Lie groups and Lie algebras

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Motivation: Lie groups appear as automorphism groups of all sorts of structures.

Defn (Lie group) A Lie group is a group G which is also a manifold. I.e. a group object in the category of manifolds, i.e. a one-object stack in manifolds.

Exercise If $m: G \times G \to G$ is smooth, smoothness of $S: G \to G$ follows automatically. Hint: Use the inverse function theorem.

Fact

metric-preserving

- For (M, g) Riemannian, $\mathrm{Diff}(M, g)$ is a (f.d.) Lie group.
- For (M, J) complex, $\mathrm{Diff}(M, J)$ ——— " ———.

complex-structure preserving

E.g. ① $GL(n,\mathbb{R}) = \{A \in Mat(n,\mathbb{R}) : \det(A) \neq 0\}$ open subset of $\mathbb{R}^{n^2}$

② $O(n) = \{A \in Mat(n,\mathbb{R}) \mid A^T A = \mathbb{1}\} = \{A : \|Ax\| = \|x\| \ \forall x \in \mathbb{R}^n\}$

Here, it's enough to show $O(n)$ is a submanifold of $Mat(n,\mathbb{R})$:

Approach 1:

- $O(n)$ is a ~~reg. val.~~ level set of $f: A \to A^T A$, and
- $\mathbb{1}$ is a reg. val. of $f: Mat(n,\mathbb{R}) \to Sym(n,\mathbb{R})$

Approach 2:

Construct manifold charts using

$$\begin{aligned} \exp: Mat_n(\mathbb{R}) &\to Mat_n(\mathbb{R}) \\ B &\mapsto \sum_{n=0}^{\infty} \frac{1}{n!} B^n \end{aligned}$$

Exercise

This is absolutely convergent.

Let's compute the derivative at 0:

$$(T_0 \exp)(B) = \frac{d}{dt}\Big|_{t=0} \exp(tB) = B$$

So $T_0\exp = id$, so by applying the inverse function theorem $\exists \epsilon > 0$ s.t. exp restricts to a diffeo

$$\exp| : \{B \mid \|B\| < \epsilon\} \to \mathcal{U} \subseteq Mat(n,\mathbb{R})$$

open neighbourhood of $\mathbb{1}$.

Since all norms are equivalent; we say, for instance

$$\|B\| = \sum_{ij} (B_{ij})^2 = tr(B^T B)$$

Let $\log : \mathcal{U} \to \{B : \|B\| < \epsilon\}$ be the inverse map.
We can regard this as a chart around $\mathbb{1} \in Mat(n, \mathbb{R})$

$\boxed{\text{Claim}}$ This is a submanifold chart for $O(n)$.
i.e. $\log(\mathcal{U} \cap O(n)) = \{B : \|B\| < \epsilon\} \cap o(n)$

$o(n) = \{B : B^T = -B\}$

Proof. Given suff. small $B \in Mat(n, \mathbb{R})$ (i.e. $\|B\| < \epsilon$) we have

$$\exp(B) \in O(n) \iff \exp(B)^T = \exp(B)^{-1}$$
$$\iff \exp(B^T) = \exp(-B)$$
$$\iff B^T = -B \qquad \blacksquare$$

So $(\mathcal{U}, \log)$ is a submanifold chart around $\mathbb{1} \in O(n)$. To get a chart near any $A \in O(n)$, use

$$\ell_A : Mat(n, \mathbb{R}) \to Mat(n, \mathbb{R})$$
$$X \mapsto AX$$

Then $(\ell_A(\mathcal{U}), \log \circ \ell_A^{-1})$ is a submanifold chart around A.

③ $SL(n, \mathbb{R}) = \{A \in Mat(n, \mathbb{R}) : \det A = 1\}$

$\boxed{\text{Remark}}$ The method in ② can be applied to ③ and all matrix groups.

