

GLOBAL SOLUTIONS OF THE GRAVITY-CAPILLARY WATER WAVE SYSTEM IN 3 DIMENSIONS, I: ENERGY ESTIMATES

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ABSTRACT. In this paper and its companion [32] we prove global regularity for the full water waves system in 3 dimensions for small data, under the influence of both gravity and surface tension. The main difficulties are the weak, and far from integrable, pointwise decay of solutions, together with the presence of a full codimension one set of quadratic resonances. To overcome these difficulties we use a combination of improved energy estimates and dispersive analysis.

In this paper we prove the energy estimates, while the dispersive estimates are proved in [32]. These energy estimates depend on several new ingredients, such as a key non-degeneracy property of the resonant hypersurfaces and some special structure of the quadratic part of the nonlinearity.

CONTENTS

1. Introduction	1
2. The main propositions	11
3. Energy estimates, I: the scalar equation and strongly semilinear structures	16
4. Energy estimates, II: setup and the main L^2 lemma	27
5. Energy estimates, III: proof of Proposition 4.1	32
6. Proof of the main L^2 lemma	42
7. The function Υ	48
8. Paradifferential calculus	54
9. The Dirichlet-Neumann operator	64
References	74

1. INTRODUCTION

The study of the motion of water waves, such as those on the surface of the ocean, is a classical question, and one of the main problems in fluid dynamics. The origins of water waves theory can be traced back¹ at least to the work of Laplace and Lagrange, Cauchy [11] and Poisson, and then Russel, Green and Airy, among others. Classical studies include those by Stokes [64], Levi-Civita [55] and Struik [62] on progressing waves, the instability analysis of Taylor [66], the works on solitary waves by Friedrichs and Hyers [33], and on steady waves by Gerber [34].

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¹We refer to the review paper of Craik [27], and references therein, for more details about these early studies on the problem.

The main questions one can ask about water waves are the typical ones for any physical evolution problem: the local-in-time wellposedness of the Cauchy problem, the regularity of solutions and the formation of singularities, the existence of special solutions (such as solitary waves) and their stability, and the global existence and long-time behavior of solutions. There is a vast body of literature dedicated to all of these aspects. As it would be impossible to give exhaustive references, we will mostly mention works that are connected to our results, and refer to various books and review papers for others.

Our main interest here is the existence of global solutions for the initial value problem. In particular, we will consider the full irrotational water waves problem for a three dimensional fluid occupying a region of infinite depth and infinite extent below the graph of a function. This is a model for the motion of waves on the surface of the deep ocean. We will consider such dynamics under the influence of the gravitational force and surface tension acting on particles at the interface. Our main result is the existence of global classical solutions for this problem, for sufficiently small initial data.

1.1. Free boundary Euler equations and water waves. The evolution of an inviscid perfect fluid that occupies a domain $\Omega_t \subset \mathbb{R}^n$, for $n \geq 2$, at time $t \in \mathbb{R}$, is described by the free boundary incompressible Euler equations. If v and p denote respectively the velocity and the pressure of the fluid (with constant density equal to 1) at time t and position $x \in \Omega_t$, these equations are

$$(\partial_t + v \cdot \nabla)v = -\nabla p - ge_n, \quad \nabla \cdot v = 0, \quad x \in \Omega_t, \quad (1.1)$$

where g is the gravitational constant. The first equation in (1.1) is the conservation of momentum equation, while the second is the incompressibility condition. The free surface $S_t := \partial\Omega_t$ moves with the normal component of the velocity according to the kinematic boundary condition

$$\partial_t + v \cdot \nabla \text{ is tangent to } \bigcup_t S_t \subset \mathbb{R}_{x,t}^{n+1}. \quad (1.2)$$

The pressure on the interface is given by

$$p(x, t) = \sigma\kappa(x, t), \quad x \in S_t, \quad (1.3)$$

where κ is the mean-curvature of S_t and $\sigma \geq 0$ is the surface tension coefficient. At liquid-air interfaces, the surface tension force results from the greater attraction of water molecules to each other than to the molecules in the air.

One can consider the free boundary Euler equations (1.1)-(1.3) in various types of domains Ω_t (bounded, periodic, unbounded) and study flows with different characteristics (rotational/irrotational, with gravity and/or surface tension), or even more complicated scenarios where the moving interface separates two fluids.

In the case of irrotational flows, $\text{curl } v = 0$, one can reduce (1.1)-(1.3) to a system on the boundary. Indeed, assume also that $\Omega_t \subset \mathbb{R}^n$ is the region below the graph of a function $h : \mathbb{R}_x^{n-1} \times I_t \rightarrow \mathbb{R}$, that is

$$\Omega_t = \{(x, y) \in \mathbb{R}^{n-1} \times \mathbb{R} : y \leq h(x, t)\} \quad \text{and} \quad S_t = \{(x, y) : y = h(x, t)\}.$$

Let Φ denote the velocity potential, $\nabla_{x,y}\Phi(x, y, t) = v(x, y, t)$, for $(x, y) \in \Omega_t$. If $\phi(x, t) := \Phi(x, h(x, t), t)$ is the restriction of Φ to the boundary S_t , the equations of motion reduce to the following system for the unknowns $h, \phi : \mathbb{R}_x^{n-1} \times I_t \rightarrow \mathbb{R}$:

$$\begin{cases} \partial_t h = G(h)\phi, \\ \partial_t \phi = -gh + \sigma \operatorname{div} \left[\frac{\nabla h}{(1 + |\nabla h|^2)^{1/2}} \right] - \frac{1}{2} |\nabla \phi|^2 + \frac{(G(h)\phi + \nabla h \cdot \nabla \phi)^2}{2(1 + |\nabla h|^2)}. \end{cases} \quad (1.4)$$

Here

$$G(h) := \sqrt{1 + |\nabla h|^2} \mathcal{N}(h), \quad (1.5)$$

and $\mathcal{N}(h)$ is the Dirichlet-Neumann map associated to the domain Ω_t . Roughly speaking, one can think of $G(h)$ as a first order, non-local, linear operator that depends nonlinearly on the domain. We refer to [65, chap. 11] or the book of Lannes [54] for the derivation of (1.4). For sufficiently small smooth solutions, this system admits the conserved energy

$$\begin{aligned} \mathcal{H}(h, \phi) &:= \frac{1}{2} \int_{\mathbb{R}^{n-1}} G(h) \phi \cdot \phi \, dx + \frac{g}{2} \int_{\mathbb{R}^{n-1}} h^2 \, dx + \sigma \int_{\mathbb{R}^{n-1}} \frac{|\nabla h|^2}{1 + \sqrt{1 + |\nabla h|^2}} \, dx \\ &\approx \left\| |\nabla|^{1/2} \phi \right\|_{L^2}^2 + \left\| (g - \sigma \Delta)^{1/2} h \right\|_{L^2}^2, \end{aligned} \quad (1.6)$$

which is the sum of the kinetic energy corresponding to the L^2 norm of the velocity field and the potential energy due to gravity and surface tension. It was first observed by Zakharov [76] that (1.4) is the Hamiltonian flow associated to (1.6).

One generally refers to (1.4) as the gravity water waves system when $g > 0$ and $\sigma = 0$, as the capillary water waves system when $g = 0$ and $\sigma > 0$, and as the gravity-capillary water waves system when $g > 0$ and $\sigma > 0$.

1.2. The main theorem. Our results in this paper and [32] concern the gravity-capillary water waves system, in the case $n = 3$. In this case h and ϕ are real-valued functions defined on $\mathbb{R}^2 \times I$.

To state our main theorem we first introduce some notation. The rotation vector-field

$$\Omega := x_1 \partial_{x_2} - x_2 \partial_{x_1} \quad (1.7)$$

commutes with the linearized system. For $N \geq 0$ let H^N denote the standard Sobolev spaces on \mathbb{R}^2 . More generally, for $N, N' \geq 0$ and $b \in [-1/2, 1/2]$, $b \leq N$, we define the norms

$$\|f\|_{H_\Omega^{N', N}} := \sum_{j \leq N'} \|\Omega^j f\|_{H^N}, \quad \|f\|_{\dot{H}^{N, b}} := \left\| (|\nabla|^N + |\nabla|^b) f \right\|_{L^2}. \quad (1.8)$$

For simplicity of notation, we sometimes let $H_\Omega^{N'} := H_\Omega^{N', 0}$. Our main theorem is the following:

Theorem 1.1 (Global Regularity). *Assume that $g, \sigma > 0$, $\delta > 0$ is sufficiently small, and N_0, N_1, N_3, N_4 are sufficiently large² (for example $\delta = 1/2000$, $N_0 := 4170$, $N_1 := 2070$, $N_3 := 30$, $N_4 := 70$, compare with Definition 2.5). Assume that the data (h_0, ϕ_0) satisfies*

$$\begin{aligned} \|\mathcal{U}_0\|_{H^{N_0} \cap H_\Omega^{N_1, N_3}} + \sup_{2m + |\alpha| \leq N_1 + N_4} \|(1 + |x|)^{1-50\delta} D^\alpha \Omega^m \mathcal{U}_0\|_{L^2} &= \varepsilon_0 \leq \bar{\varepsilon}_0, \\ \mathcal{U}_0 &:= (g - \sigma \Delta)^{1/2} h_0 + i |\nabla|^{1/2} \phi_0, \end{aligned} \quad (1.9)$$

where $\bar{\varepsilon}_0$ is a sufficiently small constant and $D^\alpha = \partial_1^{\alpha^1} \partial_2^{\alpha^2}$, $\alpha = (\alpha^1, \alpha^2)$. Then, there is a unique global solution $(h, \phi) \in C([0, \infty) : H^{N_0+1} \times \dot{H}^{N_0+1/2, 1/2})$ of the system (1.4), with $(h(0), \phi(0)) = (h_0, \phi_0)$. In addition

$$(1+t)^{-\delta^2} \|\mathcal{U}(t)\|_{H^{N_0} \cap H_\Omega^{N_1, N_3}} \lesssim \varepsilon_0, \quad (1+t)^{5/6-3\delta^2} \|\mathcal{U}(t)\|_{L^\infty} \lesssim \varepsilon_0, \quad (1.10)$$

for any $t \in [0, \infty)$, where $\mathcal{U} := (g - \sigma \Delta)^{1/2} h + i |\nabla|^{1/2} \phi$.

²The values of N_0 and N_1 , the total number of derivatives we assume under control, can certainly be decreased by reworking parts of the argument. We prefer, however, to simplify the argument wherever possible instead of aiming for such improvements. For convenience, we arrange that $N_1 - N_4 = (N_0 - N_3)/2 - N_4 = 1/\delta$.

Remark 1.2. (i) One can derive additional information about the global solution (h, ϕ) . Indeed, by rescaling we may assume that $g = 1$ and $\sigma = 1$. Let

$$\mathcal{U}(t) := (1 - \Delta)^{1/2}h + i|\nabla|^{1/2}\phi, \quad \mathcal{V}(t) := e^{it\Lambda}\mathcal{U}(t), \quad \Lambda(\xi) := \sqrt{|\xi| + |\xi|^3}. \quad (1.11)$$

Here Λ is the linear dispersion relation, and \mathcal{V} is the profile of the solution \mathcal{U} . The proof of the theorem gives the strong uniform bound

$$\sup_{t \in [0, \infty)} \|\mathcal{V}(t)\|_Z \lesssim \varepsilon_0, \quad (1.12)$$

see Definition 2.5. The pointwise decay bound in (1.10) follows from this and the linear estimates in Lemma 2.6 below.

(ii) The global solution \mathcal{U} scatters in the Z norm as $t \rightarrow \infty$, i.e. there is $\mathcal{V}_\infty \in Z$ such that

$$\lim_{t \rightarrow \infty} \|e^{it\Lambda}\mathcal{U}(t) - \mathcal{V}_\infty\|_Z = 0.$$

However, the asymptotic behavior is somewhat nontrivial since $|\widehat{\mathcal{U}}(\xi, t)| \gtrsim \log t \rightarrow \infty$ for frequencies ξ on a circle in \mathbb{R}^2 (the set of space-time resonance outputs) and for some data. This unusual behavior is due to the presence of a large set of space-time resonances.

(iii) The function $\mathcal{U} := (g - \sigma\Delta)^{1/2}h + i|\nabla|^{1/2}\phi$ is called the ‘‘Hamiltonian variable’’, due to its connection to the Hamiltonian (1.6). This variable is important in order to keep track correctly of the relative weights of the functions h and ϕ during the proof.

1.3. Background. We now discuss some background on the water waves system and review some of the history and previous work on this problem.

1.3.1. The equations and the local wellposedness theory. The free boundary Euler equations (1.1)-(1.3) are a time reversible system of evolution equations which preserve the total (kinetic plus potential) energy. Under the Rayleigh-Taylor sign condition [66]

$$-\nabla_{n(x,t)}p(x, t) < 0, \quad x \in S_t, \quad (1.13)$$

where n is the outward pointing unit normal to Ω_t , the system has a (degenerate) hyperbolic structure. This structure is somewhat hard to capture because of the moving domain and the quasilinear nature of the problem. Historically, this has made the task of establishing local wellposedness (existence and uniqueness of smooth solutions for the Cauchy problem) non-trivial.

Early results on the local wellposedness of the system include those by Nalimov [57], Yosihara [75], and Craig [22]; these results deal with small perturbations of a flat interface for which (1.13) always holds. It was first observed by Wu [72] that in the irrotational case the Rayleigh-Taylor sign condition holds without smallness assumptions, and that local-in-time solutions can be constructed with initial data of arbitrary size in Sobolev spaces [71, 72].

Following the breakthrough of Wu, in recent years the question of local wellposedness of the water waves and free boundary Euler equations has been addressed by several authors. Christodoulou–Lindblad [14] and Lindblad [56] considered the gravity problem with vorticity, Beyer–Gunther [9] took into account the effects of surface tension, and Lannes [53] treated the case of non-trivial bottom topography. Subsequent works by Coutand–Shkoller [20] and Shatah–Zeng [60, 61] extended these results to more general scenarios with vorticity and surface tension, including two-fluids systems [12, 61] where surface tension is necessary for wellposedness. For some recent papers that include surface tension and/or low regularity analysis see [8, 13, 1, 2, 28].

We remark that because of the physical relevance of the system and the aim of better describing its complex dynamics, many simplified models have been derived and studied in special

regimes. These include the KdV equation, the Benjamin–Ono equation, the Boussinesq and the KP equations, as well as the nonlinear Schrödinger equation. We refer to [22, 58, 7, 67] and to the book [54] and references therein for more about approximate/asymptotic models.

1.3.2. *Previous work on long-time existence.* The problem of long time existence of solutions is more challenging, and fewer results have been obtained so far. As in all quasilinear problems, the long-time regularity has been studied in a perturbative (and dispersive) setting, that is in the regime of small and localized perturbations of a flat interface. Large perturbations can lead to breakdown in finite time, see for example the papers on “splash” singularities [10, 21].

In the perturbative setting the main idea is to use dispersion to control the cumulative effects of nonlinear interactions. The first long-time result for the water waves system (1.4) is due to Wu [73] who showed almost global existence for the gravity problem ($g > 0$, $\sigma = 0$) in two dimensions (1d interfaces). Subsequently, Germain-Masmoudi-Shatah [36] and Wu [74] proved global existence of gravity waves in three dimensions (2d interfaces). Global regularity in 3d was also proved for the capillary problem ($g = 0$, $\sigma > 0$) by Germain-Masmoudi-Shatah [37]. See also the recent work of Wang [69, 70] on the gravity problem in 3d over a finite flat bottom.

Global regularity for the gravity water waves system in 2d (the harder case) has been proved by two of the authors in [46] and, independently, by Alazard-Delort [3, 4]. A different proof of Wu’s 2d almost global existence result was later given by Hunter-Ifrim-Tataru [40], and then complemented to a proof of global regularity in [41]. Finally, Wang [68] proved global regularity for a more general class of small data of infinite energy, thus removing the momentum condition on the velocity field that was present in all the previous 2d results. For the capillary problem in 2d, global regularity was proved by two of the authors in [48] and, independently, by Ifrim-Tataru [42] in the case of data satisfying an additional momentum condition.

1.4. **Main ideas.** The classical mechanism to establish global regularity for quasilinear equations has two main components:

- (1) Propagate control of high frequencies (high order Sobolev norms);
- (2) Prove dispersion/decay of the solution over time.

The interplay of these two aspects has been present since the seminal work of Klainerman [51, 52] on nonlinear wave equations and vector-fields, Shatah [59] on 3d Klein-Gordon and normal forms, Christodoulou-Klainerman [15] on the stability of Minkowski space, and Delort [29] on 1d Klein-Gordon. We remark that even in the weakly nonlinear regime (small perturbations of trivial solutions) smooth and localized initial data can lead to blow-up in finite time, see John [49] on quasilinear wave equations and Sideris [63] on compressible Euler.

In the last few years new methods have emerged in the study of global solutions of quasilinear evolutions, inspired by the advances in semilinear theory. The basic idea is to combine the classical energy and vector-fields methods with refined analysis of the Duhamel formula, using the Fourier transform. This is the essence of the “method of space-time resonances” of Germain-Masmoudi-Shatah [36, 37, 35], see also Gustafson-Nakanishi-Tsai [39], and of the refinements in [43, 44, 38, 45, 46, 47, 48, 31, 30], using atomic decompositions and more sophisticated norms.

The situation we consider in this paper is substantially more difficult, due to the combination of the following factors:

- Strictly less than $|t|^{-1}$ pointwise decay of solutions. In our case, the linear dispersion relation is $\Lambda(\xi) = \sqrt{g|\xi| + \sigma|\xi|^3}$ and the best possible pointwise decay, even for solutions of the linearized equation corresponding to Schwartz initial data, is $|t|^{-5/6}$ (see Fig. 1 below).

- Large set of time resonances. In certain cases one can overcome the slow pointwise decay using the method of normal forms of Shatah [59]. The critical ingredient needed is the absence of time resonances (or at least a suitable “null structure” of the quadratic nonlinearity matching the set of time resonances). Our system, however, has a full (codimension 1) set of time resonances (see Fig. 2 below) and no meaningful null structures.

We remark that *this combination was not present in any of the earlier global regularity results on water waves* described above. More precisely, in all the previous global results in 3 dimensions in [36, 74, 37, 69, 70] it was possible to prove $1/t$ pointwise decay of the nonlinear solutions. This decay allowed for high order energy estimates with slow growth.

