Module 11 Lecture Handout

This module is about the range and nullspace of a linear transformation and the analogous ideas for matrices. Note: students should work on core exercises, practice problems, and online homework (MathMatize.com) to strengthen their understanding.

1. (Definition) Range of a transformation is the set of outputs. Read the definition below.

> **Range.** The ***range*** (or ***image***) of a linear transformation $T : V \to W$ is the set of vectors that T can output. That is,
> $$\text{range}(T) = \{\vec{y} \in W \ : \ \vec{y} = T\vec{x} \text{ for some } \vec{x} \in V\}.$$

2. The range of a linear transformation is a subspace. A proof is in module 11. Using the definition above, is the range of a linear transformation a subspace of V or W?

3. Let $P : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation

$$P\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \text{vcomp}_u\left(\begin{bmatrix} x \\ y \end{bmatrix}\right)$$

for $\vec{u} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$. Find the range of P. A graph can be helpful for this problem.

4. (Definition) Please read the definition of rank of a linear transformation.

> **Rank of a Linear Transformation.** For a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$, the ***rank*** of T, denoted rank(T), is the dimension of the range of T.

Task: Suppose $T : \mathbb{R}^3 \to \mathbb{R}^3$ is a linear transformation such that the range of T is given by the set of points satisfying

$$\begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix} \cdot \vec{x} = 0$$

What is the rank of T?

5. (Definition) Please read the definitions of the null space (kernel) of a linear transformation and nullity.

> **Null Space.** The ***null space*** (or ***kernel***) of a linear transformation $T : V \to W$ is the set of vectors that get mapped to the zero vector under T. That is,
>
> $$\text{null}(T) = \{\vec{x} \in V \ : \ T\vec{x} = \vec{0}\}.$$

> **Nullity.** For a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$, the ***nullity*** of T, denoted nullity(T), is the dimension of the null space of T.

Task: The null space of a linear transformation is also a subspace. A proof is provided in the textbook in module 11. Using the notation in the definition above, is the null space a subspace of V or W?

6. Suppose the linear transformation $P:\mathbb{R}^3 \to \mathbb{R}^3$ is defined by

$$P\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ y \\ 0 \end{bmatrix}$$

Find the null space of P and the nullity of P.

Part 2: Fundamental subspace of matrices

7. (Definition) Please read the following definition.

> **Fundamental Subspaces.** Associated with any matrix M are three fundamental subspaces: the ***row space*** of M, denoted $\text{row}(M)$, is the span of the rows of M; the ***column space*** of M, denoted $\text{col}(M)$, is the span of the columns of M; and the ***null space*** of M, denoted $\text{null}(M)$, is the set of solutions to $M\vec{x} = \vec{0}$.

Task: Find the row space, column space, and null space for the matrix,

$$M = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \end{bmatrix}$$

8. This task is about connections to row reduction. We know we can simplify expressions or solve using row reduction. The question is what impact does row reduction have on the fundamental subspaces of matrices. Let

$$M = \begin{bmatrix} 2 & 2 & 3 \\ 4 & 5 & 7 \end{bmatrix}$$

(a) Apply row reduction yourself to obtain

$$\text{rref}(M) = \begin{bmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & 1 \end{bmatrix}$$

(b) Using the fact that the steps involved row reduction are linear combinations of the rows, explain why the row space of M is the same as the row space of rref(M).

(c) The null space of a matrix is the set of all vectors orthogonal to the rows of the matrix. Explain why the null space of M is the same as the null space of rref(M). Recall that the null space is also the set of solutions to $M\vec{x} = \vec{0}$.

9. Column space is affected by row operations. Apply row reduction to the matrix, M, and then explain why the column space has changed, by comparing the column space of M and the column space of rref(M).

$$M = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 2 & 2 & 1 \end{bmatrix}$$

10. (Rank of a matrix) To find the rank of a matrix, you can use row reduction. Consider the following theorems and interpretation.

Theorem. Let M be a matrix. The rank of M is equal to the number of pivots in $\text{rref}(M)$.

Takeaway. If $\mathcal{T}$ is a linear transformation and M is a corresponding matrix, $\text{range}(\mathcal{T}) = \text{col}(M)$, and answering questions about M answers questions about $\mathcal{T}$.

