## Module 14, Appendix 4 Lecture Handout

Module 14 is about determinants of linear transformations and matrices.

1. Let $S$ be the set of all points in $\mathbb{R}$ between 0 and 1 inclusive. Express the set $S$ in set builder notation.

2. This time let $S$ be the set of all points in the unit square in $\mathbb{R}^2$, where both the $x$ and $y$ coordinates are between 0 and 1 inclusive. Express $S$ in set builder notation.

3. Read the definition of the unit cube in dimension n. Note: $C_2$ is the unit square in $\mathbb{R}^2$, and $C_3$ is the unit cube in $\mathbb{R}^3$.

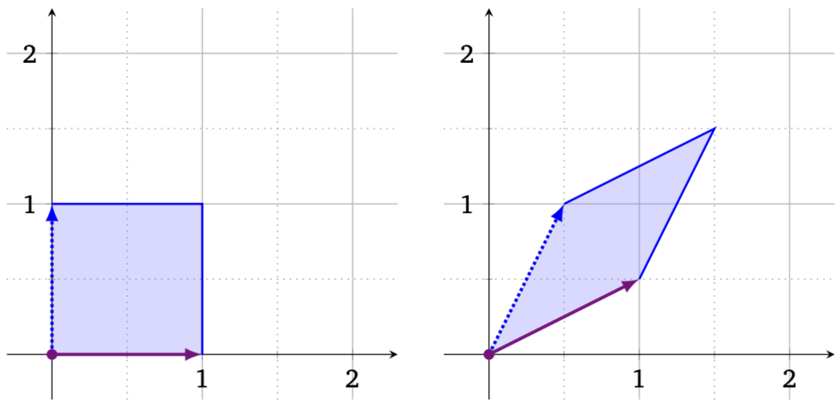
> **Unit $n$-cube.** The ***unit $n$-cube*** is the $n$-dimensional cube with sides given by the standard basis vectors and lower-left corner located at the origin. That is
>
> $$C_n = \left\{ \vec{x} \in \mathbb{R}^n : \vec{x} = \sum_{i=1}^{n} \alpha_i \vec{e}_i \text{ for some } \alpha_1, \ldots, \alpha_n \in [0,1] \right\} = [0,1]^n.$$

4. Read the definition of the determinant.

> **Determinant.** The ***determinant*** of a linear transformation $\mathcal{T} : \mathbb{R}^n \to \mathbb{R}^n$, denoted $\det(\mathcal{T})$ or $|\mathcal{T}|$, is the oriented volume of the image of the unit $n$-cube. The determinant of a square matrix is the determinant of its induced transformation.

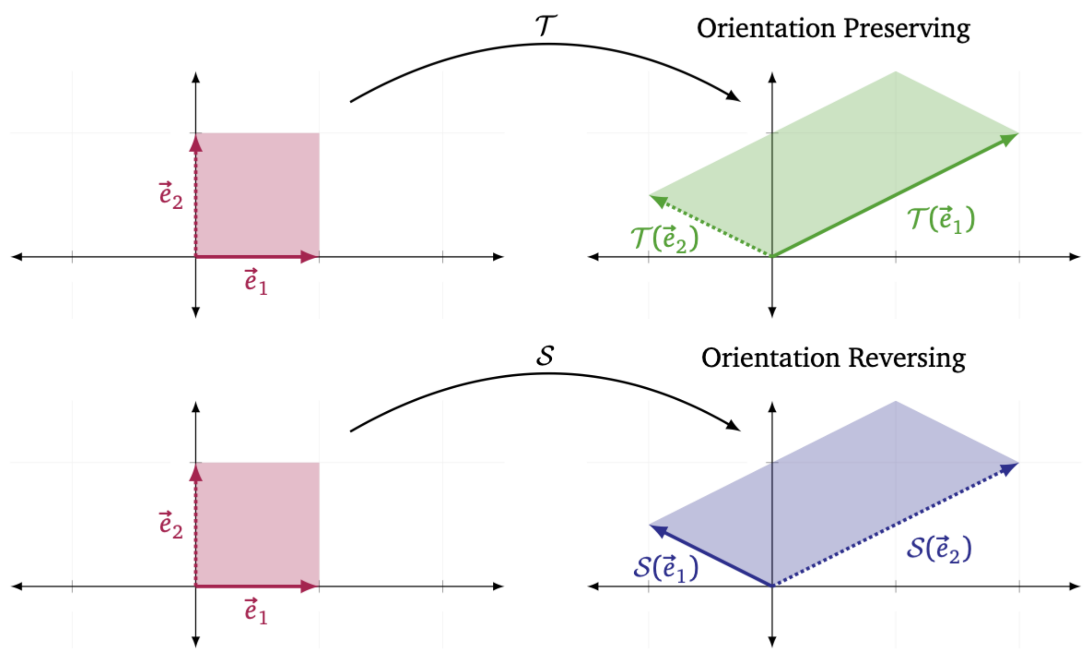
5. Find the numerical value of the determinant of the linear transformation $T$ (Core Exercise 72)

The picture shows what the linear transformation $T$ does to the unit square (i.e., the unit 2-cube).