On the other hand, in all the previous long term/global results in 2 dimensions in [73, 46, 3, 4, 40, 41, 48, 42, 68] the starting point was an identity of the form

$$\partial_t \mathcal{E}(t) = \text{quartic semilinear term,}$$

where \mathcal{E} is a suitable energy functional and the quartic expression in the right-hand side does not lose derivatives. An energy inequality of this form was first proved by Wu [73] for the gravity water wave model, and led to an almost-global existence result. Such an inequality (which is related to normal form transformations) is possible only when there are no time resonances for the quadratic terms. This is essentially the situation in all the 2D results mentioned above.³

To address these issues, in this paper we use a combination of improved energy estimates and Fourier analysis. The main components of our analysis are:

- The energy estimates, which are used to control high Sobolev norms and weighted norms (corresponding to the rotation vector-field). They rely on several new ingredients, most importantly on a *strongly semilinear structure* of the space-time integrals that control the increment of energy, and on a *restricted nondegeneracy condition* (see (1.24)) of the time resonant hypersurfaces. The strongly semilinear structure is due to an algebraic correlation (see (1.28)) between the size of the multipliers of the space-time integrals and the size of the modulation, and is related to the Hamiltonian structure of the original system.
- The dispersive estimates, which lead to decay and rely on a partial bootstrap argument in a suitable Z norm. We analyze carefully the Duhamel formula, in particular the quadratic interactions related to the slowly decaying frequencies and to the *set of space-time resonances*. The choice of the Z norm in this argument is very important; we use an atomic norm, based on a space-frequency decomposition of the profile of the solution, which depends in a significant way on the location and the shape of the space-time resonant set, thus on the quadratic part of the nonlinearity.

We discuss these main ingredients in detail in a simplified model below.

1.5. A simplified model. To illustrate these ideas, consider the initial-value problem

$$\begin{aligned} (\partial_t + i\Lambda)U &= \nabla V \cdot \nabla U + (1/2)\Delta V \cdot U, & U(0) &= U_0, \\ \Lambda(\xi) &:= \sqrt{|\xi| + |\xi|^3}, & V &:= P_{[-10,10]}\Re U. \end{aligned} \tag{1.14}$$

Compared to the full equation, this model has the same linear part and a quadratic nonlinearity leading to similar resonant sets. It is important that V is real-valued, such that solutions

³More precisely, the only time resonances are at the 0 frequency, but they are cancelled by a suitable null structure. Some additional ideas are needed in the case of capillary waves [48] where certain singularities arise. Moreover, new ideas, which exploit the Hamiltonian structure of the system as in [46], are needed to prove global (as opposed to almost-global) regularity.

of (1.14) satisfy the L^2 conservation law

$$\|U(t)\|_{L^2} = \|U_0\|_{L^2}, \quad t \in [0, \infty). \quad (1.15)$$

The model (1.14) carries many of the difficulties of the real problem and has the advantage that it is much more transparent algebraically. There are, however, significant additional issues when dealing with the full problem, see subsection 1.5.2 below for a short discussion.

The specific dispersion relation $\Lambda(\xi) = \sqrt{|\xi| + |\xi|^3}$ in (1.14) is important. It is radial and has stationary points when $|\xi| = \gamma_0 := (2/\sqrt{3} - 1)^{1/2} \approx 0.393$ (see Figure 1 below). As a result, linear solutions can only have $|t|^{-5/6}$ pointwise decay, i.e.

$$\|e^{it\Lambda}\phi\|_{L^\infty} \approx |t|^{-5/6},$$

even for Schwartz functions ϕ whose Fourier transforms do not vanish on the sphere $\{|\xi| = \gamma_0\}$.

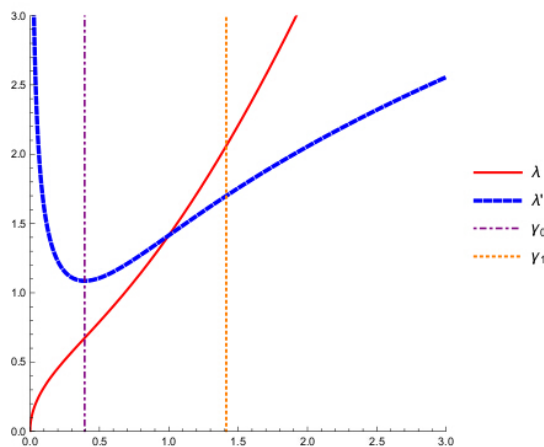


FIGURE 1. The curves represent the dispersion relation $\lambda(r) = \sqrt{r^3 + r}$ and the group velocity λ' , for $g = 1 = \sigma$. For $0 \leq |\xi| \leq \gamma_0$ the dispersion relation is well approximated by the gravity wave dispersion relation $\sqrt{|\xi|}$, while for $\gamma_0 \leq |\xi| < \infty$, the dispersion relation is well approximated by the capillary wave dispersion relation $\sqrt{|\xi|^3}$. The frequency γ_1 corresponds to the space-time resonant sphere. Notice that while the slower decay at γ_0 is due to some degeneracy in the linear problem, γ_1 is unremarkable from the point of view of the linear dispersion.

1.5.1. *Energy estimates.* We would like to control the increment of both high order Sobolev norms and weighted norms for solutions of (1.14). It is convenient to do all the estimates in the Fourier space, using a quasilinear I-method as in [47, 48, 31]. This has similarities with the well-known I-method of Colliander–Keel–Staffilani–Takaoka–Tao [16, 17] used in semilinear problems, and to the energy methods of [35, 4, 40]. Our main estimate is the following partial bootstrap bound:

$$\text{if } \sup_{t \in [0, T]} [(1+t)^{-\delta^2} \mathcal{E}(t)]^{1/2} + \|e^{it\Lambda}U(t)\|_Z \leq \varepsilon_1 \text{ then } \sup_{t \in [0, T]} (1+t)^{-\delta^2} \mathcal{E}(t)^{1/2} \lesssim \varepsilon_0 + \varepsilon_1^{3/2}, \quad (1.16)$$

where U is a solution on $[0, T]$ of (1.14), $\mathcal{E}(t) = \|U(t)\|_{H^N}^2 + \|U(t)\|_{H_\Omega^{N'}}^2$, and the initial data has small size $\sqrt{\mathcal{E}(0)} + \|U(0)\|_Z \leq \varepsilon_0$. The choice of the Z norm here is important; For simplicity, we focus on the high order Sobolev norms, and divide the argument into four steps.

Step 1. For N sufficiently large, let

$$W := W_N := \langle \nabla \rangle^N U, \quad E_N(t) := \int_{\mathbb{R}^2} |\widehat{W}(\xi, t)|^2 d\xi. \quad (1.17)$$

A simple calculation, using the equation and the fact that V is real, shows that

$$\frac{d}{dt} E_N = \int_{\mathbb{R}^2 \times \mathbb{R}^2} m(\xi, \eta) \widehat{W}(\eta) \widehat{W}(-\xi) \widehat{V}(\xi - \eta) d\xi d\eta, \quad (1.18)$$

where

$$m(\xi, \eta) = \frac{(\xi - \eta) \cdot (\xi + \eta)}{2} \frac{(1 + |\eta|^2)^N - (1 + |\xi|^2)^N}{(1 + |\eta|^2)^{N/2} (1 + |\xi|^2)^{N/2}}. \quad (1.19)$$

Notice that $|\xi - \eta| \in [2^{-11}, 2^{11}]$ in the support of the integral, due to the Littlewood-Paley operator in the definition of V . We notice that $m(\xi, \eta)$ satisfies

$$m(\xi, \eta) = \mathfrak{d}(\xi, \eta) m'(\xi, \eta), \quad \text{where} \quad \mathfrak{d}(\xi, \eta) := \frac{[(\xi - \eta) \cdot (\xi + \eta)]^2}{1 + |\xi + \eta|^2}, \quad m' \approx 1. \quad (1.20)$$

The *depletion factor* \mathfrak{d} is important in establishing energy estimates, due to its correlation with the modulation function Φ (see (1.28) below). The presence of this factor is related to the exact conservation law (1.15).

Step 2. We would like to estimate now the increment of $E_N(t)$. We use (1.18) and consider only the main case, when $|\xi|, |\eta| \approx 2^k \gg 1$, and $|\xi - \eta|$ is close to the slowly decaying frequency γ_0 . So we need to bound space-time integrals of the form

$$I := \int_0^t \int_{\mathbb{R}^2 \times \mathbb{R}^2} m(\xi, \eta) \widehat{P_k W}(\eta, s) \widehat{P_k W}(-\xi, s) \widehat{U}(\xi - \eta, s) \chi_{\gamma_0}(\xi - \eta) d\xi d\eta ds,$$

where χ_{γ_0} is a smooth cutoff function supported in the set $\{\xi : ||\xi| - \gamma_0| \ll 1\}$, and we replaced V by U (replacing V by \bar{U} leads to a similar calculation). Notice that it is not possible to estimate $|I|$ by moving the absolute value inside the time integral, due to the slow decay of U in L^∞ . So we need to integrate by parts in time; for this define the profiles

$$u(t) := e^{it\Lambda} U(t), \quad w(t) := e^{it\Lambda} W(t). \quad (1.21)$$

Then decompose the integral in dyadic pieces over the size of the modulation and over the size of the time variable. In terms of the profiles u, w , we need to consider the space-time integrals

$$I_{k,m,p} := \int_{\mathbb{R}} q_m(s) \int_{\mathbb{R}^2 \times \mathbb{R}^2} e^{is\Phi(\xi, \eta)} m(\xi, \eta) \widehat{P_k w}(\eta, s) \widehat{P_k \bar{w}}(-\xi, s) \\ \times \widehat{u}(\xi - \eta, s) \chi_{\gamma_0}(\xi - \eta) \varphi_p(\Phi(\xi, \eta)) d\xi d\eta ds, \quad (1.22)$$

where

$$\Phi(\xi, \eta) := \Lambda(\xi) - \Lambda(\eta) - \Lambda(\xi - \eta)$$

is the associated modulation (or phase), q_m is smooth and supported in the set $s \approx 2^m$ and φ_p is supported in the set $\{x : |x| \approx 2^p\}$.

Step 3. To estimate the integrals $I_{k,m,p}$ we consider several cases depending on the relative size of k, m, p . Assume that k, m are large, i.e. $2^k \gg 1, 2^m \gg 1$, which is the harder case. To deal with the case of small modulation, when one cannot integrate by parts in time, we need an L^2 bound on the Fourier integral operator

$$T_{k,m,p}(f)(\xi) := \int_{\mathbb{R}^2} e^{is\Phi(\xi, \eta)} \varphi_k(\xi) \varphi_{\leq p}(\Phi(\xi, \eta)) \chi_{\gamma_0}(\xi - \eta) f(\eta) d\eta,$$

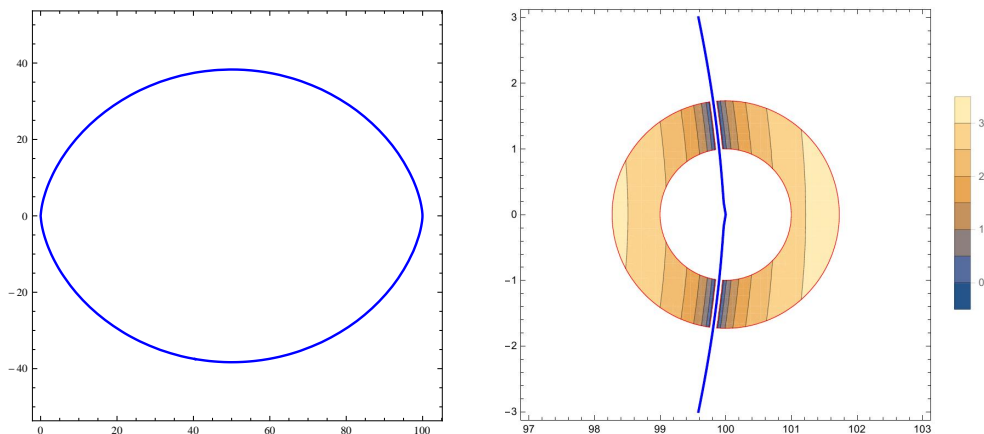


FIGURE 2. The first picture illustrates the resonant set $\{\eta : 0 = \Phi(\xi, \eta) = \Lambda(\xi) - \Lambda(\eta) - \Lambda(\xi - \eta)\}$ for a fixed large frequency ξ (in the picture $\xi = (100, 0)$). The second picture illustrates the intersection of a neighborhood of this resonant set with the set where $|\xi - \eta|$ is close to γ_0 . Note in particular that near the resonant set $\xi - \eta$ is almost perpendicular to ξ (see (1.20), (1.28)). Finally, the colors show the level sets of $\log |\Phi|$.

where $s \approx 2^m$ is fixed. The critical bound we prove in Lemma 4.6 (“the main L^2 lemma”) is

$$\|T_{k,m,p}(f)\|_{L^2} \lesssim_\epsilon 2^{\epsilon m} (2^{(3/2)(p-k/2)} + 2^{p-k/2-m/3}) \|f\|_{L^2}, \quad \epsilon > 0, \quad (1.23)$$

provided that $p - k/2 \in [-0.99m, -0.01m]$. The main gain here is the factor $3/2$ in $2^{(3/2)(p-k/2)}$ in the right-hand side (Schur’s test would only give a factor of 1).

The proof of (1.23) uses a TT^* argument, which is a standard tool to prove L^2 bounds for Fourier integral operators. This argument depends on a key nondegeneracy property of the function Φ , more precisely on what we call the *restricted nondegeneracy condition*

$$\Upsilon(\xi, \eta) = \nabla_{\xi, \eta}^2 \Phi(\xi, \eta) [\nabla_\xi^\perp \Phi(\xi, \eta), \nabla_\eta^\perp \Phi(\xi, \eta)] \neq 0 \quad \text{if} \quad \Phi(\xi, \eta) = 0. \quad (1.24)$$

This condition, which appears to be new, can be verified explicitly in our case, when $\|\xi - \eta\| - \gamma_0 \ll 1$. The function Υ does in fact vanish at two points on the resonant set $\{\eta : \Phi(\xi, \eta) = 0\}$ (where $\|\xi - \eta\| - \gamma_0 \approx 2^{-k}$), but our argument can tolerate vanishing up to order 1.

The nondegeneracy condition (1.24) can be interpreted geometrically: the nondegeneracy of the mixed Hessian of Φ is a standard condition that leads to optimal L^2 bounds on Fourier integral operators. In our case, however, we have the additional cutoff function $\varphi_{\leq p}(\Phi(\xi, \eta))$, so we can only integrate by parts in the directions tangent to the level sets of Φ . This explains the additional restriction to these directions in the definition of Υ in (1.24).

Given the bound (1.23), we can easily control the contribution of small modulations, i.e.

$$p - k/2 \leq -2m/3 - \epsilon m. \quad (1.25)$$

Step 4. In the high modulation case we integrate by parts in time in the formula (1.22). The main contribution is when the time derivative hits the high frequency terms, so we focus

on estimating the resulting integral

$$I'_{k,m,p} := \int_{\mathbb{R}} q_m(s) \int_{\mathbb{R}^2 \times \mathbb{R}^2} e^{is\Phi(\xi,\eta)} m(\xi,\eta) \frac{d}{ds} [\widehat{P}_k w(\eta, s) \widehat{P}_k \overline{w}(-\xi, s)] \\ \times \widehat{u}(\xi - \eta, s) \chi_{\gamma_0}(\xi - \eta) \frac{\varphi_p(\Phi(\xi, \eta))}{\Phi(\xi, \eta)} d\xi d\eta ds. \quad (1.26)$$

Notice that $\partial_t w$ satisfies the equation

$$\partial_t w = \langle \nabla \rangle^N e^{it\Lambda} [\nabla V \cdot \nabla U + (1/2)\Delta V \cdot U]. \quad (1.27)$$

The right-hand side of (1.27) is quadratic. We thus see that replacing w by $\partial_t w$ essentially gains a unit of decay (which is $|t|^{-5/6+}$), but loses a derivative. This causes a problem in some range of parameters, for example when $2^p \approx 2^{k/2-2m/3}$, $1 \ll 2^k \ll 2^m$, compare with (1.25).

We then consider two cases: if the modulation is sufficiently small then we can use the depletion factor \mathfrak{d} in the multiplier m , see (1.20), and the following key algebraic correlation

$$\text{if } |\Phi(\xi, \eta)| \lesssim 1 \quad \text{then } |m(\xi, \eta)| \lesssim 2^{-k}. \quad (1.28)$$

See Fig. 2. As a result, we gain one derivative in the integral $I'_{k,m,p}$, which compensates for the loss of one derivative in (1.27), and the integral can be estimated again using (1.23).

On the other hand, if the modulation is not small, $2^p \geq 1$, then the denominator $\Phi(\xi, \eta)$ becomes a favorable factor, and one can use the formula (1.27) and reiterate the symmetrization procedure implicit in the energy estimates. This symmetrization avoids the loss of one derivative and gives suitable estimates on $|I'_{k,m,p}|$ in this case. The proof of (1.16) follows.

1.5.2. *The special quadratic structure of the full water-wave system.* The model (1.14) is useful in understanding the full problem. There are, however, additional difficulties to keep in mind.

One important aspect to consider when studying the water waves is how to describe the flow, and the choice of appropriate coordinates and variables. In this paper we use *Eulerian coordinates*. The local wellposedness theory, which is nontrivial because of the quasilinear nature of the equations and the hidden hyperbolic structure, then relies on the so-called “good unknown” of Alinhac [6, 5, 1, 4].

In our problem, however, this is not enough. Alinhac’s good unknown ω is suitable for the local theory, in the sense that it prevents loss of derivatives in energy estimates. However, for the global theory, we need to adjust the main complex variable U which diagonalizes the system, using a quadratic correction of the form $T_{m'}\omega$ (see (3.4)). This way we can identify certain *special quadratic structure*, somewhat similar to the structure in the nonlinearity of (1.14). This structure, which appears to be new, is ultimately responsible for the favorable multipliers of the space-time integrals (similar to (1.20)), and leads to global energy bounds.

Identifying this structure is, unfortunately, technically involved. Our main result is in Proposition 3.1, but its proof depends on paradifferential calculus using the Weyl quantization (see section 8) and on a suitable parilinearization of the Dirichlet–Neumann operator. We include all the details of this parilinearization in section 9, mostly because its exact form has to be properly adapted to our norms and suitable for global analysis. For this we need suitable spaces: (1) the $\mathcal{O}_{m,p}$ hierarchy, which measures functions, keeping track of both multiplicity (the index m) and smoothness (the index p), and (2) the $\mathcal{M}_r^{l,m}$ hierarchy, which measures the symbols of the paradifferential operators, keeping track also of the order l .

1.5.3. *Additional remarks.* We list below some other issues one needs to keep in mind in the proof of the main theorem.

