BIRKHOFF NORMAL FORM FOR GRAVITY WATER WAVES

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Abstract. We consider the gravity water waves system with a one-dimensional periodic interface in infinite depth, and present the proof of the rigorous reduction of these equations to their cubic Birkhoff normal form [4]. This confirms a conjecture of Zakharov-Dyachenko [14] based on the formal Birkhoff integrability of the water waves Hamiltonian truncated at degree four. As a consequence, we also obtain a long-time stability result: periodic perturbations of a flat interface that are of size ε in a sufficiently smooth Sobolev space lead to solutions that remain regular and small up to times of order ε^{-3}.

1. Introduction

1.1. **The equations.** We consider an incompressible and irrotational perfect fluid, under the action of gravity, occupying at time t a two dimensional domain with infinite depth, periodic in the horizontal variable, given by

$$\mathcal{D}_\eta := \big\{(x,y)\in\mathbb{T}\times\mathbb{R}\,;\ -\infty<y<\eta(t,x)\big\}\,,\qquad \mathbb{T}:=\mathbb{R}/(2\pi\mathbb{Z})\,.$$

The time-evolution of the fluid is determined by a system of equations for the free surface $\eta(t,x)$ and the function $\psi(t,x):=\Phi(t,x,\eta(t,x))$, where Φ is the velocity potential in the fluid domain. According to [13, 6] the (η,ψ) variables evolve under

$$\partial_t\eta = G(\eta)\psi\,,\qquad \partial_t\psi = -g\eta-\frac12\psi_x^2+\frac12\frac{(\eta_x\psi_x+G(\eta)\psi)^2}{1+\eta_x^2}\,,\tag{1.1}$$

where $G(\eta)\psi := (\partial_y\Phi-\eta_x\partial_x\Phi)(t,x,\eta(t,x))$ is called the Dirichlet-Neumann operator. Without loss of generality, we set the gravity constant to $g=1$. It was first observed by Zakharov [13] that (1.1) are the Hamiltonian system $\partial_t\eta=\nabla_\psi H(\eta,\psi)$, $\partial_t\psi=-\nabla_\eta H(\eta,\psi)$ where ∇ denotes the L^2-gradient, with Hamiltonian

$$H(\eta,\psi):=\frac12\int_{\mathbb{T}}\psi\,G(\eta)\psi\,dx+\frac12\int_{\mathbb{T}}\eta^2\,dx\tag{1.2}$$

given by the sum of the kinetic and potential energy of the fluid. The mass $\int_{\mathbb{T}}\eta(x)dx$ is a prime integral and the subspace $\int_{\mathbb{T}}\eta(x)dx=\int_{\mathbb{T}}\psi(x)dx=0$ is invariant under the evolution of (1.1).

We denote by $H^s:=H^s(\mathbb{T})$, $s\in\mathbb{R}$, the standard Sobolev spaces of 2π-periodic functions of x, and, we consider the flow of (1.1) on the phase space $H_0^s\times\dot H^s$, where H_0^s is the subspace of H^s of zero average functions, and $\dot H^s$ is the homogeneous Sobolev space.

The aim of this note is to present the results obtained in [4], concerning a rigorous proof of a conjecture of Zakharov-Dyachenko [14], confirmed in Craig-Worfolk [7], on the approximate integrability of the water waves system (1.1), see Theorems 2.1 and 2.2.

1.2. **The formal Birkhoff normal form Hamiltonian of** [14, 7]**.** Consider the Hamiltonian (1.2), introduce the complex variable $u:=\frac{1}{\sqrt2}|D|^{-\frac14}\eta+\frac{\mathrm{i}}{\sqrt2}|D|^{\frac14}\psi$, where $D:=-\mathrm{i}\partial_x$, and let $H_{\mathbb{C}}$ be the Hamiltonian expressed in $(u,\overline u)$. By a Taylor expansion of the Dirichlet-Neumann operator for small η, see [6], one can write $H_{\mathbb{C}}=H_{\mathbb{C}}^{(2)}+H_{\mathbb{C}}^{(3)}+H_{\mathbb{C}}^{(4)}+\cdots$ where

