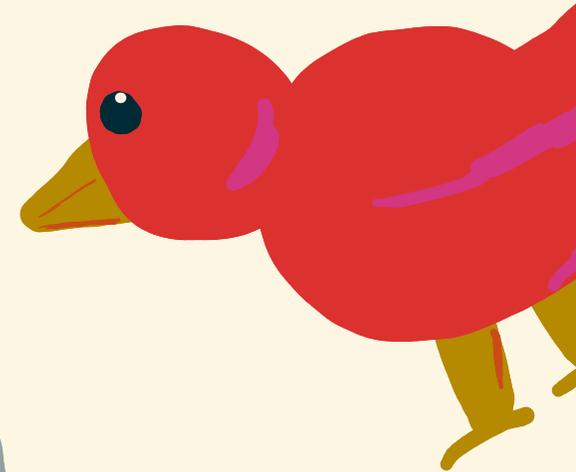
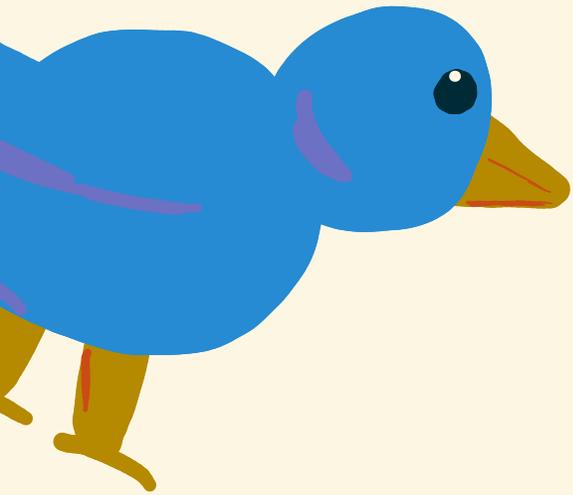
This birdsong gives
us the Jones polynomial.

(A quantum invariant)

This birdsong gives
us the Jones polynomial.

(A quantum invariant) q





Thank
you!