- (1) A significant difficulty of the full water wave system, which is not present in (1.14), is that the “linear” part of the equation is given by a more complicated paradifferential operator T_Σ , not by the simple operator Λ . The operator T_Σ includes nonlinear cubic terms that lose $3/2$ derivatives, and an additional smoothing effect is needed.
- (2) The very low frequencies $|\xi| \ll 1$ play an important role in all the global results for water wave systems. These frequencies are not captured in the model (1.14). In our case, there is a suitable *null structure* at very low frequencies: the multipliers of the quadratic terms are bounded by $|\xi| \min(|\eta|, |\xi - \eta|)^{1/2}$.
- (3) It is important to propagate energy control of both high Sobolev norms and weighted norms using many copies of the rotation vector-field, see also [31, 30]. Because of this control, we can assume that all the profiles in the dispersive part of the argument are almost radial and located at frequencies $\lesssim 1$. The linear estimates and many of the bilinear estimates in [32] are much stronger because of this almost radially property.
- (4) At many stages it is important that the four spheres, the sphere of slow decay $\{|\xi| = \gamma_0\}$, the sphere of space-time resonant outputs $\{|\xi| = \gamma_1\}$, and the sphere of space-time resonant inputs $\{|\xi| = \gamma_1/2\}$, and the sphere $\{|\xi| = 2\gamma_0\}$ are all separated from each other. Such separation conditions played an important role also in other papers, such as [35, 38, 31].

1.6. **Organization.** The rest of the paper is organized as follows: in section 2 we state the main propositions and summarize the main definitions and notation in the paper.

In sections 3–6 we prove Proposition 2.2, which is the main improved energy estimate. The key components of the proof are Proposition 3.1 (derivation of the main quasilinear scalar equation, identifying the special quadratic structure), Proposition 3.4 (the first energy estimate, including the strongly semilinear structure), Proposition 4.1 (reduction to a space-time integral bound), Lemma 4.6 (the main L^2 bound on a localized Fourier integral operator), and Lemma 5.1 (the main interactions in Proposition 4.1). The proof of Proposition 2.2 uses also the material presented in sections 8 and 9, in particular the parilinearization of the Dirichlet–Neumann operator in Proposition 9.1.

In section 7 we collect estimates on the dispersion relation Λ and the phase functions Φ . The main results are Lemmas 7.1–7.3 (the restricted nondegeneracy property of the resonant hypersurfaces), which are used in section 6 in the proof of the main L^2 bound.

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2. THE MAIN PROPOSITIONS

Recall the water-wave system with gravity and surface tension,

$$\begin{cases} \partial_t h = G(h)\phi, \\ \partial_t \phi = -gh + \sigma \operatorname{div} \left[\frac{\nabla h}{(1 + |\nabla h|^2)^{1/2}} \right] - \frac{1}{2} |\nabla \phi|^2 + \frac{(G(h)\phi + \nabla h \cdot \nabla \phi)^2}{2(1 + |\nabla h|^2)}, \end{cases} \quad (2.1)$$

where $G(h)\phi$ denotes the Dirichlet-Neumann operator associated to the water domain. Theorem 1.1 is a consequence of Propositions 2.1, 2.2, and 2.3 below.

Proposition 2.1. *(Local existence and continuity) (i) Assume that $N \geq 10$. There is $\bar{\varepsilon} > 0$ such that if*

$$\|h_0\|_{H^{N+1}} + \|\phi_0\|_{\dot{H}^{N+1/2,1/2}} \leq \bar{\varepsilon} \quad (2.2)$$

then there is a unique solution $(h, \phi) \in C([0, 1] : H^{N+1} \times \dot{H}^{N+1/2,1/2})$ of the system (2.1) with $g = 1$ and $\sigma = 1$, with initial data (h_0, ϕ_0) .

(ii) Assume $T_0 \geq 1$, $N = N_1 + N_3$, and $(h, \phi) \in C([0, T_0] : H^{N+1} \times \dot{H}^{N+1/2,1/2})$ is a solution of the system (2.1) with $g = 1$ and $\sigma = 1$. With the Z norm as in Definition 2.5 below and the profile \mathcal{V} defined as in (1.11), assume that for some $t_0 \in [0, T_0]$

$$\mathcal{V}(t_0) \in H^{N_0} \cap H_{\Omega}^{N_1, N_3} \cap Z, \quad \|\mathcal{V}(t_0)\|_{H^N} \leq 2\bar{\varepsilon}. \quad (2.3)$$

Then there is $\tau = \tau(\|\mathcal{V}(t_0)\|_{H^{N_0} \cap H_{\Omega}^{N_1, N_3} \cap Z})$ such that the mapping $t \rightarrow \|\mathcal{V}(t)\|_{H^{N_0} \cap H_{\Omega}^{N_1, N_3} \cap Z}$ is continuous on $[0, T_0] \cap [t_0, t_0 + \tau]$, and

$$\sup_{t \in [0, T_0] \cap [t_0, t_0 + \tau]} \|\mathcal{V}(t)\|_{H^{N_0} \cap H_{\Omega}^{N_1, N_3} \cap Z} \leq 2\|\mathcal{V}(t_0)\|_{H^{N_0} \cap H_{\Omega}^{N_1, N_3} \cap Z}. \quad (2.4)$$

Proposition 2.1 is a local existence result for the water waves system. We will not provide the details of its proof in the paper, but only briefly discuss it. Part (i) is a standard wellposedness statement in a sufficiently regular Sobolev space, see for example [71, 1].

Part (ii) is a continuity statement for the Sobolev norm H^{N_0} as well as for the $H_{\Omega}^{N_1, N_3}$ and Z norms⁴. Continuity for the H^{N_0} norm is standard. A formal proof of continuity for the $H_{\Omega}^{N_1, N_3}$ and Z norms and of (2.4) requires some adjustments of the arguments given in the paper, due to the quasilinear and non-local nature of the equations.

More precisely, we can define ϵ -truncations of the rotational vector-field Ω , i.e. $\Omega_{\epsilon} := (1 + \epsilon^2|x|^2)^{-1/2}\Omega$, and the associated spaces $H_{\Omega_{\epsilon}}^{N_1, N_3}$, with the obvious adaptation of the norm in (1.8). Then we notice that

$$\Omega_{\epsilon} T_a b = T_{\Omega_{\epsilon} a} b + T_a \Omega_{\epsilon} b + R$$

where R is a suitable remainder bounded uniformly in ϵ . Because of this we can adapt the arguments in Proposition 3.4 and in appendices 8 and 9 to prove energy estimates in the ϵ -truncated spaces $H_{\Omega_{\epsilon}}^{N_1, N_3}$. For the Z norm one can proceed similarly using an ϵ -truncated version Z_{ϵ} (see the proof of Proposition 2.4 in [44] for a similar argument) and the formal expansion of the Dirichlet-Neumann operator in section 6 in [32]. The conclusion follows from the uniform estimates by letting $\epsilon \rightarrow 0$.

The following two propositions summarize our main bootstrap argument.

Proposition 2.2. *(Improved energy control) Assume that $T \geq 1$ and $(h, \phi) \in C([0, T] : H^{N_0+1} \times \dot{H}^{N_0+1/2,1/2})$ is a solution of the system (2.1) with $g = 1$ and $\sigma = 1$, with initial data (h_0, ϕ_0) . Assume that, with \mathcal{U} and \mathcal{V} defined as in (1.11),*

$$\|\mathcal{U}_0\|_{H^{N_0} \cap H_{\Omega}^{N_1, N_3}} + \|\mathcal{V}_0\|_Z \leq \varepsilon_0 \ll 1 \quad (2.5)$$

and, for any $t \in [0, T]$,

$$(1+t)^{-\delta^2} \|\mathcal{U}(t)\|_{H^{N_0} \cap H_{\Omega}^{N_1, N_3}} + \|\mathcal{V}(t)\|_Z \leq \varepsilon_1 \ll 1, \quad (2.6)$$

⁴Notice that we may assume uniform in time smallness of the high Sobolev norm H^N with $N = N_1 + N_3$, thanks to the uniform control on the Z norm, see Proposition 2.2, and Definition 2.5.

where the Z norm is as in Definition 2.5. Then, for any $t \in [0, T]$,

$$(1+t)^{-\delta^2} \|\mathcal{U}(t)\|_{H^{N_0} \cap H_\Omega^{N_1, N_3}} \lesssim \varepsilon_0 + \varepsilon_1^{3/2}. \quad (2.7)$$

Proposition 2.3. (Improved dispersive control) *With the same assumptions as in Proposition 2.2 above, in particular (2.5)–(2.6), we have, for any $t \in [0, T]$,*

$$\|\mathcal{V}(t)\|_Z \lesssim \varepsilon_0 + \varepsilon_1^2. \quad (2.8)$$

It is easy to see that Theorem 1.1 follows from Propositions 2.1, 2.2, and 2.3 by a standard continuity argument and Lemma 2.6 (for the L^∞ bound on \mathcal{U} in (1.10)).

The rest of this paper is concerned with the proof of Proposition 2.2. Proposition 2.3, which is our main dispersive estimate, is proved in [32].

2.1. Definitions and notation. We summarize in this subsection some of the main definitions and notation we use in the paper.

2.1.1. The spaces $O_{m,p}$. We will need several spaces of functions, in order to properly measure linear, quadratic, cubic, and quartic and higher order terms. In addition, we also need to track the Sobolev smoothness and angular derivatives. Assume that $N_2 = 40 \geq N_3 + 10$ and N_0 (the maximum number of Sobolev derivatives) and N_1 (the maximum number of angular derivatives) and N_3 (additional Sobolev regularity) are as before.

Definition 2.4. *Assume $T \geq 1$ and let $p \in [-N_3, 10]$. For $m \geq 1$ we define $O_{m,p}$ as the space of functions $f \in C([0, T] : L^2)$ satisfying*

$$\begin{aligned} \|f\|_{O_{m,p}} := \sup_{t \in [0, T]} (1+t)^{(m-1)(5/6-20\delta^2)-\delta^2} & \left[\|f(t)\|_{H^{N_0+p}} + \|f(t)\|_{H_\Omega^{N_1, N_3+p}} \right. \\ & \left. + (1+t)^{5/6-2\delta^2} \|f(t)\|_{\widetilde{W}_\Omega^{N_1/2, N_2+p}} \right] < \infty, \end{aligned} \quad (2.9)$$

where, with P_k denoting standard Littlewood-Paley projection operators,

$$\|g\|_{\widetilde{W}^N} := \sum_{k \in \mathbb{Z}} 2^{Nk} \|P_k g\|_{L^\infty}, \quad \|g\|_{\widetilde{W}_\Omega^{N', N}} := \sum_{j \leq N'} \|\Omega^j g\|_{\widetilde{W}^N}.$$

The spaces \widetilde{W}^N are used in this paper as substitutes of the standard L^∞ based Sobolev spaces, which have the advantage of being closed under the action of singular integrals.

Note that the parameter p in $O_{m,p}$ corresponds to a gain at high frequencies and does not affect the low frequencies. We observe that, see Lemma 8.2,

$$O_{m,p} \subseteq O_{n,p} \text{ if } 1 \leq n \leq m, \quad O_{m,p} O_{n,p} \subseteq O_{m+n,p} \text{ if } 1 \leq m, n. \quad (2.10)$$

Moreover, by our assumptions (2.6) and Lemma 2.6, the main variables satisfy

$$\|(1-\Delta)^{1/2} h\|_{O_{1,0}} + \|\nabla^{1/2} \phi\|_{O_{1,0}} \lesssim \varepsilon_1. \quad (2.11)$$

The L^2 based spaces $O_{m,p}$ are used mostly in the energy estimates in this paper. However, they are not precise enough for the dispersive analysis of our evolution equation in [32]. For this we need the more precise Z -norm defined below, which is better adapted to the equation.

2.1.2. *Fourier multipliers and the Z norm.* We start by defining several multipliers that allow us to localize in the Fourier space. We fix $\varphi : \mathbb{R} \rightarrow [0, 1]$ an even smooth function supported in $[-8/5, 8/5]$ and equal to 1 in $[-5/4, 5/4]$. For simplicity of notation, we also let $\varphi : \mathbb{R}^2 \rightarrow [0, 1]$ denote the corresponding radial function on \mathbb{R}^2 . Let

$$\begin{aligned} \varphi_k(x) &:= \varphi(|x|/2^k) - \varphi(|x|/2^{k-1}) \text{ for any } k \in \mathbb{Z}, & \varphi_I &:= \sum_{m \in I \cap \mathbb{Z}} \varphi_m \text{ for any } I \subseteq \mathbb{R}, \\ \varphi_{\leq B} &:= \varphi_{(-\infty, B]}, & \varphi_{\geq B} &:= \varphi_{[B, \infty)}, & \varphi_{< B} &:= \varphi_{(-\infty, B)}, & \varphi_{> B} &:= \varphi_{(B, \infty)}. \end{aligned}$$

For any $a < b \in \mathbb{Z}$ and $j \in [a, b] \cap \mathbb{Z}$ let

$$\varphi_j^{[a, b]} := \begin{cases} \varphi_j & \text{if } a < j < b, \\ \varphi_{\leq a} & \text{if } j = a, \\ \varphi_{\geq b} & \text{if } j = b. \end{cases} \quad (2.12)$$

For any $x \in \mathbb{Z}$ let $x_+ = \max(x, 0)$ and $x_- := \min(x, 0)$. Let

$$\mathcal{J} := \{(k, j) \in \mathbb{Z} \times \mathbb{Z}_+ : k + j \geq 0\}.$$

For any $(k, j) \in \mathcal{J}$ let

$$\tilde{\varphi}_j^{(k)}(x) := \begin{cases} \varphi_{\leq -k}(x) & \text{if } k + j = 0 \text{ and } k \leq 0, \\ \varphi_{\leq 0}(x) & \text{if } j = 0 \text{ and } k \geq 0, \\ \varphi_j(x) & \text{if } k + j \geq 1 \text{ and } j \geq 1, \end{cases}$$

and notice that, for any $k \in \mathbb{Z}$ fixed, $\sum_{j \geq -\min(k, 0)} \tilde{\varphi}_j^{(k)} = 1$.

Let P_k , $k \in \mathbb{Z}$, denote the Littlewood–Paley projection operators defined by the Fourier multipliers $\xi \rightarrow \varphi_k(\xi)$. Let $P_{\leq B}$ (respectively $P_{> B}$) denote the operators defined by the Fourier multipliers $\xi \rightarrow \varphi_{\leq B}(\xi)$ (respectively $\xi \rightarrow \varphi_{> B}(\xi)$). For $(k, j) \in \mathcal{J}$ let Q_{jk} denote the operator

$$(Q_{jk}f)(x) := \tilde{\varphi}_j^{(k)}(x) \cdot P_k f(x). \quad (2.13)$$

In view of the uncertainty principle the operators Q_{jk} are relevant only when $2^j 2^k \gtrsim 1$, which explains the definitions above. For $k, k_1, k_2 \in \mathbb{Z}$ let

$$\mathcal{D}_{k, k_1, k_2} := \{(\xi, \eta) \in (\mathbb{R}^2)^2 : |\xi| \in [2^{k-4}, 2^{k+4}], |\eta| \in [2^{k_2-4}, 2^{k_2+4}], |\xi - \eta| \in [2^{k_1-4}, 2^{k_1+4}]\}. \quad (2.14)$$

Let $\lambda(r) = \sqrt{|r| + |r|^3}$, $\Lambda(\xi) = \sqrt{|\xi| + |\xi|^3} = \lambda(|\xi|)$, $\Lambda : \mathbb{R}^2 \rightarrow [0, \infty)$. Let

$$\mathcal{U}_+ := \mathcal{U}, \quad \mathcal{U}_- := \overline{\mathcal{U}}, \quad \mathcal{V}(t) = \mathcal{V}_+(t) := e^{it\Lambda}\mathcal{U}(t), \quad \mathcal{V}_-(t) := e^{-it\Lambda}\mathcal{U}_-(t). \quad (2.15)$$

Let $\Lambda_+ = \Lambda$ and $\Lambda_- := -\Lambda$. For $\sigma, \mu, \nu \in \{+, -\}$, we define the associated phase functions

$$\begin{aligned} \Phi_{\sigma\mu\nu}(\xi, \eta) &:= \Lambda_\sigma(\xi) - \Lambda_\mu(\xi - \eta) - \Lambda_\nu(\eta), \\ \tilde{\Phi}_{\sigma\mu\nu\beta}(\xi, \eta, \sigma) &:= \Lambda_\sigma(\xi) - \Lambda_\mu(\xi - \eta) - \Lambda_\nu(\eta - \sigma) - \Lambda_\beta(\sigma). \end{aligned} \quad (2.16)$$

For any set S let $\mathbf{1}_S$ denote its characteristic function. We will use two sufficiently large constants $\mathcal{D} \gg \mathcal{D}_1 \gg 1$ (\mathcal{D}_1 is only used in section 7 to prove properties of the phase functions).

Let $\gamma_0 := \sqrt{\frac{2\sqrt{3}-3}{3}}$ denote the radius of the sphere of slow decay and $\gamma_1 := \sqrt{2}$ denote the radius of the space-time resonant sphere. For $n \in \mathbb{Z}$, $I \subseteq \mathbb{R}$, and $\gamma \in (0, \infty)$ we define

$$\begin{aligned} \widehat{A_{n,\gamma}f}(\xi) &:= \varphi_{-n}(2^{100}|\xi| - \gamma) \cdot \widehat{f}(\xi), \\ A_{I,\gamma} &:= \sum_{n \in I} A_{n,\gamma}, \quad A_{\leq B,\gamma} := A_{(-\infty, B], \gamma}, \quad A_{\geq B,\gamma} := A_{[B, \infty), \gamma}. \end{aligned} \quad (2.17)$$

Given an integer $j \geq 0$ we define the operators $A_{n,\gamma}^{(j)}$, $n \in \{0, \dots, j+1\}$, $\gamma \geq 2^{-50}$, by

$$A_{j+1,\gamma}^{(j)} := \sum_{n' \geq j+1} A_{n',\gamma}, \quad A_{0,\gamma}^{(j)} := \sum_{n' \leq 0} A_{n',\gamma}, \quad A_{n,\gamma}^{(j)} := A_{n,\gamma} \text{ if } 1 \leq n \leq j. \quad (2.18)$$

These operators localize to thin annuli of width 2^{-n} around the circle of radius γ . Most of the times, for us $\gamma = \gamma_0$ or $\gamma = \gamma_1$. We are now ready to define the main Z norm.

Definition 2.5. *Assume that δ, N_0, N_1, N_4 are as in Theorem 1.1. We define*

$$Z_1 := \{f \in L^2(\mathbb{R}^2) : \|f\|_{Z_1} := \sup_{(k,j) \in \mathcal{J}} \|Q_{jk}f\|_{B_j} < \infty\}, \quad (2.19)$$

where

$$\|g\|_{B_j} := 2^{(1-50\delta)j} \sup_{0 \leq n \leq j+1} 2^{-(1/2-49\delta)n} \|A_{n,\gamma_1}^{(j)}g\|_{L^2}. \quad (2.20)$$

Then we define, with $D^\alpha := \partial_1^{\alpha^1} \partial_2^{\alpha^2}$, $\alpha = (\alpha^1, \alpha^2)$,

$$Z := \{f \in L^2(\mathbb{R}^2) : \|f\|_Z := \sup_{2m+|\alpha| \leq N_1+N_4, m \leq N_1/2+20} \|D^\alpha \Omega^m f\|_{Z_1} < \infty\}. \quad (2.21)$$

We remark that the Z norm is used to estimate the *linear profile* of the solution, which is $\mathcal{V}(t) := e^{it\Lambda} \mathcal{U}(t)$, not the solution itself. The Z norm is used extensively in the dispersive analysis in [32]. In this paper, however, we only need several simple linear estimates concerning the Z norm. These estimates, and others, are proved in Lemma 3.6 and Remark 3.7 in [32].

