Algebraic Geometry

§ Topics for part 1:

↳ Sheaves

↳ Schemes

↳ Morphisms between schemes

* Email Arul ~~the~~ your thoughts on content & speed *

This is largely a foundational course. There is no "big theorem" at the end.

There are (at least) two types of problem-solvers: people who open nuts with nutcrackers, and those who open them with water.

§ Algebraic Subsets

<u>Defn</u>. Given a set $S \subseteq \mathbb{C}[x_1, \_, x_n]$ of polynomials, define

$$V(S) := \{ v \in \mathbb{C}^n : \forall f \in S \;\; f(v) = 0 \}$$

This is called an <u>algebraic set</u>.

<u>Defn</u>. Similarly, given an algebraic set $V \subseteq \mathbb{C}^n$, define

$$I(V) := \{ f \in \mathbb{C}[x_1, \_, x_n] : \forall v \in V \;\; f(v) = 0 \}$$

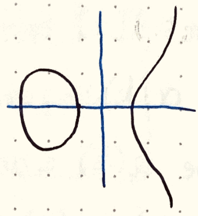
<u>Observations</u>

1. Given V, $I(V)$ is an ideal
2. The union of two algebraic sets is algebraic:
$$V(S_1) \cup V(S_2) = V(S_1 S_2)$$
3. $\bigcap_\alpha V(S_\alpha) = V\left(\bigcup_\alpha S_\alpha\right)$ (algebraic sets are closed under arbitrary intersections)
4. The functions V and I are inclusion-reversing
5. $I(V \cup V') = I(V) \cap I(V')$

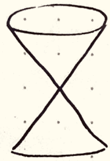
<u>Exercise</u>: Verify the above points ↑

Examples

- $V(1) = \emptyset$ • $V(0) = \mathbb{C}^n$
- any affine space is algebraic
- $S = \{ y^2 - x^3 - Ax - B \}$



- $S = \{ z^2 - y^2 - x^2 \}$



- Case $n=1$ $S = \{ f_1(x), f_2(x), \dots \}$

$V(S) = \bigcap_i \text{zeros of } f_i = \{ z_1, \dots, z_k \} \subseteq \mathbb{C}$

Define then $f(x) = (x - z_1)(x - z_2) \dots (x - z_k)$:

So $V(S) = V(f)$

$\rightsquigarrow$ Algebraic subsets of $\mathbb{C}$: $\{\text{finite sets}\} \cup \{\mathbb{C}\}$

Thm If V is an algebraic set, then $\exists$ finite $S \subseteq \mathbb{C}[x_1, \dots, x_n]$ such that $V = V(S)$.

Proof. We know $V = V(T) = V((T))$, so it is enought to prove every ideal is generated by a finite set of elements in $\mathbb{C}[x_1, \dots, x_n]$. Then $(T) = I = (S)$, so $V((T)) = V(I) = V(S)$. So, we need to show $\mathbb{C}[x_1, \dots, x_n]$ is Noetherian. This follows from the following:

$\boxed{\text{Thm (Hilbert Basis Theorem)}}$

$\boxed{\text{If } A \text{ is Noetherian, then } A[x] \text{ is Noetherian.}}$

Proof. Let $\mathfrak{a} \subseteq A[x]$ be an ideal. We want to show it's f.g.

Defn. $\mathfrak{a}(k) := \{\text{leading coefficients of deg-}k\text{ poly in } \mathfrak{a}\} \cup \{0\}$

Observe $\mathfrak{a}(k)$ is an ideal of A. Furthermore,

$$\mathfrak{a}(1) \subseteq \mathfrak{a}(2) \subseteq \dots \subseteq A$$

(simply multiply by x)

A Noetherian $\Rightarrow$ $\mathfrak{a}(k)$ stabilizes. (Say at d)

For each $k \le d$, $\mathfrak{a}(k) = (a_{k_1}, a_{k_2}, \dots, a_{k_{n_k}})$, so $\exists f_{k_1}, f_{k_2}, \dots, f_{k_{n_k}} \in \mathfrak{a}$ with corresponding leading coefficients.