Theorem. Let $\mathcal{T}$ be a linear transformation and let M be a matrix for $\mathcal{T}$. Then $\text{nullity}(\mathcal{T})$ is equal to the number of free variable columns in $\text{rref}(M)$.

Determine the rank and nullity of M.

$$M = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 2 & 2 & 1 \end{bmatrix}$$

Determine the rank and nullity of

$$M = \begin{bmatrix} 1 & 1 \\ 3 & 3 \end{bmatrix}$$

Find your own example of a matrix that has rank 3 and nullity 2.

11. In this task, we consider the matrix equation $A\vec{x} = \vec{b}$. We need to solve for both the homogeneous solutions and a particular solution. Please read the following.

Theorem. Let A be a matrix, $\vec{b}$ be a vector, and let $\vec{p}$ be a particular solution to $A\vec{x} = \vec{b}$. Then, the set of all solutions to $A\vec{x} = \vec{b}$ is

$$\text{null}(A) + \{\vec{p}\}.$$

Takeaway. To write the complete solution to $A\vec{x} = \vec{b}$, all you need is the null space of A and a particular solution to $A\vec{x} = \vec{b}$.

Find the general solution (the homogeneous solutions plus a particular solution) to the matrix equation,

$$\begin{bmatrix} 1 & -2 \\ 3 & -6 \end{bmatrix} \vec{x} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

12. The Rank Nullity Theorem is one of the main facts from module 11.

> **Theorem (Rank-nullity Theorem for Matrices).** For a matrix A,
>
> $$\text{rank}(A) + \text{nullity}(A) = \text{ \# of columns in } A.$$

> **Theorem (Rank-nullity Theorem for Linear Transformations).** Let $\mathcal{T}$ be a linear transformation. Then
>
> $$\text{rank}(\mathcal{T}) + \text{nullity}(\mathcal{T}) = \dim(\text{domain of } \mathcal{T}).$$

(a) Suppose M is a 5×6 matrix with rank 2. What is the nullity?

(b) Using row reduction find the rank and nullity of the matrix,

$$M = \begin{bmatrix} 2 & -2 & 0 \\ -4 & 4 & 0 \end{bmatrix}$$

(c) Suppose $T : \mathbb{R}^4 \to \mathbb{R}^3$ is a linear transformation. What are the possible values of the rank of T and the nullity of T?

13. (Linear Transformations, Matrices, Ordered Bases) In this task, we highlight the relationship between matrices, linear transformations, and ordered bases, and the notation need to communicate these ideas.

> **Induced Transformation.**
> Let M be an $n \times m$ matrix. We say M ***induces*** a linear transformation $\mathcal{T}_M : \mathbb{R}^m \to \mathbb{R}^n$ defined by
>
> $$[\mathcal{T}_M \vec{v}]_{\mathcal{E}'} = M[\vec{v}]_{\mathcal{E}},$$
>
> where $\mathcal{E}$ is the standard basis for $\mathbb{R}^m$ and $\mathcal{E}'$ is the standard basis for $\mathbb{R}^n$.

Unpack the meaning of the symbols in the equation below.

$$[\mathcal{T}_M \vec{v}]_{\mathcal{E}'} = M[\vec{v}]_{\mathcal{E}}$$

14. Read the proofs below and explain the connection between the proofs and the definition of subspace.

> **Subspace.** A non-empty subset $V \subseteq \mathbb{R}^n$ is called a ***subspace*** if for all $\vec{u}, \vec{v} \in V$ and all scalars k we have
>
> (i) $\vec{u} + \vec{v} \in V$; and
>
> (ii) $k\vec{u} \in V$.

> **Theorem.** Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Then $\text{range}(T) \subseteq \mathbb{R}^m$ is a subspace.

Proof. Since $\text{range}(T) = T(\mathbb{R}^n)$ and $\mathbb{R}^n$ is non-empty, we know that $\text{range}(T)$ is non-empty. Therefore, to show that $\text{range}(T)$ is a subspace, what remains to be shown is (i) that it's closed under vector addition, and (ii) that it is closed under scalar multiplication.