We emphasize that it is important in many of these estimates to take advantage of the fact that our functions are ‘‘almost radial’’ (due to the presence of the spaces H_Ω^N). The resulting bounds are much stronger than the bounds one would normally expect for general functions with the same localization properties.

Lemma 2.6. *Assume that $N \geq 10$ and*

$$\|f\|_{Z_1} + \sup_{k \in \mathbb{Z}, a \leq N} \|\Omega^a P_k f\|_{L^2} \leq 1. \quad (2.22)$$

Let $\delta' := 50\delta + 1/(2N)$. For any $(k, j) \in \mathcal{J}$ and $n \in \{0, \dots, j+1\}$ let (recall the notation (2.12))

$$f_{j,k} := P_{[k-2, k+2]} Q_{jk} f, \quad \widehat{f_{j,k,n}}(\xi) := \varphi_{-n}^{[-j-1, 0]}(2^{100}(|\xi| - \gamma_1)) \widehat{f_{j,k}}(\xi). \quad (2.23)$$

For any $\xi_0 \in \mathbb{R}^2 \setminus \{0\}$ and $\kappa, \rho \in [0, \infty)$ let $\mathcal{R}(\xi_0; \kappa, \rho)$ denote the rectangle

$$\mathcal{R}(\xi_0; \kappa, \rho) := \{\xi \in \mathbb{R}^2 : |(\xi - \xi_0) \cdot \xi_0 / |\xi_0|| \leq \rho, |(\xi - \xi_0) \cdot \xi_0^\perp / |\xi_0|| \leq \kappa\}. \quad (2.24)$$

(i) Then, for any $(k, j) \in \mathcal{J}$, $n \in [0, j+1]$, and $\kappa, \rho \in (0, \infty)$ satisfying $\kappa + \rho \leq 2^{k-10}$

$$\left\| \sup_{\theta \in \mathbb{S}^1} |\widehat{f_{j,k,n}}(r\theta)| \right\|_{L^2(rdr)} + \left\| \sup_{\theta \in \mathbb{S}^1} |f_{j,k,n}(r\theta)| \right\|_{L^2(rdr)} \lesssim 2^{(1/2-49\delta)n - (1-\delta')j}, \quad (2.25)$$

and

$$\|\widehat{f_{j,k,n}}\|_{L^\infty} \lesssim \begin{cases} 2^{(\delta+(1/2N)n)} 2^{-(1/2-\delta')(j-n)} & \text{if } |k| \leq 10, \\ 2^{-\delta'k} 2^{-(1/2-\delta')(j+k)} & \text{if } |k| \geq 10. \end{cases} \quad (2.26)$$

(ii) (Dispersive bounds) If $m \geq 0$ and $|t| \in [2^m - 1, 2^{m+1}]$ then

$$\|e^{-it\Lambda} f_{j,k,n}\|_{L^\infty} \lesssim \|\widehat{f_{j,k,n}}\|_{L^1} \lesssim 2^k 2^{-j+50\delta j} 2^{-49\delta n}, \quad (2.27)$$

$$\|e^{-it\Lambda} f_{j,k,0}\|_{L^\infty} \lesssim 2^{3k/2} 2^{-m+50\delta j}, \quad \text{if } |k| \geq 10. \quad (2.28)$$

Recall the operators A_{n,γ_0} defined in (2.17). If $j \leq (1 - \delta^2)m + |k|/2$ and $|k| + \mathcal{D} \leq m/2$ then we have the more precise bounds

$$\begin{aligned} \|e^{-it\Lambda} A_{\leq 2\mathcal{D},\gamma_0} f_{j,k}\|_{L^\infty} &\lesssim 2^{-m+2\delta^2 m} 2^k 2^{j(\delta+1/(2N))}, \\ \sum_{l \geq 1} \|e^{-it\Lambda} A_{l,\gamma_0} f_{j,k}\|_{L^\infty} &\lesssim 2^{-m+2\delta^2 m} 2^{\delta' j} 2^{(m-3j)/6}, \end{aligned} \quad (2.29)$$

and the additional bound (with no loss of $2^{2\delta^2 m}$)

$$\|e^{-it\Lambda} A_{\leq 2\mathcal{D},\gamma_0} A_{\leq 2\mathcal{D},\gamma_1} f_{j,k}\|_{L^\infty} \lesssim 2^{-m} 2^k 2^{-(1/2-\delta'-\delta)j}. \quad (2.30)$$

For all $k \in \mathbb{Z}$ we have the bound

$$\begin{aligned} \|e^{-it\Lambda} A_{\leq 0,\gamma_0} P_k f\|_{L^\infty} &\lesssim (2^{k/2} + 2^{2k}) 2^{-m} [2^{51\delta m} + 2^{m(2\delta+1/(2N))}], \\ \|e^{-it\Lambda} A_{\geq 1,\gamma_0} P_k f\|_{L^\infty} &\lesssim 2^{-5m/6+2\delta^2 m}. \end{aligned} \quad (2.31)$$

2.1.3. Paradifferential calculus. We need some elements of paradifferential calculus in order to be able to describe the Dirichlet–Neumann operator $G(h)\phi$ in (2.1). Our parilinearization relies on the *Weyl quantization*. More precisely, given a symbol $a = a(x, \zeta)$, and a function $f \in L^2$, we define the paradifferential operator $T_a f$ according to

$$\mathcal{F}(T_a f)(\xi) = \frac{1}{4\pi^2} \int_{\mathbb{R}^2} \chi\left(\frac{|\xi - \eta|}{|\xi + \eta|}\right) \tilde{a}(\xi - \eta, (\xi + \eta)/2) \widehat{f}(\eta) d\eta, \quad (2.32)$$

where \tilde{a} denotes the Fourier transform of a in the first coordinate and $\chi = \varphi_{\leq -20}$. In section 8 we prove several important lemmas related to the paradifferential calculus.

3. ENERGY ESTIMATES, I: THE SCALAR EQUATION AND STRONGLY SEMILINEAR STRUCTURES

3.1. The main propositions. In this section we assume $(h, \phi) : \mathbb{R}^2 \times [0, T] \rightarrow \mathbb{R} \times \mathbb{R}$ is a solution of (2.1) satisfying the hypothesis of Proposition 2.2; in particular, see (2.11),

$$\|\langle \nabla \rangle h\|_{O_{1,0}} + \||\nabla|^{1/2} \phi\|_{O_{1,0}} \lesssim \varepsilon_1. \quad (3.1)$$

Our goal in this section is to write the system (2.1) as a scalar equation for a suitably constructed complex-valued function, and prove energy estimates. The first result is the following:

Proposition 3.1. *Assume that (3.1) holds and let λ_{DN} be the symbol of the Dirichlet–Neumann operator defined in (9.5), let $\Lambda := \sqrt{g|\nabla| + \sigma|\nabla|^3}$, and let*

$$\ell(x, \zeta) := L_{ij}(x) \zeta_i \zeta_j - \Lambda^2 h, \quad L_{ij} := \frac{\sigma}{\sqrt{1 + |\nabla h|^2}} \left(\delta_{ij} - \frac{\partial_i h \partial_j h}{1 + |\nabla h|^2} \right), \quad (3.2)$$

be the mean curvature operator coming from the surface tension. Define the symbol

$$\Sigma := \sqrt{\lambda_{DN}(g + \ell)} \quad (3.3)$$

and the complex-valued unknown

$$U := T_{\sqrt{g+\ell}}h + iT_{\Sigma}T_{1/\sqrt{g+\ell}}\omega + iT_{m'}\omega, \quad m' := \frac{i}{2} \frac{\operatorname{div}V}{\sqrt{g+\ell}}, \quad (3.4)$$

where B, V and (the “good variable”) $\omega = \phi - T_B h$ are defined in (9.3). Then

$$U = \sqrt{g + \sigma|\nabla|^2}h + i|\nabla|^{1/2}\omega + \varepsilon_1^2 O_{2,0}, \quad (3.5)$$

and U satisfies the equation

$$(\partial_t + iT_{\Sigma} + iT_{V,\zeta})U = \mathcal{N}_U + \mathcal{Q}_S + \mathcal{C}_U, \quad (3.6)$$

where

- The quadratic term \mathcal{N}_U has the special (null) structure

$$\mathcal{N}_U := c_1[T_{\Sigma}, T_{V,\zeta}]T_{\Sigma}^{-1}U + c_2[T_{\Sigma}, T_{V,\zeta}]T_{\Sigma}^{-1}\bar{U} \quad (3.7)$$

for some constants $c_1, c_2 \in \mathbb{C}$;

- the quadratic terms \mathcal{Q}_S have a gain of one derivative, i.e. they are of the form

$$\mathcal{Q}_S = A_{++}(U, U) + A_{+-}(U, \bar{U}) + A_{--}(\bar{U}, \bar{U}) \in \varepsilon_1^2 O_{2,1}, \quad (3.8)$$

with symbols $a_{\varepsilon_1\varepsilon_2}$ satisfying, for all $k, k_1, k_2 \in \mathbb{Z}$, and $(\varepsilon_1\varepsilon_2) \in \{(++), (+-), (--) \}$,

$$\|a_{\varepsilon_1\varepsilon_2}^{k, k_1, k_2}\|_{S_{\Omega}^{\infty}} \lesssim 2^{-\max(k_1, k_2, 0)}(1 + 2^{3\min(k_1, k_2)}); \quad (3.9)$$

- \mathcal{C}_U is an $O_{3,0}$ cubic term, i.e. it satisfies for any $t \in [0, T]$

$$\|\mathcal{C}_U\|_{H^{N_0} \cap H_{\Omega}^{N_1, N_3}} \lesssim \varepsilon_1^3 \langle t \rangle^{-5/3+41\delta^2}, \quad \|\mathcal{C}_U\|_{\widetilde{W}_{\Omega}^{N_1/2, N_2}} \lesssim \varepsilon_1^3 \langle t \rangle^{-5/2+43\delta^2}. \quad (3.10)$$

Let us comment on the structure of the main equation (3.6). In the left-hand side we have the usual “quasilinear” part $(\partial_t + iT_{\Sigma} + iT_{V,\zeta})U$. In the right-hand side we have three types of terms: (1) a *strongly semilinear* quadratic term \mathcal{Q}_S , given by symbols of order -1; (2) a semilinear cubic term $\mathcal{C}_U \in \varepsilon_1^3 O_{3,0}$, whose contribution is easy to estimate; and (3) a quadratic term \mathcal{N}_U with special (null) structure, see also Remark 3.3 below. This special structure, which is a consequence of the choice of the symbol m' , allows us to obtain more favorable energy estimates in Proposition 3.4.

This proposition is the starting point of our energy analysis. Its proof is, unfortunately, technically involved, as it requires the material in sections 8 and 9. One can start by understanding the definition 8.6 of the decorated spaces of symbols $\mathcal{M}_r^{l,m}$, the simple properties (8.43)–(8.54), and the statement of Proposition 9.1 (the proof is not needed). The spaces of symbols $\mathcal{M}_r^{l,m}$ are analogous to the spaces of functions $\mathcal{O}_{m,p}$; for symbols, however, the order l is important (for example a symbol of order 2 counts as two derivatives), but its exact differentiability is less important.

In Proposition 3.1 we keep the parameters g and σ due to their physical significance.

Remark 3.2. (i) The symbols defined in this proposition can be estimated in terms of the decorated norms introduced in Definition 8.6. More precisely, using the hypothesis (3.1), the

basic bounds (8.43) and (8.45), and the definition (9.5), it is easy to verify that

$$(g + \ell) = \frac{(g + \sigma|\zeta|^2)}{\sqrt{1 + |\nabla h|^2}} \left(1 - \frac{\sigma(\zeta \cdot \nabla h)^2}{(g + \sigma|\zeta|^2)} - \frac{\Lambda^2 h}{(g + \sigma|\zeta|^2)} + \varepsilon_1^4 \mathcal{M}_{N_3-2}^{0,4} + \varepsilon_1^2 \mathcal{M}_{N_3-2}^{-2,2} \right),$$

$$\lambda_{DN} = |\zeta| \left(1 + \frac{|\zeta|^2 |\nabla h|^2 - (\zeta \cdot \nabla h)^2}{2|\zeta|^2} + \frac{|\zeta|^2 \Delta h - \zeta_j \zeta_k \partial_j \partial_k h}{2|\zeta|^3} \varphi_{\geq 0}(\zeta) + \varepsilon_1^4 \mathcal{M}_{N_3-2}^{0,4} + \varepsilon_1^3 \mathcal{M}_{N_3-2}^{-1,3} \right), \quad (3.11)$$

uniformly for every $t \in [0, T]$. Therefore we derive an expansion for Σ ,

$$\Sigma = \Lambda + \Sigma_1 + \Sigma_{\geq 2},$$

$$\Sigma_1 := \frac{1}{4} \frac{\Lambda(\zeta)}{|\zeta|} \left[\Delta h - \frac{\zeta_i \zeta_j}{|\zeta|^2} \partial_{ij} h \right] \varphi_{\geq 0}(\zeta) - \frac{1}{2} \frac{|\zeta|}{\Lambda(\zeta)} \Lambda^2 h \in \varepsilon_1 \mathcal{M}_{N_3-2}^{1/2,1}, \quad \Sigma_{\geq 2} \in \varepsilon_1^2 \mathcal{M}_{N_3-2}^{3/2,2}. \quad (3.12)$$

The formulas are slightly simpler if we disregard quadratic terms, i.e.

$$\lambda_{DN}^p = |\zeta|^p (1 + p\lambda_1^{(0)}(x, \zeta)/|\zeta| + \varepsilon_1^2 \mathcal{M}_{N_3-2}^{0,2}),$$

$$(g + \ell)^p = (g + \sigma|\zeta|^2)^p (1 - p\Lambda^2 h / (g + \sigma|\zeta|^2) + \varepsilon_1^2 \mathcal{M}_{N_3-2}^{0,2}), \quad (3.13)$$

$$\Sigma = \Lambda(1 + \Sigma_1(x, \zeta)/\Lambda + \varepsilon_1^2 \mathcal{M}_{N_3-2}^{0,2}),$$

for $p \in [-2, 2]$, where $\lambda_1^{(0)}(x, \zeta) = \frac{|\zeta|^2 \Delta h - \zeta_j \zeta_k \partial_j \partial_k h}{2|\zeta|^2} \varphi_{\geq 0}(\zeta)$ as in Remark 9.2. The identity $\partial_t h = G(h)\phi = |\nabla|\omega + \varepsilon_1^2 \mathcal{O}_{2,-1/2}$ then shows that

$$\partial_t \sqrt{g + \ell} = (g + \sigma|\zeta|^2)^{-1/2} [\Delta(g - \sigma\Delta)\omega/2] + \varepsilon_1^2 \mathcal{M}_{N_3-4}^{1,2} \in \varepsilon_1 \mathcal{M}_{N_3-4}^{-1,1} + \varepsilon_1^2 \mathcal{M}_{N_3-4}^{1,2},$$

$$\partial_t \sqrt{\lambda_{DN}} = \frac{1}{2\sqrt{|\zeta|}} \partial_t \lambda_1^{(0)} + \varepsilon_1^2 \mathcal{M}_{N_3-4}^{1/2,2} \in \varepsilon_1 \mathcal{M}_{N_3-4}^{-1/2,1} + \varepsilon_1^2 \mathcal{M}_{N_3-4}^{1/2,2}, \quad (3.14)$$

$$\partial_t \Sigma = \partial_t \Sigma_1 + \varepsilon_1^2 \mathcal{M}_{N_3-4}^{3/2,2} \in \varepsilon_1 \mathcal{M}_{N_3-4}^{1/2,1} + \varepsilon_1^2 \mathcal{M}_{N_3-4}^{3/2,2}.$$

(ii) It follows from Proposition 9.1 that $V \in \varepsilon_1 \mathcal{O}_{1,-1/2}$. Therefore $m' \in \varepsilon_1 \mathcal{M}_{N_3-2}^{-1,1}$ and the identity (3.5) follows using also Lemma 8.7. Moreover, using Proposition 9.1 again,

$$V := V_1 + V_2, \quad V_1 := |\nabla|^{-1/2} \nabla \mathfrak{S}U, \quad V_2 \in \varepsilon_1^2 \mathcal{O}_{2,-1/2}. \quad (3.15)$$

Remark 3.3. A simple computation shows that, for $F \in \{U, \bar{U}\}$,

$$[T_{V \cdot \zeta}, T_\Sigma] T_\Sigma^{-1} F = iT \frac{g+3\sigma|\zeta|^2}{2(g+\sigma|\zeta|^2)} \frac{\zeta_i \zeta_j}{|\zeta|^2} \partial_{ij} V_j F + F_2 + \varepsilon_1^3 \mathcal{O}_{3,0} = \frac{3i}{2} T_\gamma F + F_2 + \varepsilon_1^3 \mathcal{O}_{3,0}, \quad (3.16)$$

where F_2 denotes quadratic terms of the form (3.8)-(3.9) and

$$\gamma(x, \zeta) := \frac{\zeta_i \zeta_j}{|\zeta|^2} |\nabla|^{-1/2} \partial_i \partial_j (\mathfrak{S}U)(x). \quad (3.17)$$

We then see that

$$\tilde{\gamma}(\eta, \zeta) = -\frac{\zeta_i \zeta_j}{|\zeta|^2} \frac{\eta_i \eta_j}{|\eta|^{1/2}} \widehat{\mathfrak{S}U}(\eta)$$

and remark that the angle $\zeta \cdot \eta$ in this expression gives us the strongly semilinear structure we will use later (see also the factor \mathfrak{d} in (3.22)).

Proposition 3.1 is the starting point for the construction of our energy functionals. To prove it we first parilinearize and symmetrize the system (2.1) in subsection 3.2, see Lemma 3.5 and Proposition 3.7. Finally, we choose a suitable multiplier m' as in (3.4) in order to achieve the special structure (3.7), up to strongly semilinear quadratic terms and cubic terms.

