MAT224H1F

Summer 2021

Tutorial Problems

Week 1

1. Consider $\mathbb{R}_{>0} := \{a \in \mathbb{R} : a > 0\}$ equipped with the following operations: given $a, b \in \mathbb{R}_{>0}$, set $a \boxplus b = ab$, where ab is the usual product of a and b as real numbers; given $b \in \mathbb{R}_{>0}$ and $a \in \mathbb{R}$, set $a \boxdot b = b^a$. Show that $\mathbb{R}_{>0}$ with $\boxplus$ as addition and $\boxdot$ as scalar multiplication is a vector space.

2. Let V be a vector space, X a set, and $f : V \longrightarrow X$ a bijection. Use f and the vector space structure of V to make X a vector space. Is the previous question a special case of this construction?

3. The aim of this problem is to prove that the commutativity axiom does not imply the associativity axiom. Consider the set Rock, Paper, Scissors and define the operation that selects the winner out of a play. For example, Rock+Paper=Paper, Paper+Scissors=Scissors, etc. Prove that, on this framework, the axiom of commutativity holds, but the axiom of associativity fails.

4. Let V be a vector space.

(a) Show that $c \cdot 0 = 0$ for any scalar c. (Here 0 is the zero vector.)

(b) Let $v \in V$. Show that if $cv = 0$ and c is not zero, then $v = 0$.

(c) Show that for any $v \in V$, we have $v + v = 2 \cdot v$ (where the right hand side is the scalar multiplication of 2 and v).

5. In each part a vector space V and a subset W of V is given. Determine if W is a subspace of V.

(a) $V = P_2(\mathbb{R})$ (i.e. $\{ax^2 + bx + c : a, b, c \in \mathbb{R}\}$), $W = \emptyset$ (empty set)

(b) $V = P_2(\mathbb{R})$, $W = \{f \in V : f(1) = 1\}$

(c) $V = \mathbb{R}^3$, $W = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in V : x^2 + y^2 - z^2 = 0 \right\}$

(d) $V = \mathbb{R}$, $W = \mathbb{Z}$ (= the set of integers).

6. Let $V = \mathbb{R}^n$, and A, B be $n \times n$ matrices. In each part below show that $Y = W \cap U$.

(a)

$$\begin{aligned}
Z &:= \{\vec{v} \in V : AB\vec{v} = \vec{0}\} \\
Y &:= \{\vec{v} \in V : \vec{v} = B\vec{w} \text{ for some } \vec{w} \in Z\} \\
W &:= \{\vec{v} \in V : A\vec{v} = \vec{0}\} \\
U &:= \{\vec{v} \in V : \vec{v} = B\vec{w} \text{ for some } \vec{w} \in V\}
\end{aligned}$$

(Solution: For any $\vec{v} \in Y$, we have $\vec{v} = B\vec{w}$ for some $\vec{w} \in Z$, therefore, $A\vec{v} = AB\vec{w}$, because $\vec{w} \in Z$. $AB\vec{w} = \vec{0}$, so $A\vec{v} = \vec{0}$ implies $\vec{v} \in W$. On the other hand, $\vec{w} \in Z \subset V$, so $\vec{w} \in V$, therefore $\vec{v} = B\vec{w} \in U$. We proved $Y \subset W \cap U$

To prove $W \cap U \subset Y$, let $\vec{v} \in W \cap U$, therefore, $A\vec{v} = \vec{0}$ and $\vec{v} = B\vec{w}$ for some $\vec{w} \in V$. So $AB\vec{w} = \vec{0}$ for some $\vec{w} \in V$, therefore $\vec{w} \in Z$ Therefore $\vec{v} = B\vec{w} \in Y$. So $W \cap U \subset Y$.
As total, we proved $W = U \cap Y$.)

