1 Comments from problem set 10

- To show a function is integrable, use the definition of integrability. Be careful not to make any unnecessary assumptions about the function you are working with. There are a great many integrable functions which are not continuous or well-behaved by any stretch of the imagination. This is a case where your intuition may fail you. Get used to falling back on the definitions.
- Functions do not behave like real numbers. In general, given a constant c, you cannot split a set $\mathcal{S}$ of functions into the sets $\{ f \in \mathcal{S} \mid \forall x, f(x) > c \}$ and $\{ f \in \mathcal{S} \mid \forall x, f(x) \leq c \}$. If you catch yourself doing this, take another look at your work.
- Not all integrable functions are continuous! Your intuition about continuous functions will fail you if you try to come up with an argument about integrable functions. Remember to use your definitions.
- When wondering whether you should prove something, write out a proof (ideally as a lemma or a claim). This way, your proof will not be missing necessary details, and if you wrote too much, at least it's isolated in a different claim so your main argument is not cumbersomely long.

2 Practice problems

Problem 1 (PS10 Q6). Suppose f is integrable on $[a,b]$, let $g(x) := \min(f(x), 10)$. Is g integrable on $[a,b]$?

Consider the following strategies. Find a solution to this problem using each strategy, or find a flaw in the method:

a) Use the fact that f is integrable to select for each ε a partition P of $[a,b]$ such that $\mathcal{U}(f,P) - \mathcal{L}(f,P) < \varepsilon$.

b) Choose a partition P of $[a,b]$ such that on each subinterval I of P, either $g \equiv f$ or $g \equiv 10$ (i.e. $g(x) = f(x)$ for all $x \in I$ or $g(x) = 10$ for all $x \in I$).

c) Rewrite g without using the min function:

$$g(x) = \frac{f(x) + 10}{2} - \frac{\left|f(x) - 10\right|}{2} \tag{1}$$

Problem 2 (Spivak 13.5)**.** Evaluate the following. You are not permitted any computations.

$$\int_{-1}^{1} x^3\sqrt{1-x^2}\,dx \tag{2}$$

$$\int_{-1}^{1} x^5\sqrt{2+x^2-x^4}\,dx \tag{3}$$

Problem 3 (Spivak 13.6)**.** Prove that for all $x > 0$

$$\int_0^x \frac{\sin t}{t+1}\,dt > 0 \tag{4}$$

Problem 4. Evaluate the following:

$$\int_0^\infty \frac{dx}{1+x^2} = \lim_{R\to\infty}\int_0^R \frac{dx}{1+x^2} \tag{5}$$

Problem 5 (Spivak 15.1)**.** Differentiate the following functions:

1. $f(x) = \arctan\big(\arctan(\arctan x)\big)$
2. $f(x) = \arctan\big(\arctan(\arccos x)\big)$
3. $f(x) = \arctan(\tan x \cdot \arctan x)$
4. $f(x) = \arcsin\left(\frac{1}{\sqrt{1+x^2}}\right)$