From now on we set $g = 1$ and $\sigma = 1$. We can take derivatives, both Sobolev-type derivatives using the operator T_Σ and angular derivatives using Ω , to prove energy estimates. More precisely:

Proposition 3.4. *Assume that (3.1) holds. Then there is an energy functional \mathcal{E}_{tot} satisfying*

$$\|\mathcal{U}(t)\|_{H^{N_0} \cap H_\Omega^{N_1, N_3}}^2 \lesssim \mathcal{E}_{tot}(t) + \varepsilon_1^3, \quad \mathcal{E}_{tot}(t) \lesssim \|\mathcal{U}(t)\|_{H^{N_0} \cap H_\Omega^{N_1, N_3}}^2 + \varepsilon_1^3, \quad (3.18)$$

where $\mathcal{U}(t) = \langle \nabla \rangle h(t) + i|\nabla|^{1/2}\phi(t)$ as in Proposition 2.2. Moreover

$$\frac{d}{dt} \mathcal{E}_{tot} = \mathcal{B}_0 + \mathcal{B}_1 + \mathcal{B}_E, \quad |\mathcal{B}_E(t)| \lesssim \varepsilon_1^3(1+t)^{-4/3}. \quad (3.19)$$

The (bulk) terms \mathcal{B}_0 and \mathcal{B}_1 are finite sums of the form

$$\mathcal{B}_l(t) := \sum_{G \in \mathcal{G}, W, W' \in \mathcal{W}_l} \iint_{\mathbb{R}^2 \times \mathbb{R}^2} \mu_l(\xi, \eta) \widehat{G}(\xi - \eta) \widehat{W}(\eta) \widehat{W}'(-\xi) d\xi d\eta, \quad (3.20)$$

where U and Σ are defined as in Proposition 3.1, $U_+ := U$, $U_- := \bar{U}$, and

$$\begin{aligned} \mathcal{G} &:= \{\Omega^a \langle \nabla \rangle^b U_\pm : a \leq N_1/2 \text{ and } b \leq N_3 + 2\}, \\ \mathcal{W}_0 &:= \{\Omega^a T_\Sigma^m U_\pm : \text{either } (a = 0 \text{ and } m \leq 2N_0/3) \text{ or } (a \leq N_1 \text{ and } m \leq 2N_3/3)\}, \\ \mathcal{W}_1 &:= \mathcal{W}_0 \cup \{(1 - \Delta)\Omega^a T_\Sigma^m U_\pm : a \leq N_1 - 1 \text{ and } m \leq 2N_3/3\}. \end{aligned} \quad (3.21)$$

The symbols $\mu_l = \mu_{l; (G, W, W')}$, $l \in \{0, 1\}$, satisfy

$$\mu_0(\xi, \eta) = c|\xi - \eta|^{3/2} \mathfrak{d}(\xi, \eta), \quad \mathfrak{d}(\xi, \eta) := \chi\left(\frac{|\xi - \eta|}{|\xi + \eta|}\right) \left(\frac{\xi - \eta}{|\xi - \eta|} \cdot \frac{\xi + \eta}{|\xi + \eta|}\right)^2, \quad c \in \mathbb{C}, \quad (3.22)$$

$$\|\mu_1^{k, k_1, k_2}\|_{S^\infty} \lesssim 2^{-\max(k_1, k_2, 0)} 2^{3k_1^+},$$

for any $k, k_1, k_2 \in \mathbb{Z}$, see definitions (8.5)–(8.6).

This proposition is proved in subsection 3.3. Notice that the a priori energy estimates we prove here are stronger than standard energy estimates. The terms $\mathcal{B}_0, \mathcal{B}_1$ are strongly semilinear terms, in the sense that they either gain one derivative or contain the depletion factor \mathfrak{d} which gains one derivative when the modulation is small (compare with (1.28)).

3.2. Symmetrization and special quadratic structure. In this subsection we prove Proposition 3.1. We first write (2.1) as a system for h and ω , and then symmetrize it. We start by combining Proposition 9.1 on the Dirichlet-Neumann operator with a parilinearization of the equation for $\partial_t \phi$, to obtain the following:

Lemma 3.5. *[Parilinearization of the system] With the notation of Proposition 9.1 and Proposition 3.1, we can rewrite the system (2.1) as*

$$\begin{cases} \partial_t h = T_{\lambda_{DN}} \omega - \operatorname{div}(T_V h) + G_2 + \varepsilon_1^3 O_{3,1}, \\ \partial_t \omega = -gh - T_\ell h - T_V \nabla \omega + \Omega_2 + \varepsilon_1^3 O_{3,1}, \end{cases} \quad (3.23)$$

where ℓ is given in (3.2) and

$$\Omega_2 := \frac{1}{2} \mathcal{H}(|\nabla| \omega, |\nabla| \omega) - \frac{1}{2} \mathcal{H}(\nabla \omega, \nabla \omega) \in \varepsilon_1^2 O_{2,2}. \quad (3.24)$$

Proof. First, we see directly from (2.1) and Proposition 9.1 that, for any $t \in [0, T]$,

$$\begin{aligned} G(h)\phi, B, V, \partial_t h &\in \varepsilon_1 \mathcal{O}_{1,-1/2}, & \partial_t \phi &\in \varepsilon_1 \mathcal{O}_{1,-1}, \\ B &= |\nabla|\omega + \varepsilon_1^2 \mathcal{O}_{2,-1/2}, & V &= \nabla\omega + \varepsilon_1^2 \mathcal{O}_{2,-1/2}. \end{aligned} \quad (3.25)$$

The first equation in (3.23) comes directly from Propostion 9.1. To obtain the second equation, we use Lemma 8.4 (ii) with $F_l(x) = x_l/\sqrt{1+|x|^2}$ to see that

$$F_l(\nabla h) = T_{\partial_k F_l(\nabla h)} \partial_k h + \varepsilon_1^3 \mathcal{O}_{3,3}, \quad \text{hence} \quad \sigma \operatorname{div} \left[\frac{\nabla h}{\sqrt{1+|\nabla h|^2}} \right] = -T_{L_{jk}\zeta_j\zeta_k} h + \varepsilon_1^3 \mathcal{O}_{3,1}.$$

Next we parilinearize the other nonlinear terms in the second equation in (2.1). Recall the definition of V, B in (9.3). We first write

$$-\frac{1}{2}|\nabla\phi|^2 + \frac{(G(h)\phi + \nabla h \cdot \nabla\phi)^2}{2(1+|\nabla h|^2)} = -\frac{|V + B\nabla h|^2}{2} + \frac{(1+|\nabla h|^2)B^2}{2} = \frac{B^2 - 2BV \cdot \nabla h - |V|^2}{2}.$$

Using (2.1), we calculate $\partial_t h = G(h)\phi = B - V \cdot \nabla h$, and

$$\begin{aligned} \partial_t \omega &= \partial_t \phi - T_{\partial_t B} h - T_B \partial_t h \\ &= -gh - T_{L_{jk}\zeta_j\zeta_k} h + \frac{1}{2}(B^2 - 2BV \cdot \nabla h - |V|^2) - T_{\partial_t B} h - T_B B + T_B(V \cdot \nabla h) + \varepsilon_1^3 \mathcal{O}_{3,1}. \end{aligned}$$

Then, since $V = \nabla\phi - B\nabla h$, we have

$$T_V \nabla\omega = T_V \nabla\phi - T_V(\nabla T_B h) = T_V V + T_V(B\nabla h) - T_V(\nabla T_B h),$$

and we can write

$$\begin{aligned} \partial_t \omega &= -gh - T_{L_{jk}\zeta_j\zeta_k + \partial_t B} h - T_V \nabla\omega + I + II, \\ I &:= \frac{1}{2}B^2 - T_B B - \frac{1}{2}|V|^2 + T_V V = \frac{1}{2}\mathcal{H}(B, B) - \frac{1}{2}\mathcal{H}(V, V) = \Omega_2 + \varepsilon_1^3 \mathcal{O}_{3,1}, \\ II &:= -BV \cdot \nabla h + T_B(V \cdot \nabla h) + T_V(B\nabla h) - T_V(\nabla T_B h) + \varepsilon_1^3 \mathcal{O}_{3,1}. \end{aligned}$$

Using (3.25), (9.3), (2.1), and Corollary 9.7 (ii) we easily see that

$$L_{jk}\zeta_j\zeta_k + \partial_t B = L_{jk}\zeta_j\zeta_k + |\nabla|\partial_t \phi + \varepsilon_1^2 \mathcal{O}_{2,-2} = \ell + \varepsilon_1^2 \mathcal{O}_{2,-2}.$$

Moreover we can verify that II is an acceptable cubic remainder term:

$$\begin{aligned} II &= -T_{V \cdot \nabla h} B + \mathcal{H}(B, V \cdot \nabla h) + T_V(B\nabla h) - T_V T_B \nabla h - T_V T_{\nabla B} h + \varepsilon_1^3 \mathcal{O}_{3,1} \\ &= -T_{V \cdot \nabla h} B + T_V T_{\nabla h} B + T_V \mathcal{H}(B, \nabla h) - T_V T_{\nabla B} h + \varepsilon_1^3 \mathcal{O}_{3,1} \\ &= \varepsilon_1^3 \mathcal{O}_{3,1}, \end{aligned}$$

and the desired conclusion follows. \square

The symmetrization that will be performed below will allow us to write the main system in the form (3.32). Notice that the leading order operator is symmetric (the symbol is real valued) and is the same in both equations. This symbol will then be the natural notion of ‘‘derivative’’ associated to (3.32). Moreover, this will allow us to derive a single scalar equation for a single (complex-valued) unknown.

Before we proceed we observe that, using the notations of Proposition 8.5,

$$T_\Sigma T_{1/\sqrt{g+\ell}} = T_{\sqrt{\lambda_{DN}+m}} + E(\Sigma, (g+\ell)^{-1/2}), \quad m := \frac{i}{2\sqrt{g+\ell}} \{\sqrt{g+\ell}, \sqrt{\lambda_{DN}}\}. \quad (3.26)$$

Since our purpose will be to identify quadratic terms as in (3.8)-(3.9), we need a more precise notion of strongly semilinear quadratic errors.

Definition 3.6. *Given $t \in [0, T]$ we define $\varepsilon_1^2 \mathcal{O}_{2,1}^*$ to be the set of finite linear combinations of terms of the form $S[T_1, T_2]$ where $T_1, T_2 \in \{U(t), \bar{U}(t)\}$, and S satisfies*

$$\begin{aligned} \mathcal{F}(S[f, g])(\xi) &:= \frac{1}{4\pi^2} \int_{\mathbb{R}^2} s(\xi, \eta) \widehat{f}(\xi - \eta) \widehat{g}(\eta) d\eta, \\ \|s^{k, k_1, k_2}\|_{S_{\infty}^{\infty}} &\lesssim 2^{-\max(k_1, k_2, 0)} (1 + 2^{3\min(k_1, k_2)}). \end{aligned} \quad (3.27)$$

These correspond precisely to the acceptable quadratic error terms according to (3.9).

We remark that if S is defined by a symbol as in (3.27) and $p \in [-5, 5]$ then

$$S[\mathcal{O}_{m,p}, \mathcal{O}_{n,p}] \subseteq \mathcal{O}_{m+n,p+1}. \quad (3.28)$$

This follows by an argument similar to the argument used in Lemma 8.2. As a consequence, given the assumptions (3.1) and with U defined as in (3.4), we have that $\mathcal{O}_{2,1}^* \subseteq \mathcal{O}_{2,1}$.

In addition, using (3.13) and Lemma 8.7, we see if that $H := T_{\sqrt{g+\ell}}h$ and $\Psi := T_{\Sigma}T_{1/\sqrt{g+\ell}}\omega + T_{m'}\omega$ (as in (3.31) below) then

$$\begin{aligned} H &= \Re(U) + \varepsilon_1^2 \mathcal{O}_{2,0}, & \sqrt{g + \sigma|\nabla|^2}h &= \Re(U) + \varepsilon_1^2 \mathcal{O}_{2,0}, \\ \Psi &= \Im(U) + \varepsilon_1^2 \mathcal{O}_{2,0}, & |\nabla|^{1/2}\omega &= \Im(U) + \varepsilon_1^2 \mathcal{O}_{2,0}. \end{aligned} \quad (3.29)$$

As a consequence, if $T_1, T_2 \in \{U, \bar{U}, H, \Psi, (g - \sigma\Delta)^{1/2}h, |\nabla|^{1/2}\omega\}$, and S is as in (3.27), then

$$S[T_1, T_2] \in \varepsilon_1^2 \mathcal{O}_{2,1}^* + \varepsilon_1^3 \mathcal{O}_{3,0}. \quad (3.30)$$

We are now ready to isolate the main dispersion relation and quasilinear terms in the system.

Proposition 3.7 (Symmetrization). *Assume that H and Ψ are given by*

$$H := T_{\sqrt{g+\ell}}h, \quad \Psi := T_{\Sigma}T_{1/\sqrt{g+\ell}}\omega + T_{m'}\omega, \quad m' := \frac{i}{2} \frac{\operatorname{div}V}{\sqrt{g+\ell}} \in \varepsilon_1 \mathcal{M}_{N_3-2}^{-1,1}, \quad (3.31)$$

where (h, ϕ) satisfy (3.1). Then, for any $t \in [0, T]$,

$$\begin{cases} \partial_t H - T_{\Sigma}\Psi + iT_{V \cdot \zeta}H = \frac{2i}{3}[T_{V \cdot \zeta}, T_{\Sigma}]T_{\Sigma}^{-1}H - \left(\frac{1}{2}T_{\sqrt{g+\ell}}\operatorname{div}Vh + T_{m'\Sigma}\omega\right) + \varepsilon_1^2 \mathcal{O}_{2,1}^* + \varepsilon_1^3 \mathcal{O}_{3,0}, \\ \partial_t \Psi + T_{\Sigma}H + iT_{V \cdot \zeta}\Psi = \frac{i}{3}[T_{V \cdot \zeta}, T_{\Sigma}]T_{\Sigma}^{-1}\Psi - \left(T_{m'(g+\ell)}h - \frac{1}{2}T_{\sqrt{\lambda_{DN}}}\operatorname{div}V\omega\right) + \varepsilon_1^2 \mathcal{O}_{2,1}^* + \varepsilon_1^3 \mathcal{O}_{3,0}. \end{cases} \quad (3.32)$$

Proof. We compute first

$$\frac{1}{\sqrt{g+\ell}}\{V \cdot \zeta, \sqrt{g+\ell}\} = \frac{1}{2} \frac{\{V \cdot \zeta, \ell\}}{g+\ell} = \frac{\sigma|\zeta|^2}{g+\sigma|\zeta|^2} \frac{\zeta_i \zeta_j}{|\zeta|^2} \partial_i V_j + \varepsilon_1^2 \mathcal{M}_{N_3-4}^{0,2}.$$

Combining this with Lemma 8.7 and (3.16), we find that for $F \in \{H, \Psi\}$,

$$iT_{\frac{1}{\sqrt{g+\ell}}\{V \cdot \zeta, \sqrt{g+\ell}\}}F = \frac{2}{3}[T_{V \cdot \zeta}, T_{\Sigma}]T_{\Sigma}^{-1}F + \varepsilon_1^2 \mathcal{O}_{2,1}^* + \varepsilon_1^3 \mathcal{O}_{3,0}. \quad (3.33)$$

We examine now the first equation in the system (3.32). The first equation in (3.23) gives

$$\begin{aligned}
& \partial_t H - T_\Sigma \Psi + iT_{V \cdot \zeta} H - \frac{2i}{3} [T_{V \cdot \zeta}, T_\Sigma] T_\Sigma^{-1} H + \left(\frac{1}{2} T_{\sqrt{g+\ell}} \operatorname{div} V h + T_{m' \Sigma} \omega \right) \\
&= (T_{\sqrt{g+\ell}} T_{\lambda_{DN}} - T_\Sigma T_\Sigma T_{1/\sqrt{g+\ell}}) \omega - (T_\Sigma T_{m'} - T_{m' \Sigma}) \omega \\
&+ i(T_{V \cdot \zeta} H - T_{\sqrt{g+\ell}} T_{V \cdot \zeta} h - \frac{2}{3} [T_{V \cdot \zeta}, T_\Sigma] T_\Sigma^{-1} H) \\
&+ T_{\partial_t \sqrt{g+\ell}} h - \frac{1}{2} (T_{\sqrt{g+\ell}} T_{\operatorname{div} V} - T_{\sqrt{g+\ell} \operatorname{div} V}) h + T_{\sqrt{g+\ell}} G_2 + \varepsilon_1^3 T_{\sqrt{g+\ell}} \mathcal{O}_{3,1}.
\end{aligned} \tag{3.34}$$

We will treat each line separately. For the first line, we notice that the contribution of low frequencies $P_{\leq -9} \omega$ is acceptable. For the high frequencies we use Proposition 8.5 to write

$$\begin{aligned}
& (T_{\sqrt{g+\ell}} T_{\lambda_{DN}} - T_\Sigma T_\Sigma T_{1/\sqrt{g+\ell}}) P_{\geq -8} \omega \\
&= \left(T_{\lambda_{DN} \sqrt{g+\ell}} + \frac{i}{2} T_{\{\sqrt{g+\ell}, \lambda_{DN}\}} - (T_{\Sigma^2 / \sqrt{g+\ell}} + \frac{i}{2} T_{\{\Sigma^2, 1/\sqrt{g+\ell}\}}) \right) P_{\geq -8} \omega \\
&+ [E(\sqrt{g+\ell}, \lambda_{DN}) - E(\Sigma, \Sigma) T_{1/\sqrt{g+\ell}} - E(\Sigma^2, 1/\sqrt{g+\ell})] P_{\geq -8} \omega.
\end{aligned} \tag{3.35}$$

$$\tag{3.36}$$

Since

$$\lambda_{DN} \sqrt{g+\ell} = \Sigma^2 / \sqrt{g+\ell}, \quad \{\sqrt{g+\ell}, \lambda_{DN}\} = \{\Sigma^2, 1/\sqrt{g+\ell}\}$$

we observe that the expression in (3.35) vanishes. Using (3.13) and Lemma 8.8, we see that, up to acceptable cubic terms, we can rewrite the second line of (3.34) as

$$\begin{aligned}
& \left[E(\sqrt{g+\sigma|\zeta|^2}, \lambda_1^{(0)}) + E\left(-\frac{\Lambda^2 h}{2\sqrt{g+\sigma|\zeta|^2}}, |\zeta|\right) - (E(\Lambda, \Sigma_1) + E(\Sigma_1, \Lambda))(g - \sigma\Delta)^{-1/2} \right. \\
& \left. - E(\Lambda^2, \frac{\Lambda^2 h}{2(g+\sigma|\zeta|^2)^{3/2}}) - 2E(\Lambda \Sigma_1, \frac{1}{\sqrt{g+\sigma|\zeta|^2}}) - \frac{i}{2} T_{\{\Lambda, m'\}} - E(\Lambda, m') \right] P_{\geq -8} \omega + \varepsilon_1^3 \mathcal{O}_{3,0}.
\end{aligned}$$

Using (8.39) these terms are easily seen to be acceptable $\varepsilon_1^2 \mathcal{O}_{2,1}^*$ quadratic terms.