6. Determinants can be positive, negative, or zero.

> **Orientation Preserving Linear Transformation.** Let $\mathcal{T} : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation. We say $\mathcal{T}$ is ***orientation preserving*** if the ordered basis $\{\mathcal{T}(\vec{e}_1), \dots, \mathcal{T}(\vec{e}_n)\}$ is positively oriented and we say $\mathcal{T}$ is ***orientation reversing*** if the ordered basis $\{\mathcal{T}(\vec{e}_1), \dots, \mathcal{T}(\vec{e}_n)\}$ is negatively oriented. If $\{\mathcal{T}(\vec{e}_1), \dots, \mathcal{T}(\vec{e}_n)\}$ is not a basis for $\mathbb{R}^n$, then $\mathcal{T}$ is neither orientation preserving nor orientation reversing.




Show that the linear transformation $S : \mathbb{R}^2 \to \mathbb{R}^2$ has a negative determinant. Assume $S\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = \begin{bmatrix}-1\\0\end{bmatrix}$ and $S\left(\begin{bmatrix}0\\1\end{bmatrix}\right) = \begin{bmatrix}0\\2\end{bmatrix}$

7. Let $P : \mathbb{R}^2 \to \mathbb{R}^2$ be the projection onto the line $y = 2x$. Show that the determinant of $P$ is zero.

8. Let $R : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation that rotates vectors $\theta$ degrees counterclockwise. Find $\det(R)$.

9. The determinant has special properties for composition of linear transformations/matrix multiplication.

> **Theorem.** Let $\mathcal{T}:\mathbb{R}^n\to\mathbb{R}^n$ and $\mathcal{S}:\mathbb{R}^n\to\mathbb{R}^n$ be linear transformations. Then
> $$\det(\mathcal{S}\circ\mathcal{T})=\det(\mathcal{S})\det(\mathcal{T}).$$

> **Theorem.** Let $A$ and $B$ be $n\times n$ matrices. Then
> $$\det(AB)=\det(A)\det(B).$$

If $A=E_k\cdots E_2E_1$, then $\det(A)=$ ____________

10. Elementary matrices are useful when studying determinants. For we need a list of facts.

- Let $E_1$ be an elementary matrix associated with multiplying a row by the constant $\alpha$. Then $\det(E_1)=\alpha$.
- Let $E_2$ be an elementary matrix associated with switching two rows. Then $\det(E_2)=-1$.
- Let $E_3$ be an elementary matrix associated with adding a multiple of a row to another row. Then $\det(E_3)=1$

Compute the determinants using elementary matrices (not formulas from Appendix 4).

(a) $\det\left(\begin{bmatrix}3&0\\0&1\end{bmatrix}\right)$

(b) $\det\left(\begin{bmatrix}1&-2\\0&1\end{bmatrix}\begin{bmatrix}5&0\\0&1\end{bmatrix}\right)$

(c) $\det\left(\begin{bmatrix}0&1&0\\1&0&0\\0&0&1\end{bmatrix}\right)$

(d) $\det\left(\begin{bmatrix}1&0&-13\\0&1&0\\0&0&1\end{bmatrix}\right)$

(e) $\det\left(\begin{bmatrix}3&0&0\\0&5&0\\0&0&1\end{bmatrix}\right)$

11. In appendix 4, a formula is presented for computing the determinant of a $2 \times 2$ matrix.

> **Theorem.** Let $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Then,
> $$\det(M) = ad - bc.$$

Compute the determinants using the formula.

$\det\left(\begin{bmatrix} 2 & 2 \\ 3 & 1 \end{bmatrix}\right)$

$\det\left(\begin{bmatrix} 1 & 5 \\ 0 & 1 \end{bmatrix}\right)$

$\det\left(\begin{bmatrix} 5 & 0 \\ -2 & 3 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix}\right)$

12. The determinant can tell you if a matrix is invertible or not.

**Theorem.** Let $A$ be an $n \times n$ matrix. $A$ is invertible if and only if $\det(A) \neq 0$.

Further, we know that
$$\det(A^{-1}) = \frac{1}{\det(A)}$$
Give an example of (a) an invertible $2 \times 2$ matrix, and (b) a noninvertible $2 \times 2$ matrix.

13. The determinant can be used to learn the orientation of a basis. If $B = \{\vec{b}_1, \vec{b}_2, \ldots, \vec{b}_n\}$ is an ordered basis, then the sign of the determinant of the matrix $[\vec{b}_1|\vec{b}_1|\cdots\vec{b}_n]$ is orientation of the basis. Use the determinant to find the orientation of the ordered bases.