④ Let $\mathcal{A}$ be an $\mathbb{R}$-algebra, $\dim_{\mathbb{R}}(\mathcal{A}) < \infty$ with unit.
(E.g. $\mathbb{R}, \mathbb{C}, \mathbb{H}, \ldots$) and consider $Mat(n, \mathcal{A})$:

- $GL(n, \mathcal{A}) :=$ invertible matrices in $Mat(n, \mathcal{A})$ is a Lie group.
  (e.g. $GL(n, \mathbb{R})$, $GL(n, \mathbb{C})$, $GL(n, \mathbb{H})$)

- If A is commutative, we can also define $\det: \mathrm{Mat}(n,A) \to A$ and $SL(n,A)$ is defined. $\leadsto$ $SL(n,\mathbb{R})$, $SL(n,\mathbb{C})$, but not $SL(n,\mathbb{H})$.

$\boxed{\text{Exercise}}$ Why not? $\exists$ group hom $GL(n,A) \to A$?

- $\exists\ |\cdot|$ for $\mathbb{R}, \mathbb{C}$, and $\mathbb{H}$, (in $\mathbb{H}$, $|a+ib+jc+kd| = \sqrt{a^2+b^2+c^2+d^2}$)

We get $\|\cdot\|$ on $\mathbb{R}^n$, $\underset{\mathbb{R}^{2n}}{\underset{\shortparallel}{\mathbb{C}^n}}$, $\underset{\mathbb{R}^{4n}}{\underset{\shortparallel}{\mathbb{H}^n}}$, $\|x\| = \sum_i |x_i|^2$

The group of transformations $A \in \mathrm{Mat}(n,A)$ with $\|Ax\| = \|x\|$ for all x is a Lie group:

$$\underset{O(n)}{\overset{\mathbb{R}}{\downarrow}} \qquad \underset{U(n)}{\overset{\mathbb{C}}{\downarrow}} \qquad \underset{Sp(n)}{\overset{\mathbb{H}}{\downarrow}}$$

$\boxed{\text{Note}}$

$$U(n) = GL(n,\mathbb{C}) \cap O(2n)$$
$$Sp(n) = GL(n,\mathbb{H}) \cap O(4n)$$

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It's standard to define $SL(n,\mathbb{H}) := GL(n,\mathbb{H}) \cap SL(2n,\mathbb{C})$, yet whether this is a good definition is unclear; note

$$SL(n,\mathbb{C}) \neq GL(n,\mathbb{C}) \cap SL(2n,\mathbb{R})$$

Further, we have $SO(n) = O(n) \cap SL(n,\mathbb{R})$
$SU(n) = U(n) \cap SL(n,\mathbb{C})$

$GL(n,A)$ has many matrix subgroups, e.g. upper triangular matrices.

The group of affine linear transformations of $\mathbb{R}^n$ $x \mapsto Ax+b$, $b \in \mathbb{R}^n$, denoted $Aff(n)$ is $GL(n,\mathbb{R}) \rtimes \mathbb{R}^n$

Note: $Aff(1)$ is a 2-dimensional, nonabelian Lie group.

Fact: Not every Lie group is a matrix Lie group.

→ For any Lie group G, the universal cover $\tilde{G}$ is a Lie group. $\widetilde{SL(2,\mathbb{R})}$ is not a matrix Lie group.

Lie algebras

Definition (Lie algebra) A vector space $\mathfrak{g}$ together with a bracket $f \in \mathfrak{g}^* \otimes \mathfrak{g}^* \otimes \mathfrak{g}$ which is totally antisymmetric and Jacobi ∵

E.g. $\mathfrak{gl}(n,\mathbb{R}) = Mat(n,\mathbb{R})$ with bracket $[B_1,B_2] = B_1B_2 - B_2B_1$

More generally, $\mathfrak{gl}_n(A)$.