To control the terms in the second line of the right-hand side of (3.34), we observe that

$$T_{V \cdot \zeta} H - T_{\sqrt{g+\ell}} T_{V \cdot \zeta} h - \frac{2}{3} [T_{V \cdot \zeta}, T_\Sigma] T_\Sigma^{-1} H = ([T_{V \cdot \zeta}, T_{\sqrt{g+\ell}}] T_{\sqrt{g+\ell}}^{-1} - \frac{2}{3} [T_{V \cdot \zeta}, T_\Sigma] T_\Sigma^{-1}) H$$

and using (3.33), we see that

$$[T_{V \cdot \zeta}, T_{\sqrt{g+\ell}}] T_{\sqrt{g+\ell}}^{-1} H = \frac{2}{3} [T_{V \cdot \zeta}, T_\Sigma] T_\Sigma^{-1} H + \varepsilon_1^2 \mathcal{O}_{2,1}^* + \varepsilon_1^3 \mathcal{O}_{3,0}.$$

Finally, for the third line, using (3.14), (3.15), and Lemmas 8.7 and 8.8, we observe that

$$\begin{aligned}
& T_{\partial_t \sqrt{g+\ell}} h = T_{\frac{1}{2} \frac{\Delta(g-\sigma\Delta)\omega}{\sqrt{g+\sigma|\zeta|^2}}} h + \varepsilon_1^3 \mathcal{O}_{3,0}, \\
& (T_{\sqrt{g+\ell}} T_{\operatorname{div} V} - T_{\operatorname{div} V \cdot \sqrt{g+\ell}}) h = (iT_{\{\sqrt{g+\sigma|\zeta|^2}, \operatorname{div} V_1\}} + E(\sqrt{g+\sigma|\zeta|^2}, \operatorname{div} V_1)) h + \varepsilon_1^3 \mathcal{O}_{3,0}, \\
& T_{\sqrt{g+\ell}} G_2 = T_{\sqrt{g+\sigma|\zeta|^2}} G_2 + \varepsilon_1^3 \mathcal{O}_{3,0}, \\
& T_{\sqrt{g+\ell}} \mathcal{O}_{3,1} = \mathcal{O}_{3,0}.
\end{aligned}$$

Using (8.39), the bounds for G_2 in (9.6)-(9.7), and collecting all the estimates above, we obtain the identity in the first line in (3.32).

We now use (3.31) and (3.23) to compute

$$\begin{aligned}
& \partial_t \Psi + T_\Sigma H + iT_{V \cdot \zeta} \Psi - \frac{i}{3} [T_{V \cdot \zeta}, T_\Sigma] T_\Sigma^{-1} \Psi + (T_{m'(g+\ell)} h - \frac{1}{2} T_{\sqrt{\lambda_{DN} \operatorname{div}(V)}} \omega) \\
&= (T_\Sigma T_{\sqrt{g+\ell}} - T_\Sigma T_{1/\sqrt{g+\ell}} T_{g+\ell}) h + (T_{m'(g+\ell)} - T_{m'} T_{g+\ell}) h \\
&+ i(T_{V \cdot \zeta} \Psi - \frac{1}{3} [T_{V \cdot \zeta}, T_\Sigma] T_\Sigma^{-1} \Psi - (T_\Sigma T_{1/\sqrt{g+\ell}} + T_{m'}) T_{V \cdot \zeta} \omega) \\
&+ \frac{1}{2} (T_\Sigma T_{1/\sqrt{g+\ell}} T_{\operatorname{div} V} - T_{\sqrt{\lambda_{DN} \operatorname{div} V}} \omega) + \frac{1}{2} T_{m'} T_{\operatorname{div} V} \omega \\
&+ [\partial_t, T_\Sigma T_{1/\sqrt{g+\ell}} + T_{m'}] \omega + (T_\Sigma T_{1/\sqrt{g+\ell}} + T_{m'}) (\Omega_2 + \varepsilon_1^3 \mathcal{O}_{3,1}).
\end{aligned} \tag{3.37}$$

Again, we verify that all lines after the equality sign give acceptable remainders. For the terms in the first line, using Proposition 8.5, (3.13), and Lemma 8.8,

$$\begin{aligned}
& (T_\Sigma T_{\sqrt{g+\ell}} - T_\Sigma T_{1/\sqrt{g+\ell}} T_{g+\ell}) h = -T_\Sigma E(1/\sqrt{g+\ell}, g+\ell) h \\
&= \Lambda [E(\frac{\Lambda^2 h}{2(g+\sigma|\zeta|^2)^{3/2}}, g+\sigma|\zeta|^2) - E(1/\sqrt{g+\sigma|\zeta|^2}, \Lambda^2 h)] h + \varepsilon_1^3 \mathcal{O}_{3,0}.
\end{aligned}$$

Using also (8.39), this gives acceptable contributions. In addition,

$$(T_{m'} T_{g+\ell} - T_{m'(g+\ell)}) h = \frac{i}{2} T_{\{m', g+\ell\}} h + E(m', g+\ell) h = i\sigma T_{\zeta \cdot \nabla_x m'} h + E(m', \sigma|\zeta|^2) h + \varepsilon_1^3 \mathcal{O}_{3,0}.$$

This gives acceptable contributions, in view of (3.31) and (8.39).

For the terms in the second line of the right-hand side of (3.37) we observe that

$$\begin{aligned}
& T_{V \cdot \zeta} \Psi - \frac{1}{3} [T_{V \cdot \zeta}, T_\Sigma] T_\Sigma^{-1} \Psi - (T_\Sigma T_{1/\sqrt{g+\ell}} + T_{m'}) T_{V \cdot \zeta} \omega \\
&= [T_{V \cdot \zeta}, T_\Sigma T_{1/\sqrt{g+\ell}}] (T_\Sigma T_{1/\sqrt{g+\ell}})^{-1} \Psi - \frac{1}{3} [T_{V \cdot \zeta}, T_\Sigma] T_\Sigma^{-1} \Psi \\
&+ [T_\Sigma T_{1/\sqrt{g+\ell}}, T_{V \cdot \zeta}] (T_\Sigma T_{1/\sqrt{g+\ell}})^{-1} T_{m'} \omega + [T_{V \cdot \zeta}, T_{m'}] \omega \\
&= [T_{V \cdot \zeta}, T_\Sigma T_{1/\sqrt{g+\ell}}] (T_\Sigma T_{1/\sqrt{g+\ell}})^{-1} \Psi - \frac{1}{3} [T_{V \cdot \zeta}, T_\Sigma] T_\Sigma^{-1} \Psi + \varepsilon_1^3 \mathcal{O}_{3,0},
\end{aligned}$$

where we have used (3.31) and Lemma 8.8 for the last equality. Using also Lemma 8.7 we have

$$\begin{aligned}
& [T_{V \cdot \zeta}, T_\Sigma T_{1/\sqrt{g+\ell}}] (T_\Sigma T_{1/\sqrt{g+\ell}})^{-1} \Psi = [T_{V \cdot \zeta}, |\nabla|^{1/2}] |\nabla|^{-1/2} \Psi + \varepsilon_1^3 \mathcal{O}_{3,0} \\
&= iT_{\frac{1}{2} \frac{\zeta_i \zeta_j \partial_i V_j}{|\zeta|^2}} \Psi + \varepsilon_1^2 \mathcal{O}_{2,1}^* + \varepsilon_1^3 \mathcal{O}_{3,0}.
\end{aligned}$$

Using now (3.16), it follows that the sum of the terms in the second line is acceptable.

It is easy to see, using Lemma 8.8 and the definitions, that the terms the third line in the right-hand side of (3.37) are acceptable. Finally, for the last line in (3.37), we observe that

$$\begin{aligned}
& [\partial_t, T_\Sigma T_{1/\sqrt{g+\ell}} + T_{m'}] \omega = T_{\partial_t \Sigma} T_{1/\sqrt{g+\ell}} \omega + T_\Sigma T_{\partial_t(1/\sqrt{g+\ell})} \omega + T_{\partial_t m'} \omega \\
&= T_{\partial_t \Sigma_1} T_{(g+\sigma|\zeta|^2)^{-1/2}} \omega - \Lambda T_{\frac{\Delta(g-\sigma\Delta)\omega}{2(g+\sigma|\zeta|^2)^{3/2}}} \omega + \frac{i}{2} T_{\partial_t(\operatorname{div} V)(g+\sigma|\zeta|^2)^{-1/2}} \omega + \varepsilon_1^3 \mathcal{O}_{3,0},
\end{aligned}$$

where we used (3.13) and (3.14). Since $\partial_t h = |\nabla| \omega + \varepsilon_1^2 \mathcal{O}_{2,-1/2}$ and $\partial_t V = -\nabla(g + \sigma|\nabla|^2) h + \varepsilon_1^2 \mathcal{O}_{2,-2}$ (see Lemma 3.5 and Proposition 9.1), it follows that the terms in the formula above are acceptable. Finally, using the relations in Lemma 3.5,

$$(T_\Sigma T_{1/\sqrt{g+\ell}} + T_{m'}) (\Omega_2) = \varepsilon_1^3 \mathcal{O}_{3,0} + \varepsilon_1^2 \mathcal{O}_{2,1}^*, \quad (T_\Sigma T_{1/\sqrt{g+\ell}} + T_{m'}) (\varepsilon_1^3 \mathcal{O}_{3,1}) = \varepsilon_1^3 \mathcal{O}_{3,0}.$$

Therefore, the terms in the right-hand side of (3.37) are acceptable, which gives (3.32). \square

Proof of Proposition 3.1. Starting from the system (3.32) we now want to write a scalar equation for the complex unknown

$$U := T_{\sqrt{g+\ell}}h + iT_{\Sigma}T_{1/\sqrt{g+\ell}}\omega + iT_{m'}\omega = H + i\Psi, \quad m' = \frac{i}{2} \frac{\operatorname{div}V}{\sqrt{g+\ell}} \in \mathcal{M}_{N_3-2}^{-1,1}.$$

Using (3.32), we readily see that

$$\begin{aligned} \partial_t U + iT_{\Sigma}U + iT_{V,\zeta}U &= Q_U + \mathcal{N}_U + \varepsilon_1^2 \mathcal{O}_{2,1}^* + \varepsilon_1^3 \mathcal{O}_{3,0}, \\ Q_U &:= \left(-\frac{1}{2}T_{\sqrt{g+\ell}}\operatorname{div}V - iT_{m'(g+\ell)}\right)h + (-T_{m'\Sigma} + \frac{i}{2}T_{\sqrt{\lambda_{DN}}}\operatorname{div}V)\omega = 0, \\ \mathcal{N}_U &:= \frac{i}{3}[T_{V,\zeta}, T_{\Sigma}]T_{\Sigma}^{-1}(2H + i\Psi) = \frac{i}{6}[T_{V,\zeta}, T_{\Sigma}]T_{\Sigma}^{-1}(3U + \bar{U}) + \varepsilon_1^3 \mathcal{O}_{3,0}, \end{aligned}$$

where Q_U vanishes in view of our choice of m' , and \mathcal{N}_U has the null structure as claimed. \square

3.3. High order derivatives: proof of Proposition 3.4. To derive higher order Sobolev and weighted estimates for U , and hence for h and $|\nabla|^{1/2}\omega$, we need to apply (a suitable notion of) derivatives to the equation (3.6). We will then consider quantities of the form

$$W_n := (T_{\Sigma})^n U, \quad n \in [0, 2N_0/3], \quad Y_{m,p} := \Omega^p (T_{\Sigma})^m U, \quad p \in [0, N_1], \quad m \in [0, 2N_3/3], \quad (3.38)$$

for U as in (3.4) and Σ as in (3.3). We have the following consequence of Proposition 3.1:

Proposition 3.8. *With the notation above and γ as in (3.17), we have*

$$\partial_t W_n + iT_{\Sigma}W_n + iT_{V,\zeta}W_n = T_{\gamma}(c_n W_n + d_n \bar{W}_n) + \mathcal{B}_{W_n} + \mathcal{C}_{W_n}, \quad (3.39)$$

and

$$\partial_t Y_{m,p} + iT_{\Sigma}Y_{m,p} + iT_{V,\zeta}Y_{m,p} = T_{\gamma}(c_m Y_{m,p} + d_m \bar{Y}_{m,p}) + \mathcal{B}_{Y_{m,p}} + \mathcal{C}_{Y_{m,p}}, \quad (3.40)$$

for some complex numbers c_n, d_n . The cubic terms \mathcal{C}_{W_n} and $\mathcal{C}_{Y_{m,p}}$ satisfy the bounds

$$\|\mathcal{C}_{W_n}\|_{L^2} + \|\mathcal{C}_{Y_{m,p}}\|_{L^2} \lesssim \varepsilon_1^3 (1+t)^{-3/2}. \quad (3.41)$$

The quadratic strongly semilinear terms \mathcal{B}_{W_n} have the form

$$\mathcal{B}_{W_n} = \sum_{\iota_1 \iota_2 \in \{+, -\}} F_{\iota_1 \iota_2}^n [U_{\iota_1}, U_{\iota_2}], \quad (3.42)$$

where $U_+ := U$, $U_- = \bar{U}$, and the symbols $f = f_{\iota_1 \iota_2}^n$ of the bilinear operators $S_{\iota_1 \iota_2}^n$ satisfy

$$\|f^{k,k_1,k_2}\|_{S^\infty} \lesssim 2^{(3n/2-1)\max(k_1,k_2,0)} (1 + 2^{3\min(k_1,k_2)}). \quad (3.43)$$

The quadratic strongly semilinear terms $\mathcal{B}_{Y_{m,p}}$ have the form

$$\mathcal{B}_{Y_{m,p}} = \sum_{\iota_1 \iota_2 \in \{+, -\}} \{G_{\iota_1 \iota_2}^{m,p} [U_{\iota_1}, \Omega^p U_{\iota_2}] + \sum_{p_1+p_2 \leq p, \max(p_1,p_2) \leq p-1} H_{\iota_1 \iota_2}^{m,p,p_1,p_2} [\Omega^{p_1} U_{\iota_1}, \Omega^{p_2} U_{\iota_2}]\}, \quad (3.44)$$

where the symbols $g = g_{\iota_1 \iota_2}^{m,p}$ and $h = h_{\iota_1 \iota_2}^{m,p,p_1,p_2}$ of the operators $G_{\iota_1 \iota_2}^{m,p}$ and $H_{\iota_1 \iota_2}^{m,p,p_1,p_2}$ satisfy

$$\begin{aligned} \|g^{k,k_1,k_2}\|_{S^\infty} &\lesssim 2^{(3m/2-1)\max(k_1,k_2,0)} (1 + 2^{3\min(k_1,k_2)}), \\ \|h^{k,k_1,k_2}\|_{S^\infty} &\lesssim 2^{(3m/2+1)\max(k_1,k_2,0)} (1 + 2^{\min(k_1,k_2)}). \end{aligned} \quad (3.45)$$

We remark that we have slightly worse information on the quadratic terms $\mathcal{B}_{Y_{m,p}}$ than on the quadratic terms \mathcal{B}_{W_n} . This is due mainly to the commutator of the operators Ω^p and $T_{V,\zeta}$, which leads to the additional terms in (3.44). These terms can still be regarded as strongly semilinear because they do not contain the maximum number of Ω derivatives (they do contain, however, 2 extra Sobolev derivatives, but this is acceptable due to our choice of N_0 and N_1).

Proof of Proposition 3.8. In this proof we need to expand the definition of our main spaces $\mathcal{O}_{m,p}$ to exponents $p < -N_3$. More precisely, we define, for any $t \in [0, T]$,

$$\begin{aligned} \|f\|_{\mathcal{O}'_{m,p}} &:= \|f\|_{\mathcal{O}_{m,p}} \quad \text{if } p \geq -N_3, \\ \|f\|_{\mathcal{O}'_{m,p}} &:= \langle t \rangle^{(m-1)(5/6-20\delta^2)-\delta^2} [\|f\|_{H^{N_0+p}} + \langle t \rangle^{5/6-2\delta^2} \|f\|_{\widetilde{W}_\Omega^{N_2+p}}] \quad \text{if } p < -N_3, \end{aligned} \quad (3.46)$$

compare with (8.7). As in Lemmas 8.7 and 8.8, we have the basic imbeddings

$$T_a \mathcal{O}'_{m,p} \subseteq \mathcal{O}'_{m+m_1,p-l_1}, \quad (T_a T_b - T_{ab}) \mathcal{O}'_{m,p} \subseteq \mathcal{O}'_{m+m_1+m_2,p-l_1-l_2+1}, \quad (3.47)$$

if $a \in \mathcal{M}_{20}^{l_1, m_1}$ and $b \in \mathcal{M}_{20}^{l_2, m_2}$. In particular, recalling that, see (3.12),

$$\Sigma - \Lambda \in \varepsilon_1 \mathcal{M}_{N_3-2}^{3/2, 1} \quad \Sigma - \Lambda - \Sigma_1 \in \varepsilon_1^2 \mathcal{M}_{N_3-2}^{3/2, 2}, \quad (3.48)$$

it follows from (3.47) that, for any $n \in [0, 2N_0/3]$,

$$T_\Sigma^n U \in \varepsilon_1 \mathcal{O}'_{1, -3n/2}, \quad T_\Sigma^n U - \Lambda^n U = \sum_{l=0}^{n-1} \Lambda^{n-1-l} (T_{\Sigma-\Lambda})^l T_\Sigma^n U \in \varepsilon_1^2 \mathcal{O}'_{2, -3n/2}. \quad (3.49)$$

Step 1. For $n \in [0, 2N_0/3]$, we prove first that the function $W_n = (T_\Sigma)^n U$ satisfies

$$\begin{aligned} (\partial_t + iT_\Sigma + iT_{V,\zeta}) W_n &= T_\gamma (c_n W_n + d_n \overline{W_n}) + \mathcal{N}_{S,n} + \varepsilon_1^3 \mathcal{O}'_{3, -3n/2}, \\ \mathcal{N}_{S,n} &= \sum_{\iota_1, \iota_2 \in \{+, -\}} B_{\iota_1 \iota_2}^n [U_{\iota_1}, U_{\iota_2}] \in \varepsilon_1^2 \mathcal{O}'_{2, -3n/2+1}, \\ \| (b_{\iota_1 \iota_2}^n)^{k_1, k_2} \|_{S_\Omega^\infty} &\lesssim (1 + 2^{3 \min(k_1, k_2)}) \cdot (1 + 2^{\max(k_1, k_2)})^{3n/2-1}. \end{aligned} \quad (3.50)$$

Indeed, the case $n = 0$ follows from the main equation (3.6) and (3.16). Assuming that this is true for some $n < 2N_0/3$ and applying T_Σ , we find that

$$\begin{aligned} (\partial_t + iT_\Sigma + iT_{V,\zeta}) W_{n+1} &= T_\gamma (c_n W_{n+1} + d_n \overline{W_{n+1}}) + i[T_{V,\zeta}, T_\Sigma] W_n + [\partial_t, T_\Sigma] W_n \\ &\quad + [T_\Sigma, T_\gamma] (c_n W_n + d_n \overline{W_n}) + T_\Sigma \mathcal{N}_{S,n} + \varepsilon_1^3 T_\Sigma \mathcal{O}'_{3, -3n/2}. \end{aligned}$$

Using (3.47)–(3.49) and (3.14) it follows that

$$[\partial_t, T_\Sigma] W_n = T_{\partial_t \Sigma_1} \Lambda^n U + \varepsilon_1^3 \mathcal{O}'_{3, -3(n+1)/2}, \quad T_\Sigma \mathcal{N}_{S,n} = \Lambda \mathcal{N}_{S,n} + \varepsilon_1^3 \mathcal{O}'_{3, -3(n+1)/2},$$

and, using also Remark 3.3,

$$\begin{aligned} [T_\Sigma, T_\gamma] (c_n W_n + d_n \overline{W_n}) &= [T_\Lambda, T_\gamma] (c_n \Lambda^n U + d_n \Lambda^n \overline{U}) + \varepsilon_1^3 \mathcal{O}'_{3, -3(n+1)/2}, \\ [T_{V,\zeta}, T_\Sigma] W_n &= [T_{V_1, \zeta}, T_\Lambda] W_n + \varepsilon_1^3 \mathcal{O}'_{3, -3(n+1)/2} = \frac{3i}{2} T_\gamma W_{n+1} + \mathcal{N}'(\mathfrak{S}U, \Lambda^n U) + \varepsilon_1^3 \mathcal{O}'_{3, -3(n+1)/2}, \end{aligned}$$

where $\mathcal{N}'(\mathfrak{S}U, \Lambda^n U)$ is an acceptable strongly semilinear quadratic term as in (3.50). Since $\partial_t h = |\nabla|\omega + \varepsilon_1^2 \mathcal{O}_{2, -1/2}$, and recalling the formulas (3.12) and (3.29), it is easy to see that all the remaining quadratic terms are of the strongly semilinear type described in (3.50). This completes the induction step.