$$H_{\mathbb{C}}^{(2)}=\sum_{j\in\mathbb{Z}\setminus\{0\}}\omega(j)u_j\overline{u_j}\,,\quad \omega(j):=\sqrt{|j|},\qquad H_{\mathbb{C}}^{(3)}=\sum_{\sigma_1j_1+\sigma_2j_2+\sigma_3j_3=0}H^{\sigma_1,\sigma_2,\sigma_3}_{j_1,j_2,j_3}u_{j_1}^{\sigma_1}u_{j_2}^{\sigma_2}u_{j_3}^{\sigma_3}\,,$$

and $H_{\mathbb{C}}^{(4)}$ is a polynomial of order four in $(u,\overline u)$. Here u_j, $j\in\mathbb{Z}\setminus\{0\}$, denotes the j-th Fourier coefficient of u, $\sigma_j=\pm$ are signs and we denote $u_j^+=u_j, u_j^-=\overline{u_j}$. Notice that in this Taylor expansion there is a

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priori no control on the boundedness of the Hamiltonian vector fields associated to $H^{(\ell)}_{\mathbb{C}}$, $\ell = 3, 4, \ldots$. Applying the usual Birkhoff normal form procedure for Hamiltonian systems, it is possible to find a *formal symplectic transformation* Φ such that

$$H_{\mathbb{C}} \circ \Phi = H^{(2)}_{\mathbb{C}} + H^{(4)}_{ZD} + \cdots \tag{1.3}$$

where all monomials of homogeneity 3 have been eliminated due to the *absence of* 3*-waves resonant interactions*, that is, non-zero integer solutions of

$$\sigma_1\omega(j_1) + \sigma_2\omega(j_2) + \sigma_3\omega(j_3) = 0\,, \qquad \sigma_1 j_1 + \sigma_2 j_2 + \sigma_3 j_3 = 0\,, \tag{1.4}$$

and the Hamiltonian $H^{(4)}_{ZD}$ of order 4 is supported only on *Birkhoff resonant quadruples*, i.e.

$$H^{(4)}_{ZD} = \sum_{\substack{\sigma_1 j_1+\sigma_2 j_2+\sigma_3 j_3+\sigma_4 j_4=0,\\ \sigma_1\omega(j_1)+\sigma_2\omega(j_2)+\sigma_3\omega(j_3)+\sigma_4\omega(j_4)=0}} H^{\sigma_1,\sigma_2,\sigma_3,\sigma_4}_{j_1,j_2,j_3,j_4} u^{\sigma_1}_{j_1} u^{\sigma_2}_{j_2} u^{\sigma_3}_{j_3} u^{\sigma_4}_{j_4}\,, \qquad H^{\sigma_1,\sigma_2,\sigma_3,\sigma_4}_{j_1,j_2,j_3,j_4} \in \mathbb{C}\,. \tag{1.5}$$

As observed in [14], there are many solutions to the constraints for the sum in (1.5). For example, if $\sigma_1 = \sigma_3 = 1 = -\sigma_2 = -\sigma_4$, and up to permutations, there are trivial solutions of the form (k, k, j, j) which give rise to benign integrable monomials $|u_k|^2|u_j|^2$, and the two parameter family of solutions, called *Benjamin-Feir resonances*,

$$\bigcup_{\lambda\in\mathbb{Z}\setminus\{0\},\, b\in\mathbb{N}} \Big\{ j_1 = -\lambda b^2,\ j_2 = \lambda(b+1)^2\,,\ j_3 = \lambda(b^2+b+1)^2,\ j_4 = \lambda(b+1)^2 b^2 \Big\}\,. \tag{1.6}$$

As a consequence, one could expect, a priori, the presence in (1.5) of non-integrable monomials supported on the frequencies (1.6). The striking property proved in [14], see also [7], is that the coefficients $H^{\sigma_1,\sigma_2,\sigma_3,\sigma_4}_{j_1,j_2,j_3,j_4}$ in (1.5) for frequencies in (1.6) are actually all zero. The consequence of this "null condition" of the gravity water waves system in infinite depth is the following remarkable result:

Theorem 1.1. (Formal integrability at order four [14, 7]**).** *The Hamiltonian $H^{(4)}_{ZD}$ in* (1.5) *has the form* (2.8)*. The Hamiltonian $H_{ZD} := H^{(2)}_{ZD} + H^{(4)}_{ZD}$ is integrable, possesses the actions $|u_n|^2$, $n \in \mathbb{Z}\setminus\{0\}$ as prime integrals, and, in particular, the flow of H_{ZD} preserves all Sobolev norms.*

Unfortunately, this striking result is a purely formal calculation because the transformation Φ in (1.3) is not bounded and invertible, and there is no control on the higher order remainder terms. Thus, no actual relation can be established between the flow of H (which is well-posed, at least for short times) and that of $H_{\mathbb{C}} \circ \Phi$.

2. Statements of the results

We denote the horizontal and vertical components of the velocity field at the free interface by

$$V := \psi_x - \eta_x B\,, \qquad B := \frac{G(\eta)\psi + \eta_x\psi_x}{1+\eta_x^2}\,, \tag{2.1}$$

and the "good unknown" of Alinhac by

$$\omega := \psi - T_B\eta\,, \tag{2.2}$$

where $T_a b$ denotes the paraproduct operator of Bony using the Weyl quantization[1]. To state our first main result let us assume that, for some $T > 0$, we have a classical solution $(\eta,\psi) \in C^0([-T,T]; H_0^{N+\frac14} \times \dot H^{N+\frac14})$ of the Cauchy problem for (1.1). The existence of such a solution is guaranteed by the local well-posedness theorem of Alazard-Burq-Zuily [1] under the regularity assumption $(\eta,\psi,V,B)(0) \in X^{N-\frac14}$ where we denote $X^s := H_0^{s+\frac12} \times \dot H^{s+\frac12} \times H^s \times H^s$. Define the complex scalar unknown

$$u := \frac{1}{\sqrt2}|D|^{-\frac14}\eta + \frac{\mathrm{i}}{\sqrt2}|D|^{\frac14}\omega \in C^0([-T,T]; H^N_0)\,. \tag{2.4}$$

[1] More in general, for a symbol $a = a(x,\xi)$, $x\in\mathbb{T}, \xi\in\mathbb{R}$, and $u \in L^2(\mathbb{T})$, we set

$$T_{a(x,\xi)}u := Op^{\mathrm{BW}}(a)u := \frac{1}{\sqrt{2\pi}}\sum_{k\in\mathbb{Z}}\Big(\sum_{j\in\mathbb{Z}} \widehat a\big(k-j, \frac{k+j}{2}\big)\chi\big(\frac{k-j}{|k+j|}\big)\,\widehat u(j)\Big)\frac{e^{\mathrm{i}kx}}{\sqrt{2\pi}} \tag{2.3}$$

where $\widehat a$ denotes the Fourier transform in x and χ is an even smooth cutoff function supported on $[-10^{-2}, 10^{-2}]$.

Theorem 2.1. (Birkhoff normal form). *There exist* $N \gg K \gg 1$ *and* $0 < \overline{\varepsilon} \ll 1$, *such that, if*

$$\sup_{t\in[-T,T]} \sum_{k=0}^{K} \|\partial_t^k u(t)\|_{\dot H^{N-k}(\mathbb{T})} \le \overline{\varepsilon}\,, \tag{2.5}$$

then there exist a bounded and invertible transformation $\mathfrak{B} = \mathfrak{B}(u)$ *of* $\dot H^N(\mathbb{T})$, *which depends (nonlinearly) on* u, *and a constant* $C := C(N) > 0$ *such that*

$$\|\mathfrak{B}(u)\|_{\mathcal{L}(\dot H^N,\dot H^N)} + \|(\mathfrak{B}(u))^{-1}\|_{\mathcal{L}(\dot H^N,\dot H^N)} \le 1 + C\|u\|_{\dot H^N}\,, \tag{2.6}$$

and the variable $z := \mathfrak{B}(u)u$ *satisfies the equation*

$$\partial_t z = -\mathrm{i}\partial_{\overline{z}} H_{ZD}(z,\overline{z}) + \mathcal{X}_{\ge 4} \tag{2.7}$$

where:

(1) *the Hamiltonian* H_{ZD} *has the form* $H_{ZD} = H_{ZD}^{(2)} + H_{ZD}^{(4)}$ *with* $H_{ZD}^{(2)}(z,\overline{z}) := \frac{1}{2}\int_{\mathbb{T}} \big||D|^{\frac14} z\big|^2\,dx$ *and*

$$H_{ZD}^{(4)}(z,\overline{z}) := \frac{1}{4\pi}\sum_{k\in\mathbb{Z}} |k|^3\big(|z_k|^4 - 2|z_k|^2|z_{-k}|^2\big) + \frac{1}{\pi}\sum_{\substack{k_1,k_2\in\mathbb{Z},|k_2|<|k_1|\\ \mathrm{sign}(k_1)=\mathrm{sign}(k_2)}} |k_1||k_2|^2\big(-|z_{-k_1}|^2|z_{k_2}|^2 + |z_{k_1}|^2|z_{k_2}|^2\big)\,; \tag{2.8}$$

(2) $\mathcal{X}_{\ge 4} := \mathcal{X}_{\ge 4}(u,\overline{u},z,\overline{z})$ *is a quartic nonlinear term satisfying the "energy estimate"*

$$\mathrm{Re}\int_{\mathbb{T}} |D|^N \mathcal{X}_{\ge 4}\cdot\overline{|D|^N z}\,dx \lesssim_N \|z\|^5_{\dot H^N(\mathbb{T})}\,. \tag{2.9}$$

The main point of Theorem 2.1 is the construction of the *bounded* and *invertible* transformation $\mathfrak{B}(u)$ which recasts the water waves system (1.1) into the equation (2.7)-(2.9). Its main consequence is to establish a rigorous relation between the flow of the full water waves system (1.1) and the flow of (2.7), which is the sum of the explicit Hamiltonian vector field $-\mathrm{i}\partial_{\overline{z}}H_{ZD}$ plus a remainder of higher homogeneity. This remainder is under full control thanks to the energy estimate (2.9). In particular, since H_{ZD} is *integrable* (see Theorem 1.1) we deduce, by Theorem 2.1, the following long time existence result.

Theorem 2.2. (Long-time existence). *There exists* $s_0 > 0$ *such that, for all* $s \ge s_0$, *there is* $\varepsilon_0 > 0$ *such that, for any initial data* (η_0,ψ_0) *satisfying* $\|(\eta_0,\psi_0,V_0,B_0)\|_{X^s} \le \varepsilon \le \varepsilon_0$ *(the functions* $V_0 := V(\eta_0,\psi_0)$, $B_0 := B(\eta_0,\psi_0)$ *are defined by* (2.1)*), the following holds: there exists a unique classical solution* $(\eta,\psi,V,B) \in C^0([-T_\varepsilon,T_\varepsilon],X^s)$ *of the water waves system* (1.1) *with initial condition* $(\eta,\psi)(0) = (\eta_0,\psi_0)$ *and* $T_\varepsilon \gtrsim \varepsilon^{-3}$, *satisfying* $\sup_{[-T_\varepsilon,T_\varepsilon]}\big(\|(\eta,\psi)\|_{H^s\times H^s} + \|(V,B)\|_{H^{s-1}\times H^{s-1}}\big) \lesssim \varepsilon$.

The existence time $T_\varepsilon = O(\varepsilon^{-3})$ goes well beyond the time of $O(\varepsilon^{-1})$ guaranteed by the local existence theory [5, 11, 1]. It also extends past the natural time scale of $O(\varepsilon^{-2})$ which one expects for non-resonant equations, and that has indeed been achieved for (1.1) in [12, 9, 2, 8]. To our knowledge, this is the first ε^{-3} existence result for water waves in absence of external parameters. For gravity-capillary water waves, an almost global existence result of solutions even in x has been proved in Berti-Delort [3] for most values of the parameters. We remark that Theorem 2.2 is obtained by a different mechanism compared to previous works in the literature, e.g. [8, 10], as it relies on the complete conjugation of (1.1) to (2.7) and not on the use of energies.