If A is commutative, $\exists\, tr : Mat(n,A) \to A$ satisfying $tr(B_1B_2) = tr(B_2B_1)$, so we define $\mathfrak{sl}(n,A) := \ker tr$

For $\mathbb{R}, \mathbb{C}, \mathbb{H}$, we have an inner product

$$\langle x, y \rangle = \sum_i \overline{x_i}\, y_i$$

Define $(\mathfrak{o}(n), \mathfrak{u}(n), \mathfrak{sp}(n)) \leq (\mathfrak{gl}(n,\mathbb{R}), \mathfrak{gl}(n,\mathbb{C}), \mathfrak{gl}(n,\mathbb{H}))$ such that

$$\boxed{\langle Bx, y \rangle + \langle x, By \rangle = 0}$$

For $\mathbb{R}, \mathbb{C}$, we can additionally impose $\mathrm{tr} = 0$:

$$\mathfrak{so}(n) = \mathfrak{o}(n) \cap \mathfrak{sl}(n, \mathbb{R}) = \mathfrak{o}(n)$$

$$\mathfrak{su}(n) = \mathfrak{u}(n) \cap \mathfrak{sl}(n, \mathbb{C})$$

Thm (Ado) Every finite-dimensional Lie algebra is a matrix Lie algebra.

Some 3-dimensional Lie groups

Some key building-blocks of this theory are the following:

$$SO(3) \qquad SU(2) \qquad SL(2,\mathbb{R})$$

Topology of $SO(3)$: A rotation in $\mathbb{R}^3$ is determined by an axis of rotation and an angle $\theta \in [0, \pi]$. Thus, all rotations are described by the space of $\{\vec{v} \in \mathbb{R}^3 : \|\vec{v}\| \leq \pi\}$, with length corresponding to rotation angle and direction the (signed) rotation axis. Antipodal points give the same rotation, so $SO(3) \cong \mathbb{P}^3(\mathbb{R})$.

$\pi_1(SO(3)) = \mathbb{Z}_2 \rightsquigarrow$ "Dirac belt trick"

$SU(2) = \{A \in Mat(2,\mathbb{C}) \ : A^\dagger = A^{-1}, \det(A) = 1\}$

Explicitly, $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $A^{-1} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$, $A^\dagger = \begin{pmatrix} \bar{a} & \bar{c} \\ \bar{b} & \bar{d} \end{pmatrix}$

So $d = \bar{a}$, $c = -\bar{b}$

$$= \left\{ \begin{pmatrix} z & -\bar{w} \\ w & \bar{z} \end{pmatrix} : |z|^2 + |w|^2 = 1 \right\} \cong S^3 \text{ as a manifold.}$$

$su(2) = \{B \in Mat(2,\mathbb{C}) : B^\dagger = -B, tr(B) = 0\}$

$$= \left\{ \begin{pmatrix} it & -\bar{u} \\ u & -it \end{pmatrix} : t \in \mathbb{R}, u \in \mathbb{C} \right\} \cong \mathbb{R}^3 \text{ as a vector space}$$

$SU(2) \circlearrowright su(2)$ via $A.B = ABA^{-1}$, so $SU(2) \circlearrowright \mathbb{R}^3$

Note: for $B \in su(2)$, $\det B = t^2 + |u|^2 = \|(t, u_1, u_2)\|$, which is preserved by $SU(2)$. This gives us an orthogonal transformation, so:

$$\varphi : SU(2) \longrightarrow \underline{SO}(3)$$

$SU(2)$ is connected; $SO(3)$ is the connected component of $\mathbb{1}$ of $O(3)$.

$\ker \varphi = \{A \ : AB = BA$ for all $\underset{\begin{pmatrix} it & 0 \\ 0 & -it \end{pmatrix}}{B \in su(2)}\} \overset{\text{Exercise}}{=} \{A : AB = BA \ \forall B \in Mat(2,\mathbb{C})\}$

$$= SU(2) \cap \mathbb{C} \cdot \mathbb{1} = \{\pm \mathbb{1}\}$$

Further, φ is surjective:

For any Lie grp hom. $\varphi: G \to H$, rank(φ) is constant (since $\varphi \circ l_a = l_{\varphi(a)} \circ \varphi$)

In our case, since $\dim SU(2) = \dim SO(3)$, $\ker\varphi$ is discrete. φ is a local diffeomorphism. Since $SU(2)$ is compact, φ is onto.

