MAT224H1F

Summer 2021

Tutorial Problems

Week 2

$\{\vec{v}_1, \vec{v}_2, \_\_\_, \vec{v}_n\}$

Defn. A set $S \subset V$ is lineary independant if ___[fill in the blank]___.

linear dep/indep

1. Determine whether each of the following sets of vectors are linearly dependent or independent:

(a) Any collection of three vectors in $\mathbb{R}^2$.

(b) $\{e^x, \sin x, \cos x\}$ in $C(\mathbb{R})$ (the space of continuous functions $f\colon \mathbb{R} \to \mathbb{R}$).

(c) $\{1, \sin^2 x, \cos^2 x\}$ in $C(\mathbb{R})$.

(d) $\left\{\pi, e, \sqrt{2}\right\}$ in $\mathbb{R}$.

(e) $\left\{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\right\}$ in $M_2(\mathbb{R})$ (the space of 2×2 matrices with real coefficients).

2. Prove that two vectors $\vec{x}$ and $\vec{y}$ in a real vector space V are linearly dependent if and only if either $\vec{y}$ is a scalar multiple of $\vec{x}$ or $\vec{x}$ is a scalar multiple of $\vec{y}$. However, find a linearly dependent set S of three vectors in some vector space V such that no element of S is a scalar multiple of a different element of S.

3. Let V be a real vector space.

(a) Prove that a subset $\{\vec{x}, \vec{y}\}$ of V is linearly independent if and only if the set $\{\vec{x} + \vec{y}, \vec{x} - \vec{y}\}$ is linearly independent.

(b) Prove that a subset $\{\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_n\}$ of V is linearly independent if and only if the set $\{\vec{x}_1, \vec{x}_1 + \vec{x}_2, \ldots, \vec{x}_1 + \cdots + \vec{x}_n\}$ is linearly independent.

4. Let S be a subset of an n-dimensional vector space V and assume that S has n elements. Prove that S is linearly independent if and only if $\operatorname{span}(S) = V$.

5. Find a basis and compute the dimension of each of the following spaces:

(a) $W := \{f \in \operatorname{span}(e^x, e^{-x}, \sin x, \cos x, x, 1) : f(0) = f'(0) = 0\}$.

(b) For each $0 \le k \le n$, $W_{n,k} := \left\{f \in P_n(\mathbb{R})\colon f^{(k)}(0) = 0\right\}$. (*Hint:* The dimension does not depend on k).

6. Let $a_1, \ldots, a_n$ be distinct real numbers, and let $m_1, \ldots, m_n$ be arbitrary positive integers. Prove that the set

$$S := \left\{\frac{1}{x - a_1}, \ldots, \frac{1}{(x - a_1)^{m_1}}, \ldots, \frac{1}{x - a_n}, \ldots, \frac{1}{(x - a_n)^{m_n}}\right\}$$
$$= \left\{\frac{1}{(x - a_i)^{k_i}} : 1 \le i \le n \text{ and } 1 \le k_i \le m_i\right\}$$

is a basis for the subspace of rational functions

$$W := \left\{\frac{p(x)}{(x - a_1)^{m_1} \cdots (x - a_n)^{m_n}} : \deg p(x) < m_1 + \cdots + m_n\right\}$$

$\{\vec{v}_1, \vec{v}_2, \_\_\_, \vec{v}_n\}$

Defn. A set $S \subset V$ is lineary independant if [fill in the blank].

1

> ~~there exists $\vec{v}_1, \_\_\_, \vec{v}_n$ such that:~~
>
> The equation:
>
> $$x_1\vec{v}_1 + \_\_\_ + x_n\vec{v}_n = \vec{0}$$
>
> has only a trivial solution $(x_i \equiv 0)$.

2

> $$a_1\vec{v}_1 + \_\_\_ + a_n\vec{v}_n = \vec{0} \iff a_1 = a_2 = \_\_\_ = a_n = 0$$

1a A collection of 3 vectors in $\mathbb{R}^2$, $S = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$

If S were LI, then $\{\vec{v}_1, \vec{v}_2\}$ would also be L.I. Then $\text{span}\{\vec{v}_1, \vec{v}_2\} = \mathbb{R}^2$. so $\vec{v}_3 \in \text{span}\{\vec{v}_1, \vec{v}_2\}$ $\lightning$

1b $\{e^x, \sin x, \cos x\} \in C(\mathbb{R})$ is linearly ____.

$$\boxed{a_1 e^x + a_2 \sin x + a_3 \cos x = 0 \quad \text{for all } x}$$

zero function.

