A Brief Introduction to the Cross Product

This handout is optional and recommended for students as a reference for computing cross products. The cross product is used in some subsequent STEM courses. The cross product is not covered in MAT223 and will NOT appear on the MAT 223 final exam. MAT 223 students are encouraged to learn how to compute cross products, and identify the direction of the vector produced by the cross product. Save this handout as a reference.

1. The 3B1B video is an introduction to the idea of a cross product. Watch this video first. https://youtu.be/eu6i7WJeinw

2. The setting or context for the cross product is $\mathbb{R}^3$. The cross product is not defined in other spaces. Try these examples to determine the **direction** of the cross product, using the right hand rule. This Geogebra3D graph may be useful for visualizing the vectors. You can move the vectors by clicking on the endpoints of the vectors https://www.geogebra.org/3d/fymze6zk. Left click on the endpoints to toggle between horizontal movement and vertical movement.

   (a) $\vec{e}_1 \times \vec{e}_2$

   (b) $\vec{e}_2 \times \vec{e}_1$

   (c) $\vec{e}_2 \times \vec{e}_3$

   (d) $\vec{e}_3 \times \vec{e}_2$

   (e) $\vec{e}_1 \times \vec{e}_3$

   (f) $\vec{e}_3 \times \vec{e}_1$

3. The next video by the Khan Academy is more about the mechanics of computing the cross product. https://youtu.be/pJzmiywagfY. Compute the cross products (answers at the end of this handout).

   (a) $\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \times \begin{bmatrix} 3 \\ 4 \\ -2 \end{bmatrix}$

   (b) $\begin{bmatrix} -1 \\ 3 \\ 4 \end{bmatrix} \times \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

   (c) $\begin{bmatrix} 1 \\ -7 \\ 2 \end{bmatrix} \times \begin{bmatrix} 6 \\ 2 \\ 4 \end{bmatrix}$

4. This is an important fact about the cross product. If $\vec{c} = \vec{a} \times \vec{b}$, then $\vec{c}$ is **orthogonal** to both $\vec{a}$ and $\vec{b}$.

5. One use of the cross product is finding a normal vector. In MAT223, we solved a system of equations to find a vector normal to two given vectors, $\vec{d}_1$ and $\vec{d}_2$. Let P be a plane in $\mathbb{R}^3$ given in vector form by

$$\vec{x} = s\begin{bmatrix}2\\2\\3\end{bmatrix} + t\begin{bmatrix}-1\\0\\-3\end{bmatrix}$$

for some s, t in $\mathbb{R}$. Use the cross product to help you find the normal form of this plane.

6. Let $\vec{a} = \begin{bmatrix}a_1\\a_2\\a_3\end{bmatrix}$ and $\vec{b} = \begin{bmatrix}b_1\\b_2\\b_3\end{bmatrix}$

The determinant can be used to compute the cross product. Let $\vec{c} = \vec{a} \times \vec{b}$ and $\vec{c} = \begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix}$.

Create a matrix with left column $\vec{a}$ and right column $\vec{b}$ as below.

$$\begin{bmatrix}a_1 & b_1\\a_2 & b_2\\a_3 & b_3\end{bmatrix}$$

Then each component of the cross product is given by the following.

$c_1 = \det\begin{bmatrix}a_2 & b_2\\a_3 & b_3\end{bmatrix}$

$c_2 = -\det\begin{bmatrix}a_1 & b_1\\a_3 & b_3\end{bmatrix}$ (Notice this one is multiplied by -1.)

$c_3 = \det\begin{bmatrix}a_1 & b_1\\a_2 & b_2\end{bmatrix}$

Insert these numbers into $\vec{c}$ and you have computed the cross product. Notice that for c_i you are in a sense deleting row i from the matrix above and computing the determinant of the remaining matrix. Also notice that c_2 is -1 times the determinant.

Use this method to compute cross products in problem 3.

7. For learning more about determinants see the follow-up video by 3B1B, “Cross product in light of linear transformations.” https://youtu.be/BaM7OCEm3G0

Answers

Cross product directions.

(a) $\vec{e}_1 \times \vec{e}_2 = \vec{e}_3$

(b) $\vec{e}_2 \times \vec{e}_1 = -\vec{e}_3$

(c) $\vec{e}_2 \times \vec{e}_3 = \vec{e}_1$

(d) $\vec{e}_3 \times \vec{e}_2 = -\vec{e}_1$

(e) $\vec{e}_1 \times \vec{e}_3 = -\vec{e}_2$

(f) $\vec{e}_3 \times \vec{e}_1 = \vec{e}_2$

Cross product computations.

(a) $\begin{bmatrix}1\\0\\2\end{bmatrix} \times \begin{bmatrix}3\\4\\-2\end{bmatrix} = \begin{bmatrix}-8\\8\\4\end{bmatrix}$

(b) $\begin{bmatrix}-1\\3\\4\end{bmatrix} \times \begin{bmatrix}1\\1\\1\end{bmatrix} = \begin{bmatrix}-1\\5\\-4\end{bmatrix}$

(c) $\begin{bmatrix}1\\-7\\2\end{bmatrix} \times \begin{bmatrix}6\\2\\4\end{bmatrix} = \begin{bmatrix}-32\\8\\44\end{bmatrix}$

P has normal form $\begin{bmatrix}-6\\3\\2\end{bmatrix} \cdot \vec{x} = 0$. (Note: there are multiple possibilities using a nonzero scalar multiple of $\vec{n} = \begin{bmatrix}-6\\3\\2\end{bmatrix}$.