3. Sketch of the proof

In Theorem 2.1 we conjugate the water-waves system (1.1) to the equation (2.7)-(2.9) through finitely many well-defined, *bounded* and *invertible* transformations. We now outline the main steps of this procedure. For the sake of simplicity, we will use slightly different notations than in our paper [4].

3.1. **Diagonalization up to smoothing remainders.** We begin our analysis by paralinearizing (1.1), writing it as a system in the complex variable $U := (u,\overline{u})$, where u is given by (2.4). Using the paralinearization of the Dirichlet-Neumann operator in [3], the new system for U is diagonal at the highest order and has the form

$$\partial_t U = T_{A_1(U;x)}\partial_x U + \mathrm{i}T_{A_{1/2}(U;x)}|D|^{1/2}U + \cdots + R(U)U \tag{3.1}$$

where A_1 and $A_{1/2}$ are 2×2 matrices whose coefficients depend on U, with $A_1(U;x) = -\mathrm{diag}(V(U;x),V(U;x))$, $A_{1/2}(U;x) = \mathrm{diag}(-1,1) + O(U)$, "$\cdots$" denote paradifferential operators of order ≤ 0, and $R(U)$ is a matrix of smoothing operators which gain an arbitrary large number ρ of derivatives. Here we are writing $F = F(U;x)$ to emphasize the dependence (through U) on the spatial variable $x \in \mathbb{T}$.

We first diagonalize in $(u, \overline{u})$ the sub-principal operator $T_{A_{1/2}(U;x)}|D|^{1/2}$, and then use an *iterative descent procedure* to diagonalize all the operators of order 0, $-1/2$, and so on, up to order $-\rho$. The outcome of this procedure is an equation of the form

$$\partial_t u = -T_{\mathtt{V}_1+\mathtt{V}_2}\partial_x u - \mathrm{i}T_{1+\mathtt{a}_1+\mathtt{a}_2}|D|^{1/2}u + \cdots + \mathcal{R}(u,\overline{u}) + \mathcal{X}(u,\overline{u}) \tag{3.2}$$

where $\mathtt{V}_1$ and $\mathtt{a}_1$, resp. $\mathtt{V}_2$ and $\mathtt{a}_2$, are linear, resp. quadratic, functions of U, "$\cdots$" denote paradifferential operators of order ≤ 0, $\mathcal{R}$ are smoothing (quadratic and cubic) vector fields which gain ρ-derivatives, and $\mathcal{X}$ are remainder terms satisfying an energy estimate of the form (2.9). From now on we will denote generically with $\mathcal{R}$ and $\mathcal{X}$ terms with these properties.

3.2. **Reduction to constant coefficients and Poincaré-Birkhoff normal forms.** The next step is to reduce the paradifferential operators in (3.2) to be constant-in-x and *integrable*, that is of the form

$$f(U;D)u \qquad \text{with} \qquad f(U;\xi) = \sum_{n\in\mathbb{Z}\setminus\{0\}} f_{n,n}^{+-}(\xi)|u_n|^2, \quad f_{n,n}^{+-}(\xi) \in \mathbb{C}. \tag{3.3}$$

To deal with the quasilinear transport term we conjugate (3.2) by the auxiliary flow Φ^θ of the paradifferential transport equation

$$\partial_\theta \Phi^\theta = \mathcal{A}\Phi^\theta, \quad \Phi^{\theta=0} = \mathrm{Id}, \qquad \mathcal{A} := T_{b(u;x)}\partial_x, \quad b := \frac{\beta}{1+\theta\beta_x}, \tag{3.4}$$

with a real-valued function $\beta(u;x) = \beta_1(u;x) + \beta_2(u;x)$ to be determined. Here β_i, $i = 1,2$, are functions respectively linear and quadratic in u. The flow Φ^θ in (3.4), is well-posed for $\theta \in [0,1]$, bounded and invertible on Sobolev spaces. The conjugation through $\Phi^{\theta=1}$ corresponds to a paradifferential change of variable given by the paracomposition operator associated to the diffeomorphism $x \mapsto x + \beta(u;x)$ of $\mathbb{T}$. In the new variable $v := \Phi^{\theta=1}u$ we obtain an equation of the form

