Module 16 Lecture Handout

This module is about diagonalizing matrices and is also the last module of the semester!

1. First, let's define what it means for a matrix to be diagonalizable.

> **Diagonalizable.** A matrix is ***diagonalizable*** if it is similar to a diagonal matrix.

2. (Example) In this problem, you'll demonstrate explicitly what it meant by diagonalizable. The matrix A is similar to the diagonal matrix D. Confirm that the matrix $A = PDP^{-1}$ by multiplying the matrices.

   $A = \begin{bmatrix} 6 & -1 \\ 2 & 3 \end{bmatrix}$

   $D = \begin{bmatrix} 5 & 0 \\ 0 & 4 \end{bmatrix}$

   $P = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$

   $P^{-1} = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$

3. One advantage of a diagonal matrix is taking matrix powers. Find D^{100} and A^{100}, where
   $A = PDP^{-1}$
   $A = \begin{bmatrix} 6 & -1 \\ 2 & 3 \end{bmatrix}$
   $D = \begin{bmatrix} 5 & 0 \\ 0 & 4 \end{bmatrix}$
   $P = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$
   $P^{-1} = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$

4. The question is how to figure out when a linear transformation can be represented by a diagonal matrix. The following theorem answers this question.

> **Theorem.** A linear transformation $\mathcal{T}: \mathbb{R}^n \to \mathbb{R}^n$ can be represented by a diagonal matrix if and only if there exists a basis for $\mathbb{R}^n$ consisting of eigenvectors for $\mathcal{T}$. If $\mathcal{B}$ is such a basis, then $[\mathcal{T}]_{\mathcal{B}}$ is a diagonal matrix.

Hence, diagonalization is a question about finding the eigenvalues and eigenvectors of a linear transformation.

Find a basis, B, of eigenvectors for the matrix $A = \begin{bmatrix} 5 & 1 \\ 0 & 3 \end{bmatrix}$ Then find the matrix, P, such that $A = PDP^{-1}$, where $D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$.

5. Not every matrix is diagonalizable. If a matrix does not have a basis of eigenvectors, then it is not diagonalizable. We will now study some building blocks to help us answer the question of diagonalizability.

Eigenspace. Let A be an $n \times n$ matrix with eigenvalues $\lambda_1, \ldots, \lambda_m$. The ***eigenspace*** of A corresponding to the eigenvalue λ_i is the null space of $A - \lambda_i I$. That is, it is the space spanned by all eigenvectors that have the eigenvalue λ_i.
The ***geometric multiplicity*** of an eigenvalue λ_i is the dimension of the corresponding eigenspace. The ***algebraic multiplicity*** of λ_i is the number of times λ_i occurs as a root of the characteristic polynomial of A (i.e., the number of times $x - \lambda_i$ occurs as a factor).

Show that the matrix below has 7 as an eigenvalue, and find its algebraic and geometric multiplicity. $B = \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix}$

6. Let's revisit a matrix we have worked with already. You can use the eigenvalues and null spaces you found earlier. For each of the eigenvalues, find the geometric and algebraic multiplicity. Explain why the matrix is diagonalizable. $A = \begin{bmatrix} 5 & 1 \\ 0 & 3 \end{bmatrix}$

7. Find the eigenvalue(s) for the matrix and their algebraic and geometric multiplicity.

$$C = \begin{bmatrix} 7 & 2 \\ 0 & 7 \end{bmatrix}$$

8. We can now state the theorem (proof in the textbook) that tells us when a matrix is diagonalizable. Which of the following are diagonalizable? Use the theorem to explain why.

> **Theorem.** An $n \times n$ matrix A is diagonalizable if and only if the sum of its geometric multiplicities is equal to n. Further, provided complex eigenvalues are permitted, A is diagonalizable if and only if its geometric multiplicities are equal to its corresponding algebraic multiplicities.

$A = \begin{bmatrix} 5 & 1 \\ 0 & 3 \end{bmatrix}$

$B = \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix}$

$C = \begin{bmatrix} 7 & 2 \\ 0 & 7 \end{bmatrix}$

9. Sometimes the number system you are using matters. For the matrix $R = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, show that R does not have eigenvalues in $\mathbb{R}$. Explain why R is not diagonalizable as a matrix in $\mathbb{R}^2$. (This matrix would be diagonalizable in the vector space $\mathbb{C}^2$.) Also, explain why this makes sense geometrically.

10. Diagonalizable and invertible are different ideas. A matrix can be one of these, both, or neither. For each matrix, determine if the matrix is (i) diagonalizable, (ii) invertible.

$$M_1 = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

$$M_2 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$M_3 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$$M_4 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Complete the core exercises outside of class for extra practice. Solutions are posted on Quercus.