Step 2. We can now prove the proposition. The claims for W_n follow directly from (3.50). It remains to prove the claims for the functions $Y_{m,p}$. Assume $m \in [0, 2N_3/3]$ is fixed. We start from the identity (3.50) with $n = m$, and apply the rotation vector field Ω . Clearly

$$\begin{aligned} (\partial_t + iT_\Sigma + iT_{V \cdot \zeta})Y_{m,p} &= T_\gamma(c_m Y_{m,p} + d_m \overline{Y_{m,p}}) + \Omega^p \mathcal{N}_{S,m} + \varepsilon_1^3 \Omega^p \mathcal{O}'_{3,-3m/2} \\ &\quad - i[\Omega^p, T_\Sigma]W_m - i[\Omega^p, T_{V \cdot \zeta}]W_m + [\Omega^p, T_\gamma](c_m W_m + d_m \overline{W_m}). \end{aligned}$$

The terms in the first line of the right-hand side are clearly acceptable. It remains to show that the commutators in the second line can also be written as strongly semilinear quadratic terms and cubic terms. Indeed, for $\sigma \in \{\Sigma, V \cdot \zeta, \gamma\}$ and $W \in \{W_m, \overline{W_m}\}$,

$$[\Omega^p, T_\sigma]W = \sum_{p'=0}^{p-1} c_{p,p'} T_{\Omega_{x,\zeta}^{p-p'} \sigma} \Omega^{p'} W. \quad (3.51)$$

In view of (3.49),

$$\begin{aligned} \|\Omega^{N_1} W_m\|_{L^2} + \|\langle \nabla \rangle^{N_0 - N_3} W_m\|_{L^2} &\lesssim \varepsilon_1 \langle t \rangle^{\delta^2}, \\ \|\Omega^{N_1} (W_m - \Lambda^m U)\|_{L^2} + \|\langle \nabla \rangle^{N_0 - N_3} (W_m - \Lambda^m U)\|_{L^2} &\lesssim \varepsilon_1^2 \langle t \rangle^{21\delta^2 - 5/6}. \end{aligned} \quad (3.52)$$

and, for $q \in [0, N_1/2]$

$$\|\Omega^q W_m\|_{\widetilde{W}^3} \lesssim \varepsilon_1 \langle t \rangle^{3\delta^2 - 5/6}, \quad \|\Omega^q (W_m - \Lambda^m U)\|_{\widetilde{W}^3} \lesssim \varepsilon_1^2 \langle t \rangle^{23\delta^2 - 5/3}. \quad (3.53)$$

By interpolation, and using the fact that $N_0 - N_3 \geq 3N_1/2$, it follows from (3.52) that

$$\|\Omega^q \langle \nabla \rangle^{3/2} W_m\|_{L^2} \lesssim \varepsilon_1 \langle t \rangle^{\delta^2}, \quad \|\Omega^q \langle \nabla \rangle^{3/2} (W_m - \Lambda^m U)\|_{L^2} \lesssim \varepsilon_1^2 \langle t \rangle^{21\delta^2 - 5/6} \quad (3.54)$$

for $q \in [0, N_1 - 1]$. Moreover, for $\sigma \in \{\Sigma, V \cdot \zeta, \gamma\}$ and $q \in [1, N_1]$, we have

$$\|\langle \zeta \rangle^{-3/2} \Omega_{x,\zeta}^q \sigma\|_{\mathcal{M}_{20,2}} \lesssim \varepsilon_1 \langle t \rangle^{2\delta^2}, \quad \|\langle \zeta \rangle^{-3/2} \Omega_{x,\zeta}^q (\sigma - \sigma_1)\|_{\mathcal{M}_{20,2}} \lesssim \varepsilon_1^2 \langle t \rangle^{22\delta^2 - 5/6}, \quad (3.55)$$

while for $q \in [1, N_1/2]$ we also have

$$\|\langle \zeta \rangle^{-3/2} \Omega_{x,\zeta}^q \sigma\|_{\mathcal{M}_{20,\infty}} \lesssim \varepsilon_1 \langle t \rangle^{4\delta^2 - 5/6}, \quad \|\langle \zeta \rangle^{-3/2} \Omega_{x,\zeta}^q (\sigma - \sigma_1)\|_{\mathcal{M}_{20,\infty}} \lesssim \varepsilon_1^2 \langle t \rangle^{24\delta^2 - 5/3}. \quad (3.56)$$

See (8.20) for the definition of the norms $\mathcal{M}_{20,q}$. In these estimates σ_1 denotes the linear part of σ , i.e. $\sigma_1 \in \{\Sigma_1, V_1 \cdot \zeta, \gamma\}$. Therefore, using Lemma 8.7 and (3.53)–(3.56),

$$T_{\Omega_{x,\zeta}^{p-p'} \sigma} \Omega^{p'} W = T_{\Omega_{x,\zeta}^{p-p'} \sigma} \Omega^{p'} \Lambda^m U_\pm + \varepsilon_1^3 \langle t \rangle^{-8/5} L^2 = T_{\Omega_{x,\zeta}^{p-p'} \sigma_1} \Omega^{p'} \Lambda^m U_\pm + \varepsilon_1^3 \langle t \rangle^{-8/5} L^2,$$

for $p' \in [0, p - 1]$ and $W \in \{W_m, \overline{W_m}\}$. Notice that $T_{\Omega_{x,\zeta}^{p_1, \sigma_1}} \Omega^{p_2} \Lambda^m U_\pm$ can be written as $H_{\iota_1 \iota_2}^{m, p, p_1, p_2} [\Omega^{p_1} U_{\iota_1}, \Omega^{p_2} U_{\iota_2}]$, with symbols as in (3.45), up to acceptable cubic terms (the loss of 1 high derivative comes from the case $\sigma_1 = V_1 \cdot \zeta$). The conclusion of the proposition follows. \square

We are now ready to prove the energy estimates

Proof of Proposition 3.4. We define our main energy functionals

$$\mathcal{E}_{tot} := \frac{1}{2} \sum_{0 \leq n \leq 2N_0/3} \|W_n\|_{L^2}^2 + \frac{1}{2} \sum_{0 \leq m \leq 2N_3/3} \sum_{0 \leq p \leq N_1} \|Y_{m,p}\|_{L^2}^2. \quad (3.57)$$

The bound (3.18) follows from (3.5) and (3.49),

$$\|\langle \nabla \rangle h(t)\|_{H^{N_0} \cap H_\Omega^{N_1, N_3}}^2 + \|\langle \nabla \rangle^{1/2} \phi(t)\|_{H^{N_0} \cap H_\Omega^{N_1, N_3}}^2 \lesssim \|U(t)\|_{H^{N_0} \cap H_\Omega^{N_1, N_3}}^2 + \varepsilon_1^3 \lesssim \mathcal{E}_{tot}(t) + \varepsilon_1^3.$$

To prove the remaining claims we start from (3.39) and (3.40). For the terms W_n we have

$$\frac{d}{dt} \frac{1}{2} \|W_n\|_{L^2}^2 = \Re \langle T_\gamma(c_n W_n + d_n \overline{W_n}), W_n \rangle + \Re \langle \mathcal{B}_{W_n}, W_n \rangle + \Re \langle \mathcal{C}_{W_n}, W_n \rangle, \quad (3.58)$$

since, as a consequence of Lemma 8.3 (ii),

$$\Re \langle iT_\Sigma W_n + iT_{V,\zeta} W_n, W_n \rangle = 0.$$

Clearly, $|\langle \mathcal{C}_{W_n}, W_n \rangle| \lesssim \varepsilon_1^3 \langle t \rangle^{-3/2+2\delta^2}$, so the last term can be placed in $\mathcal{B}_E(t)$. Moreover, using (3.17) and the definitions, $\langle T_\gamma(c_n W_n + d_n \overline{W_n}), W_n \rangle$ can be written in the Fourier space as part of the term $\mathcal{B}_0(t)$ in (3.20).

Finally, $\langle \mathcal{B}_{W_n}, W_n \rangle$ can be written in the Fourier space as part of the term $\mathcal{B}_1(t)$ in (3.20) plus acceptable errors. Indeed, given a symbol f as in (3.43), one can write

$$f(\xi, \eta) = \mu_1(\xi, \eta) \cdot [(1 + \Lambda(\xi - \eta)^n) + (1 + \Lambda(\eta)^n)], \quad \mu_1(\xi, \eta) := \frac{f(\xi, \eta)}{2 + \Lambda(\xi - \eta)^n + \Lambda(\eta)^n}.$$

The symbol μ_1 satisfies the required estimate in (3.22). The factors $1 + \Lambda(\xi - \eta)^n$ and $1 + \Lambda(\eta)^n$ can be combined with the functions $\widehat{U}_{l_1}(\xi - \eta)$ and $\widehat{U}_{l_2}(\eta)$ respectively. Recalling that $\Lambda^n U - W_n \in \varepsilon_1^2 \mathcal{O}'_{2,-3n/2}$, see (3.49), the desired representation (3.20) follows, up to acceptable errors.

The analysis of the terms $Y_{m,p}$ is similar, using (3.44)-(3.45). This completes the proof. \square

4. ENERGY ESTIMATES, II: SETUP AND THE MAIN L^2 LEMMA

In this section we set up the proof of Proposition 2.2 and collect some of the main ingredients needed in the proof. In view of (3.18), it suffices to prove that $|\mathcal{E}_{tot}(t) - \mathcal{E}_{tot}(0)| \lesssim \varepsilon_1^2$ for any $t \in [0, T]$. In view of (3.19) it suffices to prove that, for $l \in \{0, 1\}$,

$$\left| \int_0^t \mathcal{B}_l(s) ds \right| \lesssim \varepsilon_1^3 (1+t)^{2\delta^2},$$

for any $t \in [0, T]$. Given $t \in [0, T]$, we fix a suitable decomposition of the function $\mathbf{1}_{[0,t]}$, i.e. we fix functions $q_0, \dots, q_{L+1} : \mathbb{R} \rightarrow [0, 1]$, $|L - \log_2(2+t)| \leq 2$, with the properties

$$\begin{aligned} \text{supp } q_0 &\subseteq [0, 2], \quad \text{supp } q_{L+1} \subseteq [t-2, t], \quad \text{supp } q_m \subseteq [2^{m-1}, 2^{m+1}] \text{ for } m \in \{1, \dots, L\}, \\ \sum_{m=0}^{L+1} q_m(s) &= \mathbf{1}_{[0,t]}(s), \quad q_m \in C^1(\mathbb{R}) \text{ and } \int_0^t |q'_m(s)| ds \lesssim 1 \text{ for } m \in \{1, \dots, L\}. \end{aligned} \quad (4.1)$$

It remains to prove that for $l \in \{0, 1\}$ and $m \in \{0, \dots, L+1\}$,

$$\left| \int_{\mathbb{R}} \mathcal{B}_l(s) q_m(s) ds \right| \lesssim \varepsilon_1^3 2^{2\delta^2 m}. \quad (4.2)$$

In order to be able to use the hypothesis $\|\mathcal{V}(s)\|_Z \leq \varepsilon_1$ (see (2.6)) we need to modify slightly the functions G that appear in the terms \mathcal{B}_l . More precisely, we define

$$\mathcal{G}' := \{\Omega^a \langle \nabla \rangle^b \mathcal{U}_\iota : \iota \in \{+, -\}, a \leq N_1/2 \text{ and } b \leq N_3 + 2\}, \quad (4.3)$$

where $\mathcal{U} = \langle \nabla \rangle h + i|\nabla|^{1/2} \phi$, $\mathcal{U}_+ = \mathcal{U}$ and $\mathcal{U}_- = \overline{\mathcal{U}}$. Then we define the modified bilinear terms

$$\mathcal{B}'_l(t) := \sum_{G \in \mathcal{G}', W, W' \in \mathcal{W}_l} \iint_{\mathbb{R}^2 \times \mathbb{R}^2} \mu_l(\xi, \eta) \widehat{G}(\xi - \eta, t) \widehat{W}(\eta, t) \widehat{W}'(-\xi, t) d\xi d\eta, \quad (4.4)$$

where the sets $\mathcal{W}_0, \mathcal{W}_1$ are as in (3.21), and the symbols μ_0 and μ_1 are as in (3.22). In view of (3.5), $U(t) - \mathcal{U}(t) \in \varepsilon_1^2 \mathcal{O}_{2,0}$. Therefore, simple estimates as in the proof of Lemma 8.2 show that

$$|\mathcal{B}_l(s)| \lesssim \varepsilon_1^3 (1+s)^{-4/5}, \quad |\mathcal{B}_l(s) - \mathcal{B}'_l(s)| \lesssim \varepsilon_1^3 (1+s)^{-8/5}.$$

As a result of these reductions, for Proposition 2.2 it suffices to prove the following:

Proposition 4.1. *Assume that (h, ϕ) is a solution of the system (2.1) with $g = 1$, $\sigma = 1$ on $[0, T]$, and let $\mathcal{U} = \langle \nabla \rangle h + i|\nabla|^{1/2} \phi$, $\mathcal{V}(t) = e^{it\Lambda} \mathcal{U}(t)$. Assume that*

$$\langle t \rangle^{-\delta^2} \|\mathcal{U}(t)\|_{H^{N_0} \cap H_0^{N_1, N_3}} + \|\mathcal{V}(t)\|_Z \leq \varepsilon_1, \quad (4.5)$$

for any $t \in [0, T]$, see (2.6). Then, for any $m \in [\mathcal{D}^2, L]$ and $l \in \{0, 1\}$,

$$\left| \int_{\mathbb{R}} \iint_{\mathbb{R}^2 \times \mathbb{R}^2} q_m(s) \mu_l(\xi, \eta) \widehat{G}(\xi - \eta, s) \widehat{W}(\eta, s) \widehat{W}'(-\xi, s) d\xi d\eta ds \right| \lesssim \varepsilon_1^3 2^{2\delta^2 m}, \quad (4.6)$$

where $G \in \mathcal{G}'$ (see (4.3)), and $W, W' \in \mathcal{W}' := \mathcal{W}_1$ (see (3.21)), and q_m are as in (4.1). The symbols μ_0, μ_1 satisfy the bounds (compare with (3.22))

$$\begin{aligned} \mu_0(\xi, \eta) &= |\xi - \eta|^{3/2} \mathfrak{d}(\xi, \eta), \quad \mathfrak{d}(\xi, \eta) := \chi\left(\frac{|\xi - \eta|}{|\xi + \eta|}\right) \left(\frac{\xi - \eta}{|\xi - \eta|} \cdot \frac{\xi + \eta}{|\xi + \eta|}\right)^2, \\ \|\mu_1^{k, k_1, k_2}\|_{S^\infty} &\lesssim 2^{-\max(k_1, k_2, 0)} 2^{3k_1^+}. \end{aligned} \quad (4.7)$$

The proof of this proposition will be done in several steps. We remark that both the symbols μ_0 and μ_1 introduce certain *strongly semilinear* structures. The symbol μ_0 contains the depletion factor \mathfrak{d} , which counts essentially as a gain of one high derivative in resonant situations. The symbols μ_1 clearly contain a gain of one high derivative.

We will need to further subdivide the expression in (4.6) into the contributions of “good frequencies” with optimal decay and the “bad frequencies” with slower decay. Let

$$\chi_{ba}(x) := \varphi(2^{\mathcal{D}}(|x| - \gamma_0)) + \varphi(2^{\mathcal{D}}(|x| - \gamma_1)), \quad \chi_{go}(x) := 1 - \chi_{ba}(x), \quad (4.8)$$

where $\gamma_0 = \sqrt{\frac{2\sqrt{3}-3}{3}}$ is the radius of the sphere of degenerate frequencies, and $\gamma_1 = \sqrt{2}$ is the radius of the sphere of space-time resonances. We then define for $l \in \{0, 1\}$ and $Y \in \{go, ba\}$,

$$\mathcal{A}_Y^l[F; H_1, H_2] := \iint_{\mathbb{R}^2 \times \mathbb{R}^2} \mu_l(\xi, \eta) \chi_Y(\xi - \eta) \widehat{F}(\xi - \eta) \widehat{H}_1(\eta) \widehat{H}_2(-\xi) d\xi d\eta. \quad (4.9)$$

In the proof of (4.6) we will need to distinguish between functions G and W that originate from $U = U_+$ and functions that originate from $\bar{U} = U_-$. For this we define, for $\iota \in \{+, -\}$,

$$\mathcal{G}'_\iota := \{\Omega^a \langle \nabla \rangle^b \mathcal{U}_\iota : a \leq N_1/2 \text{ and } b \leq N_3 + 2\}, \quad (4.10)$$

and

$$\begin{aligned} \mathcal{W}'_\iota &:= \{\langle \nabla \rangle^c \Omega^a T_\Sigma^m U_\iota : \text{either } (a = c = 0 \text{ and } m \leq 2N_0/3) \\ &\quad \text{or } (c \in \{0, 2\}, c/2 + a \leq N_1, \text{ and } m \leq 2N_3/3)\}. \end{aligned} \quad (4.11)$$

4.1. Some lemmas. In this subsection we collect some lemmas that are used often in the proofs in the next section. We will often use the Schur's test:

Lemma 4.2 (Schur's lemma). *Consider the operator T given by*

$$Tf(\xi) = \int_{\mathbb{R}^2} K(\xi, \eta) f(\eta) d\eta.$$

Assume that

$$\sup_{\xi} \int_{\mathbb{R}^2} |K(\xi, \eta)| d\eta \leq K_1, \quad \sup_{\eta} \int_{\mathbb{R}^2} |K(\xi, \eta)| d\xi \leq K_2.$$

Then

$$\|Tf\|_{L^2} \lesssim \sqrt{K_1 K_2} \|f\|_{L^2}.$$

We will also use a lemma about functions in \mathcal{G}'_+ and \mathcal{W}'_+ .

