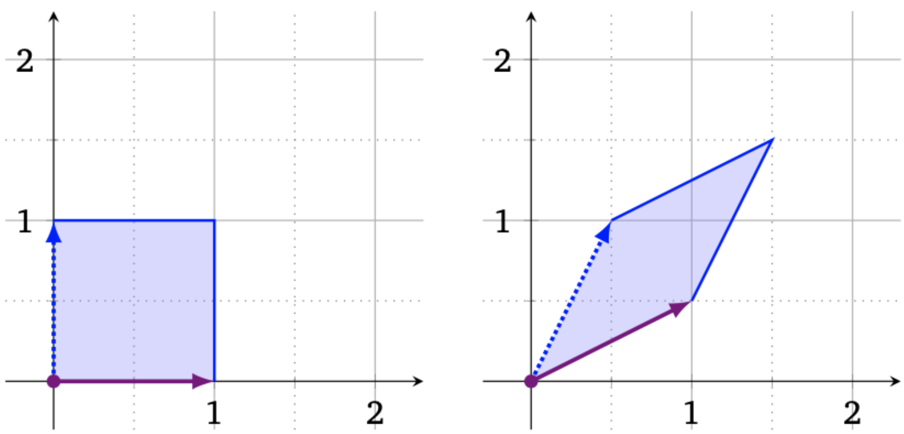
Module 15 Lecture Handout

This module is about eigenvalues and eigenvectors. These are highly important ideas that help us understand more deeply characteristics of linear transformations. Eigenvectors of a linear transformation are vectors that the linear transformation outputs a scalar multiple of the vector. The direction of the vector does not change. These vectors point in what are sometimes considered "natural" directions of the linear transformation. The associated eigenvalues are the scalar multiples and tells you the magnitude that the linear transformation stretches or shrinks the vector.

1. Read the definition of eigenvector and eigenvalue.

> **Eigenvector.** Let X be a linear transformation or a matrix. An ***eigenvector*** for X is a non-zero vector that doesn't change directions when X is applied. That is, $\vec{v} \neq \vec{0}$ is an eigenvector for X if
>
> $$X\vec{v} = \lambda\vec{v}$$
>
> for some scalar λ. We call λ the ***eigenvalue*** of X corresponding to the eigenvector $\vec{v}$.

2. (Geometry) The diagram below shows what a linear transformation, T, does to the unit square (C_2). Find a vector that is an eigenvector for T and it's associated eigenvalue. Additionally, explain why e_1 and e_2 are not eigenvectors.



3. Let T be a linear transformation associated with the matrix, $M = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$. Find the eigenvalues and an eigenvector associated to each eigenvalue by inspection.

4. (Review) Let $A = \begin{bmatrix} 1 & 5 \\ 0 & 0 \end{bmatrix}$ Find the null space of A.

5. Please read the following.

   We need a more systematic way to find eigenvectors and eigenvalues, other than observing them in simple cases like the ones above.

$$
\begin{aligned}
A\vec{v} &= \lambda\vec{v} \\
A\vec{v} - \lambda\vec{v} &= \vec{0} \\
(A - \lambda I)\vec{v} &= \vec{0}
\end{aligned}
$$

   We need to find nonzero vectors $\vec{v}$ that are in the null space of $(A - \lambda I)$. In order for the null space of $(A - \lambda I)$ to be nontrivial, it must be a non-invertible matrix. If $(A - \lambda I)$ is not invertible, then the determinant of $(A - \lambda I)$ is equal to zero.

   This brings us to the characteristic polynomial.

> **Characteristic Polynomial.**
> For a matrix A, the ***characteristic polynomial*** of A is
>
> $$\operatorname{char}(A) = \det(A - \lambda I).$$

   The way to find the eigenvalues of A, we need find the scalars λ such that $\det(A - \lambda I) = 0$. This is essentially finding the roots of a polynomial.

6. Find the eigenvalues of $M = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$ using the characteristic polynomial.

7. Find the eigenvalues of $C = \begin{bmatrix} -1 & 2 \\ 1 & 0 \end{bmatrix}$ using the characteristic polynomial.

8. Find the eigenvectors associated to the eigenvalues in the matrix C from the previous problem.

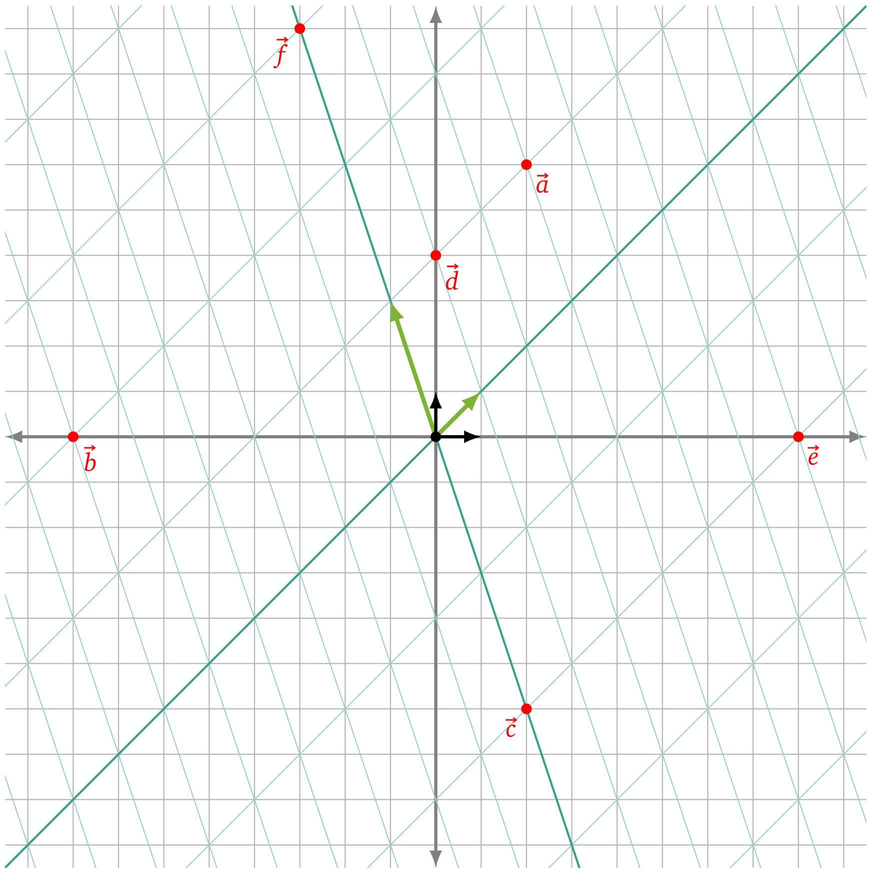
9. Find the eigenvalues and an associated eigenvector for each eigenvalue for $D = \begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix}$