$\boxed{\text{Claim}}$ $\boxed{\mathfrak{a} = (\underbrace{f_{1_1}, f_{1_2}, \dots, f_{1_{n_1}}, f_{2_1}, f_{2_2}, \dots}_{S}, f_{2_{n_2}}, \dots, f_{d_1}, f_{d_2}, \dots, f_{d_{n_d}})}$

Proof. Let $f \in \mathfrak{a}$, $f(x) = \alpha x^m + \cdots$. $\exists g \in (S)$ such that $\deg g \le \deg f$ such that g and f have the same leading coefficient, so

$$f - g \cdot x^{\deg f - \deg g}$$ has lower degree.

Proceeding by induction (**how?**), we can reduce f to ~~the~~ 0 by a series of elements of (S), so we are done. ■ ■

Q. When is $V(S)=\emptyset$?

Thm (Weak Hilbert Nullstellensatz).

Suppose $I \subsetneq \mathbb{C}[x_1, \_\_, x_n]$ is proper.
Then $V(I) \neq \emptyset$

§ An important way of thinking:

For $V = V(I) \subseteq \mathbb{C}^n$ algebraic and $P \in V \subseteq \mathbb{C}^n$, ~~alger~~ we get a map

$$\frac{\mathbb{C}[x_1, \_\_, x_n]}{I} \xrightarrow{\varphi_P} \mathbb{C}$$

Given by $\varphi_P(f) = f(P)$. φ_P is a $\mathbb{C}$-algebra homomorphism.

Conversely, given $\varphi : \mathbb{C}[x_1, \_\_, x_n]/I \longrightarrow \mathbb{C}$, we get a point $P \in V(I)$

$$P = (\varphi(x_1), \varphi(x_2), \_\_, \varphi(x_n))$$

[We will return to this correspondence later.]

Proof (WHN). Since $I \subseteq m \Rightarrow V(I) \supseteq V(m)$, we may assume I is maximal, WTS $V(I)$ has a point $\Leftrightarrow \mathbb{C}[x_1, \_, x_n]/I \to \mathbb{C}$ exists.
We know $\mathbb{C}[x_1, \_\_, x_n]/I = K/\mathbb{C}$, a field extension of $\mathbb{C}$.

Lemma (Zariski) For L/K a field extension st. L is fin. gen. as a K-algebra, then $[L:K] < \infty$

Proof. [2.12 of Milne] By induction on # generators of L/K:
$L = K[\emptyset] = K$ is obvious.

Suppose $L = K\langle x_1, \ldots, x_r \rangle$. It is enough to show that each x_i is algebraic over K. Suppose otherwise.

$$\begin{tikzcd} L = K[x_1, \overbrace{\ldots, x_r}^{r-1 \text{ elements}}] \arrow[d, dash] \\ K[x_1] \arrow[d, dash, "\leftarrow \text{not algebraic}"] \\ K \end{tikzcd}$$

By induction, $x_2, \ldots, x_r$ are algebraic over $K(x_1)$.

$$x_2^m + f_1 x_2^{m-1} + \cdots = 0 \quad , f_i \in K(x_1)$$

We can clear denominators $\Rightarrow \exists\, c \in K[x_1]$ so that

$cx_2, \ldots, cx_r$ are integral over $K[x_1]$.

Exercise: Think about this ⤴.

Let $f \in K(x_1) \subseteq K[x_1, \ldots, x_r] \Rightarrow$ for some large pwr of c,

$$c^N f \in K[x_1, cx_2, \ldots, cx_r]$$

Example: $f = x_1 x_2$, $cf = x_1(cx_2) \in K[x_1, cx_2]$

We know that $cx_2, \ldots, cx_r$ are integral over $K[x_1]$
$\Rightarrow c^N f$ is integral over $K[x_1]$ ↯
The same c cannot work for all f.

Corollary of HWN Every maximal ideal of $\mathbb{C}[x_1, \ldots, x_n]$ is $((x-a_1),(x-a_2),\ldots,(x-a_n))$ For $(a_1,\ldots,a_n)\in\mathbb{C}^n$

Proof. $(x-a_1, \ldots, x-a_n)$ is clearly maximal.