   (a) $\left\{\begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}-1\\1\end{bmatrix}\right\}$

   (b) $\left\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}1\\-1\end{bmatrix}\right\}$

   (c) $\left\{\begin{bmatrix}3\\3\end{bmatrix}, \begin{bmatrix}3\\0\end{bmatrix}\right\}$

14. Review the Rule of Sarrus, which applies specifically to $3 \times 3$ matrices only.

**Rule of Sarrus**

Let $M = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$. To compute the determinant of $M$ using the Rule of Sarrus, apply the following four steps.

**Step 1.** Augment $M$ with copies of its *first two columns*.

$$\left[\begin{array}{ccc|cc} a & b & c & a & b \\ d & e & f & d & e \\ g & h & i & g & h \end{array}\right]$$

**Step 2.** Multiply together and then *add* the entries along the three diagonals of the new matrix. These are called the *diagonal products*.

$$\left[\begin{array}{ccc|cc} a & b & c & a & b \\ d & e & f & d & e \\ g & h & i & g & h \end{array}\right]$$

$$\text{sum of diagonal products} = aei + bfg + cdh.$$

**Step 3.** Multiply together and then *subtract* the entries along the three anti-diagonals. These are called the *anti-diagonal products*.

$$\left[\begin{array}{ccc|cc} a & b & c & a & b \\ d & e & f & d & e \\ g & h & i & g & h \end{array}\right]$$

$$\text{difference of anti-diagonal products} = -gec - hfa - idb$$

**Step 4.** Add the diagonal products and subtract the anti-diagonal products to get the determinant.

$$\det(M) = aei + bfg + cdh - gec - hfa - idb.$$

Complete the following.

(a) Show $\det\left(\begin{bmatrix} 1 & 2 & 3 \\ -4 & 0 & 1 \\ 6 & -1 & 1 \end{bmatrix}\right) = 33$

(b) Show $\det\left(\begin{bmatrix} 1 & 0 & -1 \\ 2 & 4 & 3 \\ -1 & 3 & 2 \end{bmatrix}\right) = -11$

15. Suppose an $3 \times 3$ matrix is upper triangular.

$$M = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix}$$

Show in two different ways that $\det(M) = a_{11}a_{22}a_{33}$. One way you can use the Rule of Sarrus. The other way use elementary matrices and facts about the determinant. Note: the Rule of Sarrus does not apply to higher dimensions, however the method using elementary matrices does.

Complete the core exercises from module 14 outside of class. Try them before you check your answers posted on Quercus.

**Unit $n$-cube**

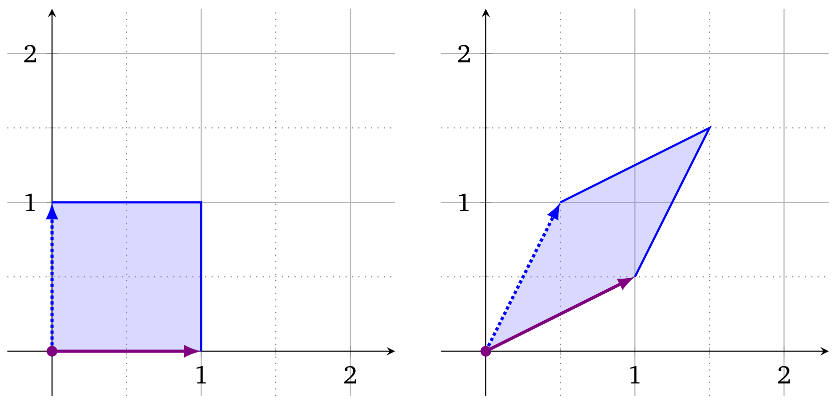
DEFINITION

The ***unit $n$-cube*** is the $n$-dimensional cube with sides given by the standard basis vectors and lower-left corner located at the origin. That is

$$C_n = \left\{ \vec{x} \in \mathbb{R}^n : \vec{x} = \sum_{i=1}^{n} \alpha_i \vec{e}_i \text{ for some } \alpha_1, \dots, \alpha_n \in [0,1] \right\} = [0,1]^n.$$

The sides of the unit $n$-cube are always length 1 and its volume is always 1.

72 The picture shows what the linear transformation $T$ does to the unit square (i.e., the unit 2-cube).




72.1 What is $T\begin{bmatrix}1\\0\end{bmatrix}$, $T\begin{bmatrix}0\\1\end{bmatrix}$, $T\begin{bmatrix}1\\1\end{bmatrix}$?

72.2 Write down a matrix for $T$.

72.3 What is the volume of the image of the unit square (i.e., the volume of $T(C_2)$)? You may use trigonometry.

**Determinant**

DEF The ***determinant*** of a linear transformation $\mathcal{T}:\mathbb{R}^n\to\mathbb{R}^n$, denoted $\det(\mathcal{T})$ or $|\mathcal{T}|$, is the oriented volume of the image of the unit $n$-cube. The determinant of a square matrix is the determinant of its induced transformation.