$\iff a_1, a_2, a_3 = 0$

Q. Does the "x" have to be the same in all three terms?

A. Yes. $f+g \in C(\mathbb{R})$ must also be a function of a single variable.

1c) $\{1, \cos^2 x, \sin^2 x\}$

Because of trig identity $\sin^2 x + \cos^2 x = 1$ for all x,
this set is L.D.

The vector space structure of $(C(\mathbb{R}), +, \cdot)$:

Given $f, g \in C(\mathbb{R})$ and a scalar $\lambda \in \mathbb{R}$,

- $(f+g)(x) = f(x) + g(x)$
- $(\lambda \cdot f)(x) = \lambda \cdot f(x)$

Strategy: choose particular values of x to get relations b/w the a_i's.

$x = 0 \Rightarrow \quad a_1 e^0 + a_2 \cancel{\sin 0}^{\,0} + a_3 \cos 0 = 0$

$= \boxed{a_1 + a_3 = 0}$

$x = \frac{\pi}{2} \Rightarrow \quad a_1 e^{\pi/2} + a_2 + a_3 \cdot 0 = 0 \rightsquigarrow \boxed{a_1 \cdot e^{\pi/2} + a_2 = 0}$

$x = \pi \Rightarrow \quad a_1 e^{\pi} + a_2 \cdot 0 + a_3 \cdot (-1) \rightsquigarrow \boxed{a_1 e^{\pi} = a_3}$

$a_1 (\underbrace{1 + e^{\pi}}_{\approx 30}) = 0 \Rightarrow a_1 = 0 \qquad a_3 = 0 \qquad a_2 = 0$

$e^{\pi} \neq -1 \Leftarrow \underline{e^{x} > 0 \text{ for all } x}$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ e^{\pi/2} & 1 & 0 & 0 \\ e^{\pi} & 0 & 1 & 0 \end{array}\right]$$

7. Let $W_1, \dots, W_n$ be subspaces of a vector space V. Prove that

$$\dim(W_1 + \cdots + W_n) \leq \dim(W_1) + \cdots + \dim(W_n),$$

and that the equality is attained if and only if

$$W_i \cap (W_1 + \cdots + W_{i-1}) = \{\vec{0}\}$$

for all $1 < i \leq n$.

8. Determine which of the following maps $T\colon V \to W$ are linear:

(a) With $V = W = \mathbb{R}$, $T(a) = a + 1$.
(b) With $V = C(\mathbb{R}) \times \mathbb{R}$ and $W = \mathbb{R}$, $T(f, a) = f'(a)$.
(c) With $V = \mathbb{R}^2$ and $W = \mathbb{R}^3$, $T(a, b) = (a, b, ab)$.

9. On $(0, \infty)$ define the operations as in Problem **1.** of last week's problem set, and in $\mathbb{R}^2$ define the operations

$$(a, b) \boxplus (c, d) := (b + d, a + c) \text{ and } a \boxdot (c, d) := (ac, ad) :$$

(a) Is the exponential function $\exp\colon \mathbb{R} \to (0, \infty)$ given by $\exp(x) = e^x$, with respect to the operations described above linear?
(b) Is the identity $\mathrm{id}\colon (\mathbb{R}^2, +, \cdot) \to (\mathbb{R}^2, \boxplus, \boxdot)$ linear?

10. If a is a real number, observe that the map $T_a\colon \mathbb{R} \to \mathbb{R}$ given by $T_a(b) = ab$ is linear.

(a) Prove that every linear transformation $T\colon \mathbb{R} \to \mathbb{R}$ is of the form T_a for some a. (*Hint:* Compute $T(1)$).
(b) Prove that the map $T\colon \mathbb{R} \to \mathbb{R}$ given by $T(a) = a \cdot a$ is not linear. Why does this not contradict the statement above?