$$\partial_t v = -T_{\mathtt{V}_1+\mathtt{V}_2}\partial_x v - [\partial_t, \mathcal{A}]v + \cdots = -T_{\mathtt{V}_1+\mathtt{V}_2+\beta_t+Q(\beta,\mathtt{V}_1)}\partial_x v + \cdots \tag{3.5}$$

where $Q(\beta, \mathtt{V}_1)$ is a quadratic expression in β and $\mathtt{V}_1$, the "$\cdots$" denote paradifferential operators of order $\leq 1/2$, smoothing remainders and vector fields satisfying (2.9). Notice that the highest order contribution comes from the conjugation of ∂_t because the dispersion relation $-\mathrm{i}|D|^{1/2}$ has sub-linear growth. This creates several difficulties in our Birkhoff normal form reduction compared, for example, to [3] where the dispersion relation is super-linear. In light of (3.5) we look for β_1, β_2 solving $\partial_t(\beta_1+\beta_2) + \mathtt{V}_1 + \mathtt{V}_2 + Q(\beta_1, \mathtt{V}_1) = \zeta(u) + O(u^3)$, where $\zeta(u)$ is constant-in-x. However, in general it is only possible to obtain

$$\partial_t(\beta_1+\beta_2) + \mathtt{V}_1 + \mathtt{V}_2 + Q(\beta_1, \mathtt{V}_1) = \sum_{n\in\mathbb{Z}\setminus\{0\}} \mathtt{V}_{n,n}^{+-}|u_n|^2 + \sum_{n\in\mathbb{Z}\setminus\{0\}} \mathtt{V}_{n,-n}^{+-} u_n \overline{u_{-n}} e^{\mathrm{i}2nx} + O(u^3),$$

where $\mathtt{V}_{n_1 n_2}^{+-}$ are some coefficients depending on V. We then verify the essential cancellation[2] $\mathtt{V}_{n,-n}^{+-} \equiv 0$, and reduce the equation (3.5) to the form

$$\partial_t v = -\zeta(u)\partial_x v - \mathrm{i}T_{1+\mathtt{a}_2^{(1)}}|D|^{1/2}v + \cdots + \mathcal{R} + \mathcal{X}, \qquad \zeta(u) := \frac{1}{\pi}\sum_{n\in\mathbb{Z}\setminus\{0\}} n|n||u_n|^2, \tag{3.6}$$

that, at highest order, has only Birkhoff resonant cubic vector field monomials.

Using the flow (bounded and invertible) generated by a paradifferential "semi-Fourier integral operator" $\mathcal{A} = \mathrm{i}T_{\beta(u)}|D|^{\frac{1}{2}}$, for a suitable real function β, we also reduce to constant coefficients – and in Birkhoff normal form – the dispersive term. Additional algebraic cancellations, which appear to be intrinsic to the water waves system (1.1), show that the new dispersive term is exactly $-\mathrm{i}|D|^{\frac{1}{2}}$. All paradifferential operators of order ≤ 0 are also reduced to constant coefficients – and in Poincaré-Birkhoff normal form – using flows generated by Banach space ODEs. Eventually we obtain the equation

$$\partial_t z = -\zeta(z)\partial_x z - \mathrm{i}|D|^{\frac{1}{2}}z + r_{-1/2}(z;D)[z] + \mathcal{R} + \mathcal{X} \tag{3.7}$$

[2] This can also be deduced using invariance properties of (1.1) such as the reversibility and preservation of the subspace of even functions.

where $r_{-1/2}$ is an integrable symbol of order $-1/2$. Notice that (3.7) is in cubic Poincaré-Birkhoff-normal form (it is not Hamiltonian, since we performed non-symplectic transformations) up to the smoothing (quadratic and cubic) vector fields $\mathcal{R}$, and an admissible remainder $\mathcal{X}$ which satisfies (2.9).