10. Suppose a matrix E has one eigenvalue equal to zero. Is the matrix invertible? Explain.

Practice working on the core exercises from the textbook. Solutions are posted on Quercus.

82 The subway system of Oronto is laid out in a skewed grid. All tracks run parallel to one of the green lines shown. Compass directions are given by the black lines.

While studying the subway map, you decide to pick two bases to help: the green basis $\mathcal{G} = \{\vec{g}_1, \vec{g}_2\}$, and the black basis $\mathcal{B} = \{\vec{e}_1, \vec{e}_2\}$.



1. Write each point above in both the green and the black bases.

2. Find a change-of-basis matrix X that converts vectors from a green basis representation to a black basis representation. Find another matrix Y that converts vectors from a black basis representation to a green basis representation.

3. The city commission is considering renumbering all the stops along the $y = -3x$ direction. You deduce that the commission's proposal can be modeled by a linear transformation.

   Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation that stretches in the $y = -3x$ direction by a factor of 2 and leaves vectors in the $y = x$ direction fixed.

   Describe what happens to the vectors $\vec{u}$, $\vec{v}$, and $\vec{w}$ when T is applied given that

$$[\vec{u}]_{\mathcal{G}} = \begin{bmatrix} 6 \\ 1 \end{bmatrix} \qquad [\vec{v}]_{\mathcal{G}} = \begin{bmatrix} 4 \\ -3 \end{bmatrix} \qquad [\vec{w}]_{\mathcal{B}} = \begin{bmatrix} -8 \\ -7 \end{bmatrix}.$$

4. When working with the transformation T, which basis do you prefer vectors be represented in? What coordinate system would you propose the city commission use to describe their plans?

Eigenvectors

DEFINITION

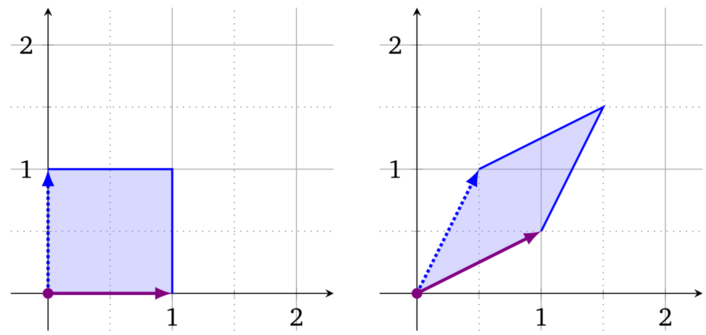
Eigenvector

Let X be a linear transformation or a matrix. An ***eigenvector*** for X is a non-zero vector that doesn't change directions when X is applied. That is, $\vec{v} \neq \vec{0}$ is an eigenvector for X if

$$X\vec{v} = \lambda\vec{v}$$

for some scalar λ. We call λ the ***eigenvalue*** of X corresponding to the eigenvector $\vec{v}$.

83 The picture shows what the linear transformation T does to the unit square (i.e., the unit 2-cube).



83.1 Give an eigenvector for T. What is the eigenvalue?

83.2 Can you give another?

84 For some matrix A,

$$A\begin{bmatrix}3\\3\\1\end{bmatrix}=\begin{bmatrix}2\\2\\2/3\end{bmatrix} \qquad \text{and} \qquad B=A-\tfrac{2}{3}I.$$

84.1 Give an eigenvector and a corresponding eigenvalue for A.

84.2 What is $B\begin{bmatrix}3\\3\\1\end{bmatrix}$?

84.3 What is the dimension of $\mathrm{null}(B)$?

84.4 What is $\det(B)$?

85 Let $C = \begin{bmatrix} -1 & 2 \\ 1 & 0 \end{bmatrix}$ and $E_\lambda = C - \lambda I$.

85.1 For what values of λ does E_λ have a non-trivial null space?

85.2 What are the eigenvalues of C?

85.3 Find the eigenvectors of C.

DEFINITION

Characteristic Polynomial

For a matrix A, the ***characteristic polynomial*** of A is

$$\mathrm{char}(A) = \det(A - \lambda I).$$

86 Let $D = \begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix}$.

86.1 Compute $\mathrm{char}(D)$.

86.2 Find the eigenvalues of D.

87 Suppose $\mathrm{char}(E) = -\lambda(2-\lambda)(-3-\lambda)$ for some unknown 3×3 matrix E.

87.1 What are the eigenvalues of E?

87.2 Is E invertible?

87.3 What can you say about $\mathrm{nullity}(E)$, $\mathrm{nullity}(E-3I)$, $\mathrm{nullity}(E+3I)$?