(i) Let $\vec{x}, \vec{y} \in \text{range}(T)$. By definition, there exist $\vec{u}, \vec{v} \in \mathbb{R}^n$ such that $\vec{x} = T(\vec{u})$ and $\vec{y} = T(\vec{v})$. Since T is linear,

$$\vec{x} + \vec{y} = T(\vec{u}) + T(\vec{v}) = T(\vec{u} + \vec{v}),$$

and so $\vec{x} + \vec{y} \in \text{range}(T)$.

(ii) Let $\vec{x} \in \text{range}(T)$ and let α be a scalar. By definition, there exists $\vec{u} \in \mathbb{R}^n$ such that $\vec{x} = T(\vec{u})$, and so by the linearity of T,

$$\alpha\vec{x} = \alpha T(\vec{u}) = T(\alpha\vec{u}).$$

Therefore $\alpha\vec{x} \in \text{range}(T)$.

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> **Theorem.** Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Then $\text{null}(T) \subseteq \mathbb{R}^n$ is a subspace.

Proof. Since T is linear, $T(\vec{0}) = \vec{0}$ and so $\vec{0} \in \text{null}(T)$ which shows that $\text{null}(T)$ is non-empty. Therefore, to show that $\text{null}(T)$ is a subspace, we only need to show (i) that it's closed under vector addition, and (ii) that it is closed under scalar multiplication.

(i) Let $\vec{x}, \vec{y} \in \text{null}(T)$. By definition, $T(\vec{x}) = T(\vec{y}) = \vec{0}$. By linearity we see

$$T(\vec{x} + \vec{y}) = T(\vec{x}) + T(\vec{y}) = \vec{0} + \vec{0} = \vec{0},$$

and so $\vec{x} + \vec{y} \in \text{null}(T)$.

(ii) Let $\vec{x} \in \text{null}(T)$ and let α be a scalar. By definition, $T(\vec{x}) = \vec{0}$, and so by the linearity of T,

$$T(\alpha\vec{x}) = \alpha T(\vec{x}) = \alpha\vec{0} = \vec{0}.$$

Therefore $\alpha\vec{x} \in \text{null}(T)$.

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It is recommended that you work on the core exercises from the textbook from module 11 for additional practice. Solutions are posted on Quercus.

DEF

Range

The ***range*** (or ***image***) of a linear transformation $T : V \to W$ is the set of vectors that T can output. That is,

$$\text{range}(T) = \{\vec{y} \in W \;:\; \vec{y} = T\vec{x} \text{ for some } \vec{x} \in V\}.$$

DEFINITION

Null Space

The ***null space*** (or ***kernel***) of a linear transformation $T : V \to W$ is the set of vectors that get mapped to the zero vector under T. That is,

$$\text{null}(T) = \{\vec{x} \in V \;:\; T\vec{x} = \vec{0}\}.$$

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51 Let $\mathcal{P} : \mathbb{R}^2 \to \mathbb{R}^2$ be projection onto span$\{\vec{u}\}$ where $\vec{u} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ (like before).

51.1 What is the range of $\mathcal{P}$?

51.2 What is the null space of $\mathcal{P}$?

52 Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be an arbitrary linear transformation.

52.1 Show that the null space of T is a subspace.

52.2 Show that the range of T is a subspace.

Induced Transformation

DEFINITION

Let M be an $n \times m$ matrix. We say M ***induces*** a linear transformation $\mathcal{T}_M : \mathbb{R}^m \to \mathbb{R}^n$ defined by

$$[\mathcal{T}_M \vec{v}]_{\mathcal{E}'} = M[\vec{v}]_{\mathcal{E}},$$

where $\mathcal{E}$ is the standard basis for $\mathbb{R}^m$ and $\mathcal{E}'$ is the standard basis for $\mathbb{R}^n$.

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53 Let $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, let $\vec{v} = \vec{e}_1 + \vec{e}_2 \in \mathbb{R}^2$, and let $\mathcal{T}_M$ be the transformation induced by M.