Conversely, $\mathfrak{m}$ maximal $\Rightarrow$ $\mathbb{C}[x_1,\ldots,x_n]/\mathfrak{m}$ is a field, hence $\mathbb{C}$. So $V(\mathfrak{m})$ is a point

There is only one $\mathbb{C}$-alg hom $\mathbb{C}\to\mathbb{C}$, so $\exists!$ point $P=(a_1,\ldots,a_n)$

So $I(\{P\})$

Corollary $\exists$ natural bijection $\mathbb{C}^n \rightleftharpoons \{\text{max. ideals of } \mathbb{C}[x_1,\ldots,x_n]\}$

$P \longmapsto$ all functions which vanish at P

Corollary For $V=V(I)$, I an ideal, $\exists$ natural bijection

$$V(I) \rightleftharpoons \{\text{max ideals of } \mathbb{C}[x_1,\ldots,x_n]/I\}$$

Proof. Restrict to points of $V(I)$:

$$\begin{aligned} V(I) &\rightleftharpoons \{\text{max ideals of } \mathbb{C}[x_1,\ldots,x_n] \text{ which vanish on } P\in V(I)\} \\ &\rightleftharpoons \{\text{max ideals of } \mathbb{C}[x_1,\ldots,x_n] \text{ which contain } I\} \\ &\rightleftharpoons \{\text{max ideals of } \mathbb{C}[x_1,\ldots,x_n]/I\} \end{aligned}$$ ∎

Foreshadowing: These correspondences allow us to understand algebraic sets by understanding quotients of polynomial rings.

Thm Let $W \subseteq \mathbb{C}^n$ any subset. Then $V(I(W))$ is the smallest algebraic set containing W. So W alg. $\Rightarrow V(I(W))$ alg.

Proof. Suppose $V(\mathfrak{a}) \supseteq W$. Then $\mathfrak{a} \subseteq I(W) \Rightarrow V(I(W)) \subseteq V(\mathfrak{a})$ as required. ■

Q. What about $I(V(I))$?

For $I \subseteq \mathbb{C}[x_1, \ldots, x_n]$ an ideal, $I(V(I)) \supseteq I$ exercise

Example $n=1$, $f(x) = x^2$, $V((f)) = \{0\}$, $I(V((f))) = I(\{0\}) = (x)$

and $(x) \supsetneq (x^2)$

Thm (Hilbert's Strong Nullstellensatz) 2.16 in Milne

$$I(V(I)) = \sqrt{I} = \{f \in \mathbb{C}[x_1, \ldots, x_n] : f^N \in I \text{ for some } N\}$$

Proof. Let's call the ideal $\mathfrak{a}$. $I(V(\mathfrak{a})) \supseteq \sqrt{\mathfrak{a}}$

$f \in \sqrt{\mathfrak{a}} \Rightarrow \exists n \; f^n \in \mathfrak{a} \Rightarrow f^n$ vanishes on $V(\mathfrak{a}) \Rightarrow f$ vanishes on $V(\mathfrak{a})$ (the complex numbers have no nilpotents).

WTS. If h vanishes on $V(\mathfrak{a})$, then $h^N \in \mathfrak{a}$ for some N.

By HBT, $\mathfrak{a} = (g_1, \ldots, g_m)$.

Consider the system on $\mathbb{C}^{n+1}$, $\mathbb{C}[x_1, \ldots, x_n, y]$

$$\begin{cases} g_i(x_1, \ldots, x_n) = 0 & \forall i \\ 1 - yh(x_1, \ldots, x_n) = 0 \end{cases}$$

If $(a_1, \ldots, a_n, b)$ satisfies the system except the last eqn, then $(a_1, \ldots, a_n) \in V(\mathfrak{a})$, $h(a_1, \ldots, a_n) = 0$, so the last eqn cannot be satisfied.

So $V((g_1, \ldots, g_m, 1 - yh)) = \emptyset$ By WHN:

$\exists f_1, \ldots, f_{m+1}$ such that $1 = \sum_{i=1}^{m} f_i g_i + f_{m+1}(1-yh)$

Send $y \mapsto h^{-1}$, $\mathbb{C}[x_1, \ldots, x_n, y] \to \mathbb{C}(x_1, \ldots, x_n)$

$$1 = \sum \underbrace{f_i(x_1, \ldots, x_n, h^{-1})}_{\frac{F_i(x_1, \ldots, x_n)}{h^{N_i}}} \cdot g_i(x_1, \ldots, x_n)$$

Multiplying out we get $h^N = \sum F_i g_i$ ∎

Corollary $\mathfrak{a} \leftrightarrow V(\mathfrak{a})$ gives a nat. bij.
$\{\text{alg. subsets of } \mathbb{C}^n\} \rightleftarrows \{\text{radical ideals } \mathbb{C}[x_1, \ldots, x_n]\}$