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73 We know the following about the linear transformation $A$:

$$A\begin{bmatrix}1\\0\end{bmatrix}=\begin{bmatrix}2\\0\end{bmatrix}\qquad\text{and}\qquad A\begin{bmatrix}0\\1\end{bmatrix}=\begin{bmatrix}1\\1\end{bmatrix}.$$

73.1 Draw $C_2$ and $A(C_2)$, the image of the unit square under $A$.

73.2 Compute the area of $A(C_2)$.

73.3 Compute $\det(A)$.

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74 Suppose $R$ is a rotation counter-clockwise by $30°$.

74.1 Draw $C_2$ and $R(C_2)$.

74.2 Compute the area of $R(C_2)$.

74.3 Compute $\det(R)$.

75 We know the following about the linear transformation $F$:

$$F\begin{bmatrix}1\\0\end{bmatrix}=\begin{bmatrix}0\\1\end{bmatrix} \qquad \text{and} \qquad F\begin{bmatrix}0\\1\end{bmatrix}=\begin{bmatrix}1\\0\end{bmatrix}.$$

75.1 What is $\det(F)$?

THM **Volume Theorem I**

For a square matrix $M$, $\det(M)$ is the oriented volume of the parallelepiped ($n$-dimensional parallelogram) given by the column vectors of $M$.

THM **Volume Theorem II**

For a square matrix $M$, $\det(M)$ is the oriented volume of the parallelepiped ($n$-dimensional parallelogram) given by the row vectors of $M$.

76

76.1 Explain Volume Theorem I using the definition of determinant.

76.2 Based on Volume Theorems I and II, how should $\det(M)$ and $\det(M^T)$ relate for a square matrix $M$?

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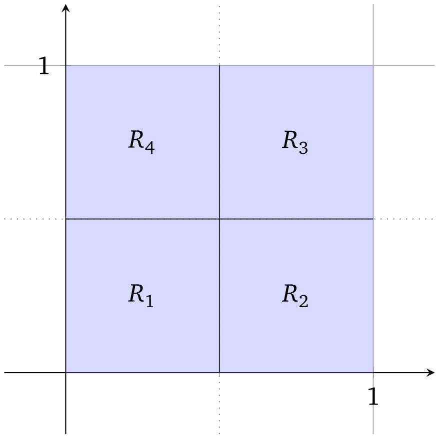


Let $R = R_1 \cup R_2 \cup R_3 \cup R_4$. You know the following about the linear transformations $M$, $T$, and $S$.

$$M \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2x \\ y \end{bmatrix}$$

$T : \mathbb{R}^2 \to \mathbb{R}^2$ has determinant 2

$S : \mathbb{R}^2 \to \mathbb{R}^2$ has determinant 3

77.1 Find the volumes (areas) of $R_1$, $R_2$, $R_3$, $R_4$, and $R$.

77.2 Compute the oriented volume of $M(R_1)$, $M(R_2)$, and $M(R)$.

77.3 Do you have enough information to compute the oriented volume of $T(R_2)$? What about the oriented volume of $T(R + \{\vec{e}_2\})$?

77.4 What is the oriented volume of $S \circ T(R)$? What is $\det(S \circ T)$?

- $E_f$ is $I_{3\times 3}$ with the first two rows swapped.
- $E_m$ is $I_{3\times 3}$ with the third row multiplied by 6.
- $E_a$ is $I_{3\times 3}$ with $R_1 \mapsto R_1 + 2R_2$ applied.

78.1 What is $\det(E_f)$?
78.2 What is $\det(E_m)$?
78.3 What is $\det(E_a)$?
78.4 What is $\det(E_f E_m)$?
78.5 What is $\det(4I_{3\times 3})$?
78.6 What is $\det(W)$ where $W = E_f E_a E_f E_m E_m$?

 

79
$$U = \begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & 3 & -2 & 4 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}$$

79.1 What is $\det(U)$?

79.2 $V$ is a square matrix and $\mathrm{rref}(V)$ has a row of zeros. What is $\det(V)$?

80

80.1 $V$ is a square matrix whose columns are linearly dependent. What is $\det(V)$?

80.2 $P$ is projection onto $\operatorname{span}\left\{\begin{bmatrix}-1\\-1\end{bmatrix}\right\}$. What is $\det(P)$?

81 Suppose you know $\det(X) = 4$.

81.1 What is $\det(X^{-1})$?

81.2 Derive a relationship between $\det(Y)$ and $\det(Y^{-1})$ for an arbitrary matrix $Y$.

81.3 Suppose $Y$ is not invertible. What is $\det(Y)$?