Unless otherwise specified, $f\colon \mathbb{R} \to \mathbb{R}$ denotes a real-valued function.

Problem 1. For each of the following statements, determine what they mean, and find a function which satisfies the condition.

1. $\forall \varepsilon > 0$, $\forall \delta > 0$, if $0 < |x-a| < \delta$ then $\left|f(x) - L\right| < \varepsilon$.
2. $\exists \varepsilon > 0$, $\forall \delta > 0$, if $0 < |x-a| < \delta$ then $\left|f(x) - L\right| < \varepsilon$.
3. $\forall \varepsilon > 0$, $\exists \delta > 0$, if $0 < |x-a| < \delta$ then $\left|f(x) - L\right| < \varepsilon$.
4. $\exists \varepsilon > 0$, $\exists \delta > 0$, if $0 < |x-a| < \delta$ then $\left|f(x) - L\right| < \varepsilon$.
5. $\forall \delta > 0$, $\exists \varepsilon > 0$, if $0 < |x-a| < \delta$ then $\left|f(x) - L\right| < \varepsilon$.
6. $\exists \delta > 0$, $\forall \varepsilon > 0$, if $0 < |x-a| < \delta$ then $\left|f(x) - L\right| < \varepsilon$.

Problem 2. One of the above is the statement "$\lim\limits_{x \to a} f(x) = L$". Write out the negation of this statement with quantifiers.

Problem 3 (Spivak 5.9)**.** Prove that $\lim\limits_{x \to a} f(x) = \lim\limits_{h \to 0} f(h+a)$.

Problem 4. For the following, determine whether the limit $\lim\limits_{x \to a} f(x)$ exists using the ε-δ definition of the limit. If it does, compute the value.

a) $f(x) = x - 2$, with $a \in \mathbb{R}$

b) $f(x) = x^2$, with $a = 1$

c) $f(x) = \begin{cases} x \sin\left(\frac{1}{x}\right), & x \neq 0 \\ 0, & x = 0 \end{cases}$, with $a = 0$

d) $f(x) = \begin{cases} x^2, & x \in \mathbb{Q} \\ \sqrt{|x|}, & \text{else} \end{cases}$, with $a \in \mathbb{R}$

Problem 5 (Monotonicity of the limit)**.**

a) Suppose $f(x) \leq g(x)$ for all x. If the limits exist, prove that

$$\lim_{x \to a} f(x) \leq \lim_{x \to a} g(x) \tag{1}$$

b) What happens if $f(x) < g(x)$ for all x?

Problem 6 (Squeeze Theorem)**.** Suppose that there exists a $c > 0$ so that $f(x) \leq g(x) \leq h(x)$ for all x with $|x - a| < c$. Suppose that $\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L$. Prove that $\lim_{x \to a} g(x)$ exists and equals L.

Problem 7. We say f is *Lipschitz* if for all x and y, $\big|f(x) - f(y)\big| \leq k|x - y|$ for some $k > 0$. Prove that f is continuous.

Problem 8. Let $f \colon [a, b] \to [a, b]$ be a continuous function. Show that f has a fixed point. That is, there exists some $c \in [a, b]$ such that $f(c) = c$.

Problem 9 (Bonus)**.** We say a set M is a *metric space* if there is a map $d \colon M \times M \to \mathbb{R}$ such that for all x, y, $z \in M$,

- $d(x, y) \geq 0$, with equality if and only if $x = y$.
- $d(x, y) = d(y, x)$ (Symmetry)
- $d(x, z) \leq d(x, y) + d(y, z)$ (Triangle inequality)

Generalize the definition of a limit to a metric space, and prove that limits are unique.