(b)

$$\begin{aligned} Y &:= \{\vec{v} \in V : (\mu A + \lambda B)\vec{v} = \vec{0} \text{ for all } \lambda, \mu \in F\} \\ W &:= \{\vec{v} \in V : A\vec{v} = \vec{0}\} \\ U &:= \{\vec{v} \in V : B\vec{v} = \vec{0}\} \end{aligned}$$

(Solution: Firstly we prove $Y \subset W \cap U$, for any $\vec{v} \in Y$, then

$$(\mu A + \lambda B)\vec{v} = \vec{0} \tag{1}$$

for any $\lambda, \mu \in F$, let $\mu = 1$, $\lambda = 0$ in (1) we have

$$A\vec{v} = \vec{0}$$

this implies $\vec{v} \in W$. Not let $\mu = 0$, $\lambda = 1$ in (1) we have

$$B\vec{v} = \vec{0}$$

this implies $\vec{v} \in U$. Since $\vec{v} \in W$ and $\vec{v} \in U$, we have

$$\vec{v} \in W \cap U$$

This prove $Y \subset U \cap W$
Secondly, we prove $Y \supset U \cap W$, for any $\vec{v} \in U \cap W$, we both have $\vec{v} \in U$ and $\vec{v} \in W$. Since $\vec{v} \in U$, we have $A\vec{v} = \vec{0}$. Since $\vec{v} \in W$, we have $B\vec{v} = \vec{0}$. Therefore for any $\lambda, \mu \in F$, we have

$$(\lambda A + \mu B)\vec{v} = \lambda A\vec{v} + \mu B\vec{v} = \lambda\vec{0} + \mu\vec{0} = \vec{0}$$

Therefore $\vec{v} \in Y$, this proves $Y \supset U \cap W$.
As whole, we showed $Y = U \cap W$.)

7. (a) Consider the following elements of $\mathbb{R}^3$:

$$v_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \qquad v_2 = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}, \qquad w = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

Is w in the span of $\{v_1, v_2\}$? If yes, write w as a linear combination of $\{v_1, v_2\}$.
(b) Consider the polynomials $f_1, f \ldots, f_4$ defined below:

$$f_1(x) = 1 + x, \quad f_2(x) = x, \quad f_3(x) = 2 + x, \quad f_4(x) = x^3 + 1.$$

Determine if the polynomial g defined by $g(x) = x^3 + 2x + 1$ is in the span of $\{f_1, f_2, f_3, f_4\}$. If yes, find *all* $\underline{a_1, \ldots, a_4}$ $\in \mathbb{R}$ such that $g = \sum_{i=1}^{4} a_i f_i$. (Note: Two polynomials are equal if and only if their corresponding coefficients coincide; that is, $\sum_i b_i x^i$ and $\sum_i c_i x^i$ are equal as functions if and only if $b_i = c_i$ for all i.)

7a) No!

$$\vec{w} = s\vec{v}_1 + t\vec{v}_2$$

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = s\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + t\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} \rightsquigarrow \left(\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 2 & 0 \end{array}\right)$$

Q. Which vector space are we working in?

7b) Yes: $g(x) = f_4(x) + 2\cdot f_2(x) \Rightarrow g \in \text{span}\{f_1, \_\_, f_4\}$

General solution

Method 1 $V_1 = P_3(\mathbb{R}) = \text{span}\{1, x, x^2, x^3\}$

$$(1, x, x^2, x^3) \quad \left[\begin{array}{cccc|c} 1 & 0 & 2 & 1 & 1 \\ 1 & 1 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right] \qquad (1, x, x^2, x^3) \quad \begin{bmatrix} 1 & 0 & 2 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix}$$

Method 2 $V_2 = \text{span}\{x^3, x, 1\}$

$$\left[\begin{array}{cccc|c} 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 2 \\ 1 & 0 & 2 & 1 & 1 \end{array}\right] \qquad \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 2 & 1 \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

Method 3 I know each of the $f_i(x)$ and $g(x)$ are elements of $V_3 = P_5(\mathbb{R})$

$$\left[\begin{array}{cccc|c} 1 & 0 & 2 & 1 & 1 \\ 1 & 1 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$V_2 \subseteq V_1 \subseteq V_3$$

$$\left\{ a_3 \begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ -2 \\ 0 \\ 1 \end{bmatrix} : a_3 \in \mathbb{R} \right\} \neq \left\{ a_3 \begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \\ 0 \\ 1 \end{bmatrix} : a_3 \in \mathbb{R} \right\}$$

One of these is incorrect; error-searching comes next!

8. Find a spanning set for the subspace

$$W := \{f \in P_3(\mathbb{R}) : f(2) = 0\}$$

of $P_3(\mathbb{R})$.