Problem 1. Find all the points where f is continuous, where

$$f(x) = \begin{cases} \frac{1}{q}, & \text{if } x = \frac{p}{q} \in \mathbb{Q} \text{ in lowest terms} \\ 0, & \text{if } x \notin \mathbb{Q} \end{cases} \tag{1}$$

Problem 2 (Spivak 6-7)**.** Suppose that f satisfies $f(x+y) = f(x) + f(y)$ and that f is continuous at 0. Prove that f is continuous at a for all a.

Problem 3 (Spivak 6-15)**.** Prove that if f is continuous at a, then for any $\varepsilon > 0$ there is a $\delta > 0$ so that whenever $|x-a| < \delta$ and $|y-a| < \delta$, we have $\bigl|f(x) - f(y)\bigr| < \varepsilon$.

Problem 4 (Spivak 7-5)**.** Suppose f is continuous and $f(x) \in \mathbb{Q}$ for all x. What can you say about f?

Problem 5 (Spivak 7-10)**.** Suppose f and g are continuous on $[a,b]$ and $f(a) < g(a)$, yet $f(b) > g(b)$. Prove $f(x) = g(x)$ for some $x \in [a,b]$.

Problem 6 (Spivak 7-18)**.** Suppose $f\colon \mathbb{R} \to \mathbb{R}$ is continuous, positive ($f(x) > 0$ for all x), and

$$\lim_{x\to\infty} f(x) = 0 = \lim_{x\to-\infty} f(x) \tag{2}$$

Prove that f attains a maximum on $\mathbb{R}$ (i.e. there exists a y such that $f(y) \leq f(x)$ for all x).

Problem 7. Let $f\colon [a,b] \to [a,b]$ be a continuous function. Show that f has a fixed point. That is, there exists some $c \in [a,b]$ such that $f(c) = c$.