53.1 What is the difference between "$M\vec{v}$" and "$M[\vec{v}]_{\mathcal{E}}$"?

53.2 What is $[\mathcal{T}_M \vec{e}_1]_{\mathcal{E}}$?

53.3 Can you relate the columns of M to the range of $\mathcal{T}_M$?

Fundamental Subspaces

DEF

Associated with any matrix M are three fundamental subspaces: the ***row space*** of M, denoted $\mathrm{row}(M)$, is the span of the rows of M; the ***column space*** of M, denoted $\mathrm{col}(M)$, is the span of the columns of M; and the ***null space*** of M, denoted $\mathrm{null}(M)$, is the set of solutions to $M\vec{x} = \vec{0}$.

54 Consider $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$.

54.1 Describe the row space of A.

54.2 Describe the column space of A.

54.3 Is the row space of A the same as the column space of A?

54.4 Describe the set of all vectors orthogonal to the rows of A.

54.5 Describe the null space of A.

54.6 Describe the range and null space of T_A, the transformation induced by A.

55

$$B = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \end{bmatrix} \qquad C = \mathrm{rref}(B) = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \end{bmatrix}$$

55.1 How does the row space of B relate to the row space of C?

55.2 How does the null space of B relate to the null space of C?

55.3 Compute the null space of B.

56

$$P = \begin{bmatrix} 0 & 0 \\ 1 & 2 \end{bmatrix} \qquad Q = \operatorname{rref}(P) = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$$

56.1 How does the column space of P relate to the column space of Q?

56.2 Describe the column space of P and the column space of Q.

Rank

DEF

For a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$, the ***rank*** of T, denoted $\operatorname{rank}(T)$, is the dimension of the range of T.

For an $m \times n$ matrix M, the ***rank*** of M, denoted $\operatorname{rank}(M)$, is the dimension of the column space of M.

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57 Let $\mathcal{P} : \mathbb{R}^2 \to \mathbb{R}^2$ be projection onto $\operatorname{span}\{\vec{u}\}$ where $\vec{u} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$, and let $\mathcal{R} : \mathbb{R}^2 \to \mathbb{R}^2$ be rotation counter-clockwise by 90°.

57.1 Describe $\operatorname{range}(\mathcal{P})$ and $\operatorname{range}(\mathcal{R})$.

57.2 What is the rank of $\mathcal{P}$ and the rank of $\mathcal{R}$?

57.3 Let P and R be the matrices corresponding to $\mathcal{P}$ and $\mathcal{R}$. What is the rank of P and the rank of R?

57.4 Make a conjecture about how the rank of a transformation and the rank of its corresponding matrix relate. Can you justify your claim?

58

58.1 Determine the rank of (a) $\begin{bmatrix}1 & 1\\ 2 & 2\end{bmatrix}$ (b) $\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix}$ (c) $\begin{bmatrix}1 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$ (d) $\begin{bmatrix}3\\ 3\\ 2\end{bmatrix}$ (e) $\begin{bmatrix}1 & 0 & 1\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$.

59

Consider the homogeneous system

$$\begin{array}{rrrl} x & +2y & +z & =0\\ x & +2y & +3z & =0\\ -x & -2y & +z & =0 \end{array} \tag{11}$$

and the non-augmented matrix of coefficients $A=\begin{bmatrix}1 & 2 & 1\\ 1 & 2 & 3\\ -1 & -2 & 1\end{bmatrix}$.

59.1 What is rank(A)?

59.2 Give the general solution to system (11).

59.3 Are the column vectors of A linearly independent?

59.4 Give a non-homogeneous system with the same coefficients as (11) that has

(a) infinitely many solutions

(b) no solutions.

60

60.1 The rank of a 3×4 matrix A is 3. Are the column vectors of A linearly independent?

60.2 The rank of a 4×3 matrix B is 3. Are the column vectors of B linearly independent?

Rank-nullity Theorem

THEOREM

The ***nullity*** of a matrix is the dimension of the null space.

The rank-nullity theorem for a matrix A states

$$\text{rank}(A) + \text{nullity}(A) = \#\text{ of columns in } A.$$

61

61.1 Is there a version of the rank-nullity theorem that applies to linear transformations instead of matrices? If so, state it.

62

The vectors $\vec{u}, \vec{v} \in \mathbb{R}^9$ are linearly independent and $\vec{w} = 2\vec{u} - \vec{v}$. Define $A = [\vec{u}|\vec{v}|\vec{w}]$.

62.1 What is the rank and nullity of A?

62.2 What is the rank and nullity of A^T?