Exercise Fill in the above details.

So, $\varphi: SU(2) \to SO(3)$ is a double cover.

Similarly $SO(4) = \dfrac{SU(2)^{\times 2}}{\{\pm 1\}}$

To see this, view $SU(2) \hookrightarrow Mat(2,\mathbb{C}) = \mathbb{H}$

$SU(2) \times SU(2) \circlearrowright Mat(2,\mathbb{C})$

$(A_1, A_2).B = A_1 B A_2^{-1}$

$SL(2,\mathbb{R}) = \{A \in Mat(2,\mathbb{R}) : \det(A) = 1\}$

$SL(2,\mathbb{R}) \circlearrowright \mathbb{R}^2$, $\quad \overset{S^1}{\overset{\wr\wr}{SO(2)}} \subseteq SL(2,\mathbb{R})$.

Polar decomposition Every $A \in SL(2,\mathbb{R})$ can be uniquely written as $O \cdot P$, where $O \in SO(2)$, and P is pos. def, so $\exists \log P = B$, B symmetric, $tr(B) = 0$.

$\Rightarrow SL(2,\mathbb{R}) \cong S^1 \times \mathbb{R}^2$

We can also think of $SL(2,\mathbb{R})$ as the interior of the solid torus

Consider rank-1 matrices: $\{A \in Mat(2,\mathbb{R}),\ A \neq 0,\ \det(A) = 0\} \subseteq \mathbb{R}^4 \setminus \{0\}$

Exercise The image of this set under $\pi : \mathbb{R}^4 \setminus \{0\} \to S^3$ is a 2-torus, decomposing S^3 into 2 solid 2-tori.

Review of differential geometry

Let us work in the ~~algebra~~ category of manifolds with morphisms smooth maps.

For $m \in M$, $T_m M := \{ v : C^\infty(\mathbb{R}) \to \mathbb{R} \mid v(fg) = v(f) g|_m + f|_m v(g) \}$

Fact This definition is local: for $U \subset M$, $T_m U = T_m M$

Define tangent maps in terms of $((T_m \varphi)(v))(g) = v(g \circ \varphi)$

$\mathfrak{X}(M)$, vector fields of M, are families of tangent vectors X_m varying smoothly with $m \in M$, i.e. $X : C^\infty(M) \to C^\infty(M)$ with $X(f)|_m = X_m(f)$ such that $X(fg) = X(f)g + fX(g)$, i.e. $\mathfrak{X}(M) = \mathrm{Der}(C^\infty(M))$:

This is also local: For $U \subseteq M$ open, $X|_U$ defined so $X|_U(f|_U) = X(f)|_U$.

The flow of a complete vector field X is

$$\Phi : \mathbb{R} \times M \longrightarrow M$$
$$(t,m) \longmapsto \Phi_t(m)$$

where $\Phi_{t}(m)$ is an integral curve in t with initial condition $\Phi_0(m) = m$.

$\exists$ property $\Phi_{t_1+t_2} = \Phi_{t_1} \circ \Phi_{t_2}$

So $t \longmapsto \Phi_t$ is a group morphism $\mathbb{R} \longrightarrow \mathrm{Diff}(M)$

$\mathfrak{X}(M)$ is a Lie algebra under commutation: $[X_1, X_2] = X_1 \circ X_2 - X_2 \circ X_1$

Fact $[X_1, X_2] = 0 \iff$ The flows Φ^{X_1}, Φ^{X_2} commute.

