Module 13 Lecture Handout

The module is about changing bases, connecting the idea of change of basis to matrix multiplication, and representing linear transformations in different bases. One overarching idea in math is mathematical equivalence, where the same object has many different looking, yet equivalent representations.

1. (Review) In module 8 we studied change of basis. Assume E is the standard basis, and let C be the ordered basis

$$\{\vec{c}_1, \vec{c}_2\} = \left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix}_E, \begin{bmatrix} -2 \\ 1 \end{bmatrix}_E \right\}$$

Find the following.

(a) $[\vec{c}_1]_C =$

(b) $[\vec{c}_2]_C =$

(c) $[\vec{c}_1]_E =$

(d) $[\vec{c}_2]_E =$

(e) The change of basis matrix $[E \leftarrow C]$.

2. Compute the matrix inverse of $[E \leftarrow C]$ in the previous problem, and explain what the inverse does in terms of change of basis.

3. Read the following.

> **Change of Basis Matrix.** Let $\mathcal{A}$ and $\mathcal{B}$ be bases for $\mathbb{R}^n$. The matrix M is called a ***change of basis*** matrix (which converts from $\mathcal{A}$ to $\mathcal{B}$) if for all $\vec{x} \in \mathbb{R}^n$
>
> $$M[\vec{x}]_{\mathcal{A}} = [\vec{x}]_{\mathcal{B}}.$$
>
> Notationally, $[\mathcal{B} \leftarrow \mathcal{A}]$ stands for the change of basis matrix converting from $\mathcal{A}$ to $\mathcal{B}$, and we may write $M = [\mathcal{B} \leftarrow \mathcal{A}]$.

Notice what is different in module 13 compared to module 8. In module 13 we are changing between two arbitrary bases $\mathcal{A}$ and $\mathcal{B}$. In Module 8, one of the bases was the standard basis.

4. In this task we revisit one of the examples in the textbook with different perspective. Please briefly review the example below taken from the textbook.

> **Example.** Let $\mathcal{A} = \{\vec{a}_1, \vec{a}_2\}$ where $\vec{a}_1 = \begin{bmatrix}1\\1\end{bmatrix}_{\mathcal{E}}$ and $\vec{a}_2 = \begin{bmatrix}1\\-1\end{bmatrix}_{\mathcal{E}}$ and let $\mathcal{B} = \{\vec{b}_1, \vec{b}_2\}$ where $\vec{b}_1 = \begin{bmatrix}2\\1\end{bmatrix}_{\mathcal{E}}$ and

> $\vec{b}_2 = \begin{bmatrix}5\\3\end{bmatrix}_{\mathcal{E}}$ be bases for $\mathbb{R}^2$. Find the change of basis matrix $[\mathcal{B} \leftarrow \mathcal{A}]$.
>
> We know $[\mathcal{B} \leftarrow \mathcal{A}]$ will be a 2×2 matrix and that
>
> $$[\mathcal{B} \leftarrow \mathcal{A}][\vec{a}_1]_{\mathcal{A}} = [\vec{a}_1]_{\mathcal{B}} \qquad \text{and} \qquad [\mathcal{B} \leftarrow \mathcal{A}][\vec{a}_2]_{\mathcal{A}} = [\vec{a}_2]_{\mathcal{B}}.$$
>
> Therefore, we need to compute $[\vec{a}_1]_{\mathcal{B}}$ and $[\vec{a}_2]_{\mathcal{B}}$. Repeating the procedure from the previous example, we find
>
> $$[\vec{a}_1]_{\mathcal{B}} = \begin{bmatrix}-2\\1\end{bmatrix} \qquad \text{and} \qquad [\vec{a}_2]_{\mathcal{B}} = \begin{bmatrix}8\\-3\end{bmatrix},$$
>
> and so
>
> $$[\mathcal{B} \leftarrow \mathcal{A}] = \begin{bmatrix}-2 & 8\\1 & -3\end{bmatrix}.$$

Task: find $[\mathcal{B} \leftarrow \mathcal{A}]$ by finding two "building blocks" and using the inverse. That is, find a way to compute $[\mathcal{B} \leftarrow \mathcal{A}]$ by finding $[\mathcal{E} \leftarrow \mathcal{A}]$ and $[\mathcal{E} \leftarrow \mathcal{B}]$ and using the inverse appropriately.

5. Suppose $T:\mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation. Let $\mathcal{C}$ be an ordered basis, and let $\mathcal{E}$ be the standard ordered basis. Further, assume $T(\vec{c}_1) = 3\vec{c}_1$ and $T(\vec{c}_2) = 2\vec{c}_2$. Find the matrix representation of T with respect to $\mathcal{C}$, which we denote with the symbol $[T]_C$.

6. Suppose T is the linear transformation in the previous problem, and further assume that $\mathcal{C}$ is an ordered basis $\{\vec{c}_1, \vec{c}_2\}$, where $\vec{c}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}_{\mathcal{E}}$, $\vec{c}_2 = \begin{bmatrix} -1 \\ 3 \end{bmatrix}_{\mathcal{E}}$. Let $\vec{a} = \begin{bmatrix} 4 \\ -6 \end{bmatrix}_{\mathcal{C}}$. Express $T(\vec{a})$ in both $\mathcal{C}$ and $\mathcal{E}$ bases.