88 Consider

$$A=\begin{bmatrix}1&0&1\\0&1&1\\1&1&0\end{bmatrix}\qquad \vec v_1=\begin{bmatrix}1\\1\\1\end{bmatrix}\qquad \vec v_2=\begin{bmatrix}1\\1\\-2\end{bmatrix}\qquad \vec v_3=\begin{bmatrix}-1\\1\\0\end{bmatrix}$$

and notice that $\vec v_1, \vec v_2, \vec v_3$ are eigenvectors for A. Let T_A be the transformation induced by A.

88.1 Find the eigenvalues of T_A.

88.2 Find the characteristic polynomial of T_A.

88.3 Compute $T_A\vec w$ where $\vec w = 2\vec v_1 - \vec v_2$.

88.4 Compute $T_A\vec u$ where $\vec u = a\vec v_1 + b\vec v_2 + c\vec v_3$ for unknown scalar coefficients a, b, c.

Notice that $\mathcal{V} = \{\vec v_1, \vec v_2, \vec v_3\}$ is a basis for $\mathbb{R}^3$.

88.5 If $[\vec x]_{\mathcal{V}} = \begin{bmatrix}1\\3\\4\end{bmatrix}$ is $\vec x$ written in the $\mathcal{V}$ basis, compute $T_A\vec x$ in the $\mathcal{V}$ basis.

89 Recall from Problem 88 that

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} \qquad \vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \qquad \vec{v}_2 = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} \qquad \vec{v}_3 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

and $\mathcal{V} = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$. Let T_A be the transformation induced by A and let $P = [\vec{v}_1|\vec{v}_2|\vec{v}_3]$ be the matrix with columns $\vec{v}_1$, $\vec{v}_2$, and $\vec{v}_3$ (written in the standard basis).

89.1 Describe in words what P and P^{-1} do in terms of change-of-basis.

89.2 If you were asked to compute $T_A\vec{y}$ for some $\vec{y} \in \mathbb{R}^3$, which basis would you prefer to do your computations in? Explain.

89.3 Given a vector $\vec{y} \in \mathbb{R}^3$ written in the standard basis, is there a way to compute $T_A\vec{y}$ without using the matrix A? (You may use P and P^{-1}, just not A.) Explain.

89.4 Can you find a matrix D so that

$$PDP^{-1} = A?$$

89.5 $[\vec{x}]_{\mathcal{V}} = \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}$. Compute $T_A^{100}\vec{x}$. Express your answer in both the $\mathcal{V}$ basis and the standard basis.

Diagonalizable

DEF A matrix is ***diagonalizable*** if it is similar to a diagonal matrix.

90 Let B be an $n \times n$ matrix and let T_B be the induced transformation. Suppose T_B has eigenvectors $\vec{v}_1, \ldots, \vec{v}_n$ which form a basis for $\mathbb{R}^n$, and let $\lambda_1, \ldots, \lambda_n$ be the corresponding eigenvalues.

90.1 How do the eigenvalues and eigenvectors of B and T_B relate?

90.2 Is B diagonalizable (i.e., similar to a diagonal matrix)? If so, explain how to obtain its diagonalized form.

90.3 What if one of the eigenvalues of T_B is zero? Would B be diagonalizable?

90.4 What if the eigenvectors of T_B did not form a basis for $\mathbb{R}^n$. Would B be diagonalizable?

Eigenspace

DEFINITION

Let A be an $n \times n$ matrix with eigenvalues $\lambda_1, \dots, \lambda_m$. The ***eigenspace*** of A corresponding to the eigenvalue λ_i is the null space of $A - \lambda_i I$. That is, it is the space spanned by all eigenvectors that have the eigenvalue λ_i.

The ***geometric multiplicity*** of an eigenvalue λ_i is the dimension of the corresponding eigenspace. The ***algebraic multiplicity*** of λ_i is the number of times λ_i occurs as a root of the characteristic polynomial of A (i.e., the number of times $x - \lambda_i$ occurs as a factor).

---

91 Let $F = \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}$ and $G = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$.

91.1 Is F diagonalizable? Why or why not?

91.2 Is G diagonalizable? Why or why not?

91.3 What are the geometric and algebraic multiplicities of each eigenvalue of F? What about the multiplicities for each eigenvalue of G?

91.4 Suppose A is a matrix where the geometric multiplicity of one of its eigenvalues is smaller than the algebraic multiplicity of the same eigenvalue. Is A diagonalizable? What if all the geometric and algebraic multiplicities match?