Defn (related vector fields)

Given $\varphi : M \longrightarrow N$ smooth and v.f. $X \in \mathfrak{X}(M)$, $Y \in \mathfrak{X}(N)$ are φ-related ($X \sim_\varphi Y$) if

$$\forall m \; (T_m \varphi) X_m = Y_{\varphi(m)} \iff X \circ \varphi^* = \varphi^* \circ Y$$

where $\varphi^* : C^\infty(N) \longrightarrow C^\infty(M)$
$g \longmapsto g \circ \varphi$

Note $X_1 \sim_\varphi X_2,\ Y_1 \sim_\varphi Y_2 \implies [X_1, X_2] \sim_\varphi [Y_1, Y_2]$

The Lie functor

For G a Lie group, put $\mathfrak{g} = T_eG$. For $a \in G$,

$$\left.\begin{array}{l} L_a : G \longrightarrow G,\ g \longmapsto ag \\ R_a : G \longrightarrow G,\ g \longmapsto ga \end{array}\right\} \text{diffeomorphisms.}$$

So, we get $T_gL_a : T_gG \longrightarrow T_{ag}G$

In particular, $T_eL_g : \mathfrak{g} \xrightarrow{\sim} T_gG$

Taken together, we get a vector bundle isomorphism:

$$\left.\begin{array}{rcl} G \times \mathfrak{g} & \longrightarrow & TG \\ (g, \xi) & \longmapsto & (T_eL_g)(\xi) \end{array}\right\} \text{"left trivialization"}$$

This is smooth because it is a restriction of

$$\begin{array}{c} T\mathrm{Mult} : TG \times TG \longrightarrow TG \\ \cup\!\! \mid \\ G \times T_eG \end{array}$$

$X \in \mathfrak{X}(G)$ is left-invariant

$\iff$ it corresponds to the constant section of $G \times \mathfrak{g}$ under left translation

$\iff X_g = (T_eL_g)(X_e) \quad \forall g \in G$

$\iff (T_gL_a)(X_g) = X_{ag} \quad \forall a, g \in G$

$\iff \forall a \quad X \sim_{L_a} X$

Note $\mathfrak{X}(G)^L$ (the space of left-invariant vector fields) is closed under $[\cdot,\cdot]$, hence is a Lie subalgebra.

But $\mathfrak{X}(G)^L \longrightarrow \mathfrak{g} = T_eG$ is a v.s. iso.
$X \longmapsto X_e$

Defn The Lie algebra of G is the vector space $\mathfrak{g} = T_eG$ with bracket such that

$$\begin{aligned} \mathfrak{X}(G)^L &\longrightarrow \mathfrak{g} \\ X &\longmapsto X_e \end{aligned}$$

For $\xi \in \mathfrak{g}$, let $\xi^L \in \mathfrak{X}(G)$ be the corresponding v.f.
By def $[\xi_1, \xi_2]^L = [\xi_1^L, \xi_2^L]$

Remark we say $\Phi_{-t}(m)$ must be an integral curve so that

$$\begin{aligned} \mathrm{Diff}(M) &\longrightarrow \mathrm{Aut}(C^\infty(M)) \\ \Phi &\longmapsto (\Phi^{-1})^* \end{aligned}$$

is a homomorphism (esp. covariant)

Thm (Functoriality) $\forall$ Lie grp hom $\varphi: G \longrightarrow G'$,
$T_e\varphi: \mathfrak{g} \longrightarrow \mathfrak{g}'$ is a Lie algebra hom.

Proof. Given $\xi \in \mathfrak{g}$, let $\xi' = (T_e\varphi)\xi$

$\boxed{\text{Claim}}$ $\boxed{\xi^L \sim_\varphi (\xi')^L}$

From this we get $\xi_i^L \sim_\varphi (\xi_i')^L$ for $i = 1, 2$, so

$$[\xi_1, \xi_2]^L = [\xi_1^L, \xi_2^L] \sim_\varphi [(\xi_1')^L, (\xi_2')^L] = [\xi_1', \xi_2']^L \quad \blacksquare$$