Lemma 4.3. (i) *Assume $G \in \mathcal{G}'_+$, see (4.10). Then*

$$\sup_{|\alpha|+2a \leq 30} \|D^\alpha \Omega^a [e^{it\Lambda} G(t)]\|_{Z_1} \lesssim \varepsilon_1, \quad \|G(t)\|_{H^{N_1-2} \cap H^{N_1/2-1,0}} \lesssim \varepsilon_1 \langle t \rangle^{\delta^2}, \quad (4.12)$$

for any $t \in [0, T]$. Moreover, G satisfies the equation

$$(\partial_t + i\Lambda)G = \mathcal{N}_G, \quad \|\mathcal{N}_G(t)\|_{H^{N_1-4} \cap H^{N_1/2-2,0}} \lesssim \varepsilon_1^2 \langle t \rangle^{-5/6+\delta}. \quad (4.13)$$

(ii) *Assume $W \in \mathcal{W}'_+$, see (4.11). Then*

$$\|W(t)\|_{L^2} \lesssim \varepsilon_1 \langle t \rangle^{\delta^2}, \quad (4.14)$$

for any $t \in [0, T]$. Moreover, W satisfies the equation

$$(\partial_t + i\Lambda)W = \mathcal{Q}_W + \mathcal{E}_W, \quad (4.15)$$

where, with $\Sigma_{\geq 2} := \Sigma - \Lambda - \Sigma_1 \in \varepsilon_1^2 \mathcal{M}_{N_3-2}^{3/2,2}$ as in (3.12),

$$\mathcal{Q}_W = -iT_{\Sigma_{\geq 2}}W - iT_{V \cdot \zeta}W, \quad \|\langle \nabla \rangle^{-1/2} \mathcal{E}_W\|_{L^2} \lesssim \varepsilon_1^2 \langle t \rangle^{-5/6+\delta}. \quad (4.16)$$

Using Lemma 8.3 we see that for all $k \in \mathbb{Z}$ and $t \in [0, T]$

$$\begin{aligned} \|(P_k T_{V \cdot \zeta} W)(t)\|_{L^2} &\lesssim \varepsilon_1 2^{k+} \langle t \rangle^{-5/6+\delta} \|P_{[k-2, k+2]} W(t)\|_{L^2}, \\ \|(P_k T_{\Sigma_{\geq 2}} W)(t)\|_{L^2} &\lesssim \varepsilon_1^2 2^{3k+/2} \langle t \rangle^{-5/3+\delta} \|P_{[k-2, k+2]} W(t)\|_{L^2}. \end{aligned} \quad (4.17)$$

Proof. The claims in (4.12) follow from Definition 2.5, the assumptions (4.5), and interpolation (recall that $N_0 - N_3 = 2N_1$). The identities (4.13) follow from (3.4)–(3.6), since $(\partial_t + i\Lambda)\mathcal{U} \in \varepsilon_1^2 \mathcal{O}_{2,-2}$. The inequalities (4.14) follow from (3.49). The identities (4.15)–(4.16) follow from Proposition 3.8, since all quadratic terms that lose up to 1/2 derivatives can be placed into \mathcal{E}_W . Finally, the bounds (4.17) follow from (8.22) and (8.48). \square

Next we summarize some properties of the linear profiles of the functions in \mathcal{G}'_+ .

Lemma 4.4. *Assume $G \in \mathcal{G}'_+$ as before and let $f = e^{it\Lambda} G$. Recall the operators Q_{jk} and $A_{n,\gamma}, A_{n,\gamma}^{(j)}$ defined in (2.13)–(2.18). For $(k, j) \in \mathcal{J}$ and $n \in \{0, \dots, j+1\}$ let*

$$f_{j,k} := P_{[k-2, k+2]} Q_{jk} f, \quad f_{j,k,n} := A_{n,\gamma_1}^{(j)} f_{j,k}.$$

Then, if $m \geq 0$, for all $t \in [2^m - 1, 2^{m+1}]$ we have

$$\begin{aligned} \sup_{|\alpha|+2a \leq 30} \|D^\alpha \Omega^a f\|_{Z_1} &\lesssim \varepsilon_1, & \|f\|_{H^{N_1-2} \cap H_\Omega^{N_1/2-1,0}} &\lesssim \varepsilon_1 2^{\delta^2 m}, \\ \|P_k \partial_t f\|_{L^2} &\lesssim \varepsilon_1^2 2^{-8k+2^{-5m/6+\delta m}}, & \|P_k e^{-it\Lambda} \partial_t f\|_{L^\infty} &\lesssim \varepsilon_1^2 2^{-5m/3+\delta m}. \end{aligned} \quad (4.18)$$

Also, the following L^∞ bounds hold, for any $k \in \mathbb{Z}$ and $s \in \mathbb{R}$ with $|s - t| \leq 2^{m-\delta m}$,

$$\begin{aligned} \|e^{-is\Lambda} A_{\leq 2\mathcal{D}, \gamma_0} P_k f\|_{L^\infty} &\lesssim \varepsilon_1 \min(2^{k/2}, 2^{-4k}) 2^{-m} 2^{5\delta m}, \\ \|e^{-is\Lambda} A_{\geq 2\mathcal{D}+1, \gamma_0} P_k f\|_{L^\infty} &\lesssim \varepsilon_1 2^{-5m/6+3\delta^2 m}. \end{aligned} \quad (4.19)$$

Moreover, we have

$$\begin{aligned} \|e^{-is\Lambda} f_{j,k}\|_{L^\infty} &\lesssim \varepsilon_1 \min(2^k, 2^{-4k}) 2^{-j+50\delta j}, \\ \|e^{-is\Lambda} f_{j,k}\|_{L^\infty} &\lesssim \varepsilon_1 \min(2^{3k/2}, 2^{-4k}) 2^{-m+50\delta j} \quad \text{if } |k| \geq 10. \end{aligned} \quad (4.20)$$

Away from the bad frequencies, we have the stronger bound

$$\|e^{-is\Lambda} A_{\leq 2\mathcal{D}, \gamma_0} A_{\leq 2\mathcal{D}, \gamma_1} f_{j,k}\|_{L^\infty} \lesssim \varepsilon_1 2^{-m} \min(2^k, 2^{-4k}) 2^{-j/4}, \quad (4.21)$$

provided that $j \leq (1 - \delta^2)m + |k|/2$ and $|k| + D \leq m/2$.

Finally, for all $n \in \{0, 1, \dots, j\}$ we have

$$\begin{aligned} \|\widehat{f_{j,k,n}}\|_{L^\infty} &\lesssim \varepsilon_1 2^{2\delta^2 m} 2^{-4k^+} 2^{3\delta n} \cdot 2^{-(1/2-55\delta)(j-n)}, \\ \|\sup_{\theta \in \mathbb{S}^1} |\widehat{f_{j,k,n}}(r\theta)|\|_{L^2(rdr)} &\lesssim \varepsilon_1 2^{2\delta^2 m} 2^{-4k^+} 2^{n/2} 2^{-j+55\delta j}. \end{aligned} \quad (4.22)$$

Proof. The estimates in the first line of (4.18) follow from (4.12). The estimates (4.19), (4.20), and (4.22) then follow from Lemma 2.6, while the estimate (4.21) follows from (2.30). Finally, the estimates on $\partial_t f$ in (4.18) follows from the Lemma 4.1 in [32]. \square

We prove now a lemma that is useful when estimating multilinear expression containing a localization in the modulation Φ .

Lemma 4.5. *Assume that $k, k_1, k_2 \in \mathbb{Z}$, $m \geq \mathcal{D}$, $\bar{k} := \max(k, k_1, k_2)$, $|k| \leq m/2$, $p \geq -m$. Assume that $\underline{\mu}_0$ and $\underline{\mu}_1$ are symbols supported in the set $\mathcal{D}_{k_2, k, k_1}$ and satisfying*

$$\begin{aligned} \underline{\mu}_0(\xi, \eta) &= \mu_0(\xi, \eta) n(\xi, \eta), & \underline{\mu}_1(\xi, \eta) &= \mu_1(\xi, \eta) n(\xi, \eta), & \|n\|_{S^\infty} &\lesssim 1, \\ \mu_0(\xi, \eta) &= |\xi - \eta|^{3/2} \mathfrak{d}(\xi, \eta), & \|\mu_1(\xi, \eta)\|_{S^\infty} &\lesssim 2^{3k^+ - \bar{k}^+}, \end{aligned} \quad (4.23)$$

compare with (4.7). For $l \in \{0, 1\}$ and $\Phi = \Phi_{\sigma\mu\nu}$ as in (7.1) let

$$\mathcal{I}_p^l[F; H_1, H_2] = \iint_{(\mathbb{R}^2)^2} \underline{\mu}_l(\xi, \eta) \psi_p(\Phi(\xi, \eta)) \widehat{P_k F}(\xi - \eta) \widehat{P_{k_1} H_1}(\eta) \widehat{P_{k_2} H_2}(-\xi) d\xi d\eta,$$

where $\psi \in C_0^\infty(-1, 1)$ and $\psi_p(x) := \psi(2^{-p}x)$. Then

$$\begin{aligned} |\mathcal{I}_p^0[F; H_1, H_2]| &\lesssim 2^{3k/2} \min(1, 2^{-\bar{k}^+} 2^{\max(2p, 3k^+)}) 2^{-2k} N(P_k F) \cdot \|P_{k_1} H_1\|_{L^2} \|P_{k_2} H_2\|_{L^2}, \\ |\mathcal{I}_p^1[F; H_1, H_2]| &\lesssim 2^{3k^+ - \bar{k}^+} N(P_k F) \cdot \|P_{k_1} H_1\|_{L^2} \|P_{k_2} H_2\|_{L^2}, \end{aligned} \quad (4.24)$$

where

$$N(P_k F) := \sup_{|\rho| \leq 2^{-p} 2^{\delta m}} \|e^{i\rho\Lambda} P_k F\|_{L^\infty} + 2^{-10m} \|P_k F\|_{L^2}. \quad (4.25)$$

In particular, if $2^k \approx 1$ then

$$\begin{aligned} |\mathcal{I}_p^0[F; H_1, H_2]| &\lesssim \min(1, 2^{2p^+ - \bar{k}^+}) N(P_k F) \cdot \|P_{k_1} H_1\|_{L^2} \|P_{k_2} H_2\|_{L^2}, \\ |\mathcal{I}_p^1[F; H_1, H_2]| &\lesssim 2^{-\bar{k}^+} N(P_k F) \cdot \|P_{k_1} H_1\|_{L^2} \|P_{k_2} H_2\|_{L^2}. \end{aligned} \quad (4.26)$$

Proof. The proof when $l = 1$ is easy. We start from the formula

$$\psi_p(\Phi(\xi, \eta)) = C \int_{\mathbb{R}} \widehat{\psi}(s) e^{is2^{-p}\Phi(\xi, \eta)} ds. \quad (4.27)$$

Therefore

$$\mathcal{I}_p^1[F; H_1, H_2] = C \int_{\mathbb{R}} \widehat{\psi}(s) \iint_{(\mathbb{R}^2)^2} e^{is2^{-p}\Phi(\xi, \eta)} \underline{\mu}_1(\xi, \eta) \widehat{P_k F}(\xi - \eta) \widehat{P_{k_1} H_1}(\eta) \widehat{P_{k_2} H_2}(-\xi) d\xi d\eta.$$

Using Lemma 8.1 (i) and (4.23), it follows that

$$|\mathcal{I}_p^1[F; H_1, H_2]| \lesssim \int_{\mathbb{R}} |\widehat{\psi}(s)| 2^{3k^+ - \bar{k}^+} \|e^{-is2^{-p}\Lambda^\mu} P_k F\|_{L^\infty} \|P_{k_1} H_1\|_{L^2} \|P_{k_2} H_2\|_{L^2} ds.$$

The bound for $l = 1$ in (4.26) follows.

In the case $l = 0$, the desired bound follows in the same way unless

$$\bar{k}^+ + 2k \geq \max(2p, 3k^+) + \mathcal{D}. \quad (4.28)$$

On the other hand, if (4.28) holds then we need to take advantage of the depletion factor \mathfrak{d} . The main point is that if (4.28) holds and

$$\text{if } |\Phi(\xi, \eta)| \lesssim 2^p \quad \text{then} \quad \mathfrak{d}(\xi, \eta) \lesssim \frac{2^{-\bar{k}}(2^{2p} + 2^{3k^+})}{2^{2k}}. \quad (4.29)$$

Indeed, if (4.28) holds then $\bar{k} \geq \mathcal{D}$ and $p \leq 3\bar{k}/2 - \mathcal{D}/4$, and we estimate

$$\mathfrak{d}(\xi, \eta) \lesssim \left(\frac{|\xi| - |\eta|}{|\xi - \eta|} \right)^2 \lesssim \left(\frac{2^{-\bar{k}/2} |\lambda(|\xi|) - \lambda(|\eta|)|}{2^k} \right)^2 \lesssim \frac{2^{-\bar{k}} (|\Phi(\xi, \eta)| + \lambda(|\xi - \eta|))^2}{2^{2k}}$$

in the support of the function \mathfrak{d} , which gives (4.29). To continue the proof, we fix a function $\chi \in C_0^\infty(\mathbb{R}^2)$ supported in the ball of radius $2^{k^+ + 1}$ with the property that $\sum_{v \in (2^{k^+} \mathbb{Z})^2} \chi(x - v) = 1$ for any $x \in \mathbb{R}^2$. For any $v \in (2^{k^+} \mathbb{Z})^2$, consider the operator Q_v defined by

$$\widehat{Q_v f}(\xi) = \chi(\xi - v) \widehat{f}(\xi).$$

In view of the localization in $(\xi - \eta)$, we have

$$\mathcal{I}_p^0[F; H_1, H_2] = \sum_{|v_1 + v_2| \lesssim 2^{k^+}} \mathcal{I}_{p; v_1, v_2}^0, \quad \mathcal{I}_{p; v_1, v_2}^0 := \mathcal{I}_p^0[F; Q_{v_1} H_1, Q_{v_2} H_2]. \quad (4.30)$$

Moreover, using (4.29) we can insert a factor of $\varphi_{\leq \mathcal{D}}(2^{-X}(\xi - \eta) \cdot v_1)$ in the integral defining $\mathcal{I}_p^l[F; Q_{v_1} H_1, Q_{v_2} H_2]$ without changing the integral, where $2^X \approx (2^p + 2^{3k^+/2})2^{\bar{k}/2}$. Let

$$m_{v_1}(\xi, \eta) := \underline{\mu}_0(\xi, \eta) \cdot \varphi_{[k_2 - 2, k_2 + 2]}(\xi) \varphi_{[k - 2, k + 2]}(\xi - \eta) \varphi_{\leq k_+ + 2}(\eta - v_1) \varphi_{\leq \mathcal{D}}(2^{-X}(\xi - \eta) \cdot v_1).$$

We will show below that for any $v_1 \in \mathbb{R}^2$ with $|v_1| \approx 2^{\bar{k}}$

$$\|\mathcal{F}^{-1}(m_{v_1})\|_{L^1(\mathbb{R}^2 \times \mathbb{R}^2)} \lesssim 2^{3k/2} \cdot 2^{2X} 2^{-2k} 2^{-2\bar{k}}. \quad (4.31)$$

Assuming this, the desired bound follows as before. To prove (4.31) we recall that $\|\mathcal{F}^{-1}(ab)\|_{L^1} \lesssim \|\mathcal{F}^{-1}(a)\|_{L^1} \|\mathcal{F}^{-1}(b)\|_{L^1}$. Then we write

$$(\xi - \eta) \cdot (\xi + \eta) = 2(\xi - \eta) \cdot v_1 + |\xi - \eta|^2 + 2(\xi - \eta) \cdot (\eta - v_1).$$

The bound (4.31) follows by analyzing the contributions of the 3 terms in this formula. \square

Our next lemma concerns a linear L^2 estimate on certain localized Fourier integral operators.

Lemma 4.6. *Assume that $k \geq -100$, $m \geq \mathcal{D}^2$,*

$$-(1 - \delta)m \leq p - k/2 \leq -\delta m, \quad 2^{m-2} \leq |s| \leq 2^{m+2}. \quad (4.32)$$

Given $\chi \in C_0^\infty(\mathbb{R})$ supported in $[-1, 1]$, introduce the operator $L_{p,k}$ defined by

$$L_{p,k}f(\xi) := \varphi_{\geq -100}(\xi) \int_{\mathbb{R}^2} e^{is\Phi(\xi,\eta)} \chi(2^{-p}\Phi(\xi,\eta)) \varphi_k(\eta) a(\xi,\eta) f(\eta) d\eta, \quad (4.33)$$

where, for some $\mu, \nu \in \{+, -\}$,

$$\begin{aligned} \Phi(\xi, \eta) &= \Lambda(\xi) - \Lambda_\mu(\xi - \eta) - \Lambda_\nu(\eta), & a(\xi, \eta) &= A(\xi, \eta) \chi_{ba}(\xi - \eta) \widehat{g}(\xi - \eta), \\ \|D^\alpha A\|_{L_{x,y}^\infty} &\lesssim_{|\alpha|} 2^{|\alpha|m/3}, & \|g\|_{Z_1 \cap H_\Omega^{N_1/3,0}} &\lesssim 1. \end{aligned} \quad (4.34)$$

Then

$$\|L_{p,k}f\|_{L^2} \lesssim 2^{30\delta m} (2^{(3/2)(p-k/2)} + 2^{p-k/2-m/3}) \|f\|_{L^2}.$$

Remark 4.7. *(i) Lemma 4.6, which is proved in section 6 below, plays a central role in the proof of Proposition 4.1. A key role in its proof is played by the “curvature” component*

$$\Upsilon(\xi, \eta