7. Restating the assumptions from the previous problem for convenience... Let $\mathcal{E}$ be the standard ordered basis. Let $\mathcal{C}$ be an ordered basis $\{\vec{c}_1, \vec{c}_2\}$, where $\vec{c}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}_{\mathcal{E}}$, $\vec{c}_2 = \begin{bmatrix} -1 \\ 3 \end{bmatrix}_{\mathcal{E}}$. Assume T is as above, where $T(\vec{c}_1) = 3\vec{c}_1$ and $T(\vec{c}_2) = 2\vec{c}_2$. Find the matrix representation of T in the $\mathcal{E}$ basis, which is denoted by the symbol $[T]_{\mathcal{E}}$.

The following are notes in response to some of the questions from students.

A matrix A is similar to B if

$$A = XBX^{-1}$$

For some matrix X. One result in module 13 is that for a linear transformation, T, every basis is associated to a unique matrix representation of T. In effect there exist infinitely many different matrices associated to T, one for each basis. How are these matrices related to each other? They are all similar to each other via the change of basis matrix (and the inverse of the change of basis matrix).

One student asked if we can still use the symbol, M, to represent matrices. The answer is yes! The symbols such as $[A \leftarrow B]$ are useful ways to help us keep track of what basis we changing from and to. But we can use the same notation for matrices as we have been using.

(Practice at home) Work on the core exercise from the textbook. Solutions are posted on Quercus.

69 Let $\mathcal{B}=\{\vec{b}_1,\vec{b}_2\}$ where $\vec{b}_1=\begin{bmatrix}1\\1\end{bmatrix}_{\mathcal{E}}$, $\vec{b}_2=\begin{bmatrix}1\\-1\end{bmatrix}_{\mathcal{E}}$ and let $X=[\vec{b}_1|\vec{b}_2]$ be the matrix whose columns are $[\vec{b}_1]_{\mathcal{E}}$ and $[\vec{b}_2]_{\mathcal{E}}$.

69.1 Write down X.

69.2 Compute $[\vec{e}_1]_{\mathcal{B}}$ and $[\vec{e}_2]_{\mathcal{B}}$.

69.3 Compute $X[\vec{e}_1]_{\mathcal{B}}$ and $X[\vec{e}_2]_{\mathcal{B}}$. What do you notice?

69.4 Find the matrix X^{-1}. How does X^{-1} relate to change of basis?

70 Let $\mathcal{E}=\{\vec{e}_1,\vec{e}_2,\dots,\vec{e}_n\}$ be the standard basis for $\mathbb{R}^n$ and let $\mathcal{B}=\{\vec{b}_1,\vec{b}_2,\dots,\vec{b}_n\}$ be another basis for $\mathbb{R}^n$. Define the matrix $X=[\vec{b}_1|\vec{b}_2|\cdots|\vec{b}_n]$ to be the matrix whose columns are the $\vec{b}_i$ vectors written in the standard basis. Notice that X converts vectors from the $\mathcal{B}$ basis into the standard basis. In other words,

$$X[\vec{v}]_{\mathcal{B}}=[\vec{v}]_{\mathcal{E}}.$$

70.1 Should X^{-1} exist? Explain.

70.2 Consider the equation

$$X^{-1}[\vec{v}]_{?}=[\vec{v}]_{?}.$$

Can you fill in the "?" symbols so that the equation makes sense?

70.3 What is $[\vec{b}_1]_{\mathcal{B}}$? How about $[\vec{b}_2]_{\mathcal{B}}$? Can you generalize to $[\vec{b}_i]_{\mathcal{B}}$?

71 Let $\vec{c}_1 = \begin{bmatrix}2\\1\end{bmatrix}_{\mathcal{E}}$, $\vec{c}_2 = \begin{bmatrix}5\\3\end{bmatrix}_{\mathcal{E}}$, $\mathcal{C} = \{\vec{c}_1, \vec{c}_2\}$, and $A = \begin{bmatrix}2 & 5\\1 & 3\end{bmatrix}$. Note that $A^{-1} = \begin{bmatrix}3 & -5\\-1 & 2\end{bmatrix}$ and that A changes vectors from the $\mathcal{C}$ basis to the standard basis and A^{-1} changes vectors from the standard basis to the $\mathcal{C}$ basis.

71.1 Compute $[\vec{c}_1]_{\mathcal{C}}$ and $[\vec{c}_2]_{\mathcal{C}}$.

Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation that stretches in the $\vec{c}_1$ direction by a factor of 2 and doesn't stretch in the $\vec{c}_2$ direction at all.

71.2 Compute $T\begin{bmatrix}2\\1\end{bmatrix}_{\mathcal{E}}$ and $T\begin{bmatrix}5\\3\end{bmatrix}_{\mathcal{E}}$.

71.3 Compute $[T\vec{c}_1]_{\mathcal{C}}$ and $[T\vec{c}_2]_{\mathcal{C}}$.

71.4 Compute the result of $T\begin{bmatrix}\alpha\\\beta\end{bmatrix}_{\mathcal{C}}$ and express the result in the $\mathcal{C}$ basis (i.e., as a vector of the form $\begin{bmatrix}?\\?\end{bmatrix}_{\mathcal{C}}$).

71.5 Find $[T]_{\mathcal{C}}$, the matrix for T in the $\mathcal{C}$ basis.

71.6 Find $[T]_{\mathcal{E}}$, the matrix for T in the standard basis.

DEFINITION

Similar Matrices

The matrices A and B are called ***similar matrices***, denoted $A \sim B$, if A and B represent the same linear transformation but in possibly different bases. Equivalently, $A \sim B$ if there is an invertible matrix X so that

$$A = XBX^{-1}.$$