Proof. $\xi^L \sim_\varphi (\xi')^L$

$$\iff (T_a\varphi)(\xi^L)_a = (\xi')^L_{\varphi(a)}$$

$$\iff (T_a\varphi)(T_e L_a \xi) = (T_e L_{\varphi(a)})(T_e\varphi(\xi))$$

$$\iff T_e(\varphi \circ L_a)(\xi) = T_e(L_{\varphi(a)} \circ \varphi)(\xi)$$

$$\Leftarrow \varphi \circ L_a = L_{\varphi(a)} \circ \varphi$$

$$\Leftarrow \varphi(ab) = \varphi(a)\varphi(b) \quad \blacksquare$$

Special cases

① For V an $\mathbb{R}$-v.s., a G-rep is a Lie group morphism

$G \to GL(V) = \mathrm{Aut}(V)$

A $\mathfrak{g}$-rep is a Lie algebra morphism $\mathfrak{g} \to \mathfrak{gl}(V) = \mathrm{End}(V)$

By above the Lie Functor also translates representations!

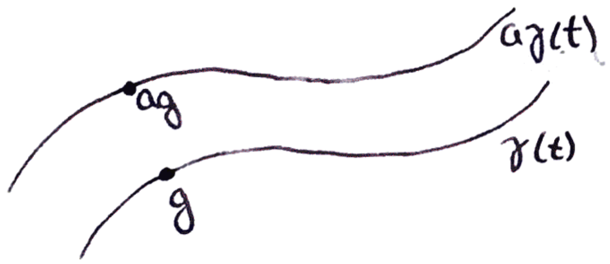
② Automorphisms $G \xrightarrow{\sim} G$ induce automorphisms $\mathfrak{g} \xrightarrow{\sim} \mathfrak{g}$

Properties of left/right-invariant vector fields

Defn A 1-parameter subgroup of G is a Lie group morphism $\gamma : \mathbb{R} \to G$

Thm For $\xi \in \mathfrak{g}$, the v.f. ξ^L is <u>complete</u>. The unique integral curve $\gamma^\xi(t)$ with $\gamma^\xi(0) = e$ is a 1-parameter subgroup and the flow of ξ^L is by right translations $(t, g) \mapsto g\gamma^\xi(-t)$.

<u>Proof</u>. Since ξ^L is left invariant, the left translate $a \cdot \gamma(t)$ of any integral curve $\gamma(t)$ is again an integral curve.



$\gamma^\xi(t)$ is an integral curve with $\gamma^\xi(0) = e$ for small t. By left-translating along itself, it is seen to be complete.

The integral curve $\gamma(t)$ with $\gamma(0) = a$ is the left-translate $\gamma(t) = a\gamma^\xi(t)$ hence exists for all $t \in \mathbb{R}$

THUS $\Phi^\xi_t(a) = a\gamma^\xi(-t)$. ∎

Similarly, ξ^R is complete with flow given by left-translation:

$$a \longmapsto \gamma^{\xi}(-t)a$$

In particular, $\gamma^{\xi}(t)$ is also an integral curve for ξ^R.

Lemma $\xi^L \sim_\iota -\xi^R$, or $\iota^*\xi^L = -\xi^R$

where $\iota(g) = g^{-1}$.

Proof. ι exchanges left- and right-translation:

$$\iota \circ L_a = R_{a^{-1}} \circ \iota$$

So $\iota^*\xi^L = \zeta^R$ for some $\zeta \in \mathfrak{g}$

$\xi^L = \iota_* \zeta^R$ evaluate at e

$\xi = (T_e\iota)\zeta = -\zeta$. ■

Proposition $\forall \xi, \zeta \in \mathfrak{g}$,

ⓐ $[\xi^L, \zeta^L] = [\xi, \zeta]^L$

ⓑ $[\xi^R, \zeta^R] = -[\xi, \zeta]^R$

ⓒ $[\xi^L, \zeta^R] = 0$

Proof.