3.3. **Poincaré-Birkhoff normal forms.** Next, we apply *Poincaré-Birkhoff normal form transformations*, generated by the flow of Banach space ODEs, to eliminate the non-resonant quadratic and cubic nonlinear terms in $\mathcal{R}$, arriving at

$$\begin{aligned} &\partial_t z = -\zeta(z)\partial_x z - \mathrm{i}|D|^{\frac{1}{2}} z + r_{-1/2}(z;D)[z] + \mathcal{R}^{\mathrm{res}}(z) + \mathcal{X} \\ &\mathcal{R}^{\mathrm{res}}(z) := \sum_{\substack{\sigma_1 n_1 + \sigma_2 n_2 + \sigma_3 n_3 = n, \\ \sigma_1\omega(n_1)+\sigma_2\omega(n_2)+\sigma_3\omega(n_3)=\omega(n)}} c^{\sigma_1\sigma_2,\sigma_3}_{n_1,n_2,n_3} \, z^{\sigma_1}_{n_1} z^{\sigma_2}_{n_2} z^{\sigma_3}_{n_3} e^{\mathrm{i}nx} \,. \end{aligned} \tag{3.8}$$

In the construction of these transformations we see the appearance of the divisor $\sigma_1\omega(n_1)+\sigma_2\omega(n_2)+\sigma_3\omega(n_3)-\omega(n) \neq 0$. Notice that this expression may degenerate rapidly close to a resonances, such as in the case $\sigma_1 = 1 = \sigma_3$, $\sigma_2 = -1$, and $n_1 = k$, $n_2 = -k$, $n_3 = j$, $n = j+2k$, with $j \gg k$, which gives $|\omega(n_1)-\omega(n_2)+\omega(n_3)-\omega(n)| \approx j^{-1/2}$. The loss of derivatives induced by these near resonances is compensated by the smoothing nature of the remainder $\mathcal{R}$. Also notice that the presence of the non-trivial 4-waves Benjamin-Feir resonances (1.6) in the normal form (3.8) constitutes a potentially strong obstruction to controlling the dynamics for times of $O(\varepsilon^{-3})$.

3.4. **Normal form identification.** One could expect, in analogy with Theorem 1.1, to be able to check by direct computations that the coefficients $c^{\sigma_1\sigma_2\sigma_3}_{n_1,n_2,n_3}$ in (3.8) vanish on the Benjamin-Feir resonances. However, after having performed all the (non-symplectic) reductions described above, such a computation appears rather involved. We then prove this vanishing property through a novel *uniqueness argument for the cubic Poincaré-Birkhoff normal form.* This argument, which relies on the uniqueness of solutions of the quadratic homological equation (1.4), shows that the cubic terms in (3.8) coincide with the Hamiltonian vector field of (2.8):

$$-\zeta(z)\partial_x z + r_{-1/2}(z;D)[z] + \mathcal{R}^{\mathrm{res}}(z) = -\mathrm{i}\partial_{\overline{z}} H^{(4)}_{ZD} \,. \tag{3.9}$$

In particular $\mathcal{R}^{\mathrm{res}}(z)$ is supported only on trivial resonances. Finally, the boundedness properties of all the transformations that we constructed in order to arrive at (3.8), and the identity (3.9), lead to Theorem 2.1.

3.5. **Long-time existence.** Theorem 2.2 follows by the *quintic energy estimate*

$$\|u(t)\|^2_{\dot{H}^N} \leq C_N \|u(0)\|^2_{\dot{H}^N} + C_N \int_0^t \|u(\tau)\|^5_{\dot{H}^N} \, d\tau \,, \tag{3.10}$$

combined with the local existence theory [1] and a standard boostrap argument. The energy estimate (3.10) is obtained by the boundedness of $\mathfrak{B}, \mathfrak{B}^{-1}$ in (2.6), which give $\|z\|_{\dot{H}^N} \approx \|u\|_{\dot{H}^N}$ (provided $\|u\|_{\dot{H}^N} \ll 1$), the equation (2.7) for z, the integrability of the Hamiltonian (2.8), and the control (2.9) on the remainders.

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