ⓐ By def. of $[\cdot,\cdot]$ on $\mathfrak{g}$

ⓑ Apply ι^* to both sides of ⓐ

ⓒ The flows of left/right translation commute by associativity! ■

The exponential map

Defn (exponential map)

$$\exp : \mathfrak{g} \longrightarrow G$$
$$\xi \longmapsto \gamma^{\xi}(1)$$

Proposition If $\xi, \zeta \in \mathfrak{g}$, $[\xi, \zeta] = 0$, then

$$\exp(\xi + \zeta) = \exp(\xi)\exp(\zeta)$$

Proof. $[\xi, \zeta] = 0 \Rightarrow [\xi^L, \zeta^L] = [\xi, \zeta]^L = 0$, hence their flows commute. So $t \mapsto \Phi_t^{\xi} \circ \Phi_t^{\zeta}$ is again a flow. Taking derivatives we see that this is $(\xi + \zeta)^L$.

$$\Phi^{\xi}_{-t} \circ \Phi^{\zeta}_{-t} = \Phi^{\xi+\zeta}_{-t}$$

Evaluate both sides at e:

Prop exp is functorial.

$$\begin{tikzcd} G \arrow[r, "\varphi"] & H \\ \mathfrak{g} \arrow[u, "\exp"] \arrow[r, "T_e\varphi"'] \arrow[ur, phantom, "\circlearrowright"] & \mathfrak{h} \arrow[u, "\exp"'] \end{tikzcd}$$

Proof. Given $\xi \in \mathfrak{g}$, let $\xi' = (T_e\varphi)(\xi)$. We have $\xi^L \sim_\varphi (\xi')^L$

So φ takes integral curves for ξ^L to those for $(\xi')^L$:

$$\varphi(\gamma^{\xi^L}(t)) = \gamma^{(\xi')^L}(t)$$

$$\underset{ev_1}{\rightsquigarrow} \varphi(\exp(\xi)) = \exp(\xi').$$ ■

Thm If $G \subseteq GL(n,\mathbb{R})$ is a matrix Lie group, then $\exp : \mathfrak{g} = T_eG \to G$ is given by

$$\exp(\xi) = \sum_{n=0}^{\infty} \frac{1}{n!}\xi^n$$

Proof. It is sufficient by functoriality applied to inclusion to prove only when $G = GL(n,\mathbb{R})$. Note $t \mapsto \sum_{n=0}^{\infty} \frac{t^n}{n!}\xi^n = \exp_{mat}(t\xi)$ is a one-parameter subgroup, $\varphi^\zeta(t)$ for some $\zeta \in \mathfrak{g}$. Taking the derivative at 0, we see $\dot\varphi^\zeta(0) = \frac{d}{dt}\big|_{t=0} \exp_{mat}(t\xi) = \xi$.

So $\varphi^\xi(t) = \exp_{mat}(t\xi)$. Setting $t=1$ finishes the proof. ■

Lemma $\boxed{\gamma^{\xi}(t) = \exp(t\xi)}$

Proof. Fix $s \in \mathbb{R}$. $t \mapsto \gamma^{\xi}(st)$ is a 1-parameter subgroup so equals $\gamma^{\zeta}(t)$ for some ζ. $\frac{d}{dt}\big|_{t=0} \gamma^{\xi}(st) = s \cdot \frac{d}{dt}\big|_{r=0} \gamma^{\xi}(r) = s\xi$, so $\gamma^{\xi}(st) = \gamma^{s\xi}(t)$, so $\gamma^{\xi}(t) = \gamma^{t\xi}(1) = \exp(t\xi)$. ■

Thm For $G \leq GL(n,\mathbb{R})$, the Lie bracket on $\mathfrak{g} = T_eG$ is the commutator of matrices

Proof. Again, it is enough to show for $G = GL(n,\mathbb{R})$

We will use the "group commutator" of $g,h \in G$:

$$ghg^{-1}h^{-1}$$

which is trivial if and only if g and h commute.

For matrices $\xi, \eta \in \mathfrak{g} \leq GL(n,\mathbb{R})$ and $t,s \in \mathbb{R}$

$$[\exp(t\xi), \exp(s\eta)] = \exp(t\xi)\exp(s\eta)\exp(-t\xi)\exp(-s\eta)$$

$$= \mathbb{1} + t(\;0\;) + s(\;0\;) + t^2(\;0\;) + s^2(\;0\;)$$

$$+\, ts(\xi\eta - \eta\xi) + \text{h.o.t.}$$

$\xi\eta - \xi\eta - \eta\xi + \xi\eta$

$\frac{t^2\xi^2}{2} + \frac{(-t)^2\xi^2}{2} - t^2\xi^2$

For general G, consider the representation of G on $C^\infty(G)$

$$\rho(g) = R_g^* : C^\infty(G) \longrightarrow C^\infty(G)$$

$$(\rho(g)f)(a) = f(a \cdot g)$$

Recall: the flow of ξ^L is $t \mapsto R_{\exp(-t\xi)}$, i.e.

$$\xi^L = \frac{d}{dt}\Big|_{t=0} R^*_{\exp(t\xi)} = \frac{d}{dt}\Big|_{t=0} \rho(\exp(t\xi))$$

$$\Rightarrow \frac{d}{dt}\rho(\exp(t\xi)) = \rho(\exp(t\xi)) \circ \xi^L \quad (\text{on } C^\infty(G))$$

$$\Rightarrow \rho(\exp(t\xi)) = \mathbb{1} + t\xi^L + \frac{t^2}{2}(\xi^L)^2 + \cdots$$

$$\Rightarrow \rho(\exp(t\xi)\exp(s\eta)\exp(-t\xi)\exp(-s\eta))$$

$$= \mathbb{1} + st(\xi^L\eta^L - \eta^L\xi^L) + \text{h.o.t}$$

$$= \mathbb{1} + st\,[\xi^L, \eta^L] + \cdots$$

$$= \exp(st[\xi,\eta]^L) + \cdots$$

Assuming $G = GL(n,\mathbb{R})$, we have

$$\rho(\exp(t\xi)\exp(s\eta)\exp(-t\xi)\exp(-s\eta)) = \mathbb{1} + st\,[\xi,\eta]^L + \text{h.o.t.}$$

On the other hand, this is also $\rho(\exp(st(\xi\eta - \eta\xi)) + \text{cubic})$

$$= \mathbb{1} + st(\xi\eta - \eta\xi)^L + \text{cubic}$$

$$\Rightarrow [\xi,\eta] = \xi\eta - \eta\xi. \quad \blacksquare$$

In particular, for $G \subseteq GL(n,\mathbb{R})$, $\mathfrak{g} = T_eG$ is closed under the commutator.

"Canonical" Charts

Just as for $GL(n,\mathbb{R})$, we can use exp to construct charts. This follows from:

Proposition The differential of $\exp: \mathfrak{g} \to G$ at 0 is the identity: $T_0\exp : T_0\mathfrak{g} \cong \mathfrak{g} \xrightarrow{\sim} \mathfrak{g}$

Proof. Let $\gamma(t) = t\xi$ represent $\xi \in \mathfrak{g} \cong T_0\mathfrak{g}$. Then

$$\exp(\gamma(t)) = \exp(t\xi) = \gamma^{\xi}(t)$$

By definition, $\left.\frac{d}{dt}\right|_{t=0} \exp(\gamma(t)) = \left.\frac{d}{dt}\right|_{t=0} \gamma^{\xi}(t) = \left.\xi^L\right|_e = \xi.$ ∎

More facts about the exponential map

- For $G = SU(n), SO(n), U(n)$, exp is surjective (the same for any compact, connected group).
- Also for $GL(n,\mathbb{C})$
- $G \subseteq GL(n,\mathbb{C})$ is <u>unipotent</u> if $\forall g \in G$, $(g - \mathbb{1})^m = 0$ for some $m \in \mathbb{N}$.

E.g. Upper triangular matrices with 1's on the diagonal.
In this case, $\exp: \mathfrak{g} \to G$ is a diffeomorphism.

- $G = SL(2,\mathbb{R})$, $\exp: \mathfrak{sl}(2,\mathbb{R}) \to SL(2,\mathbb{R})$ is not surjective.

Ex $g = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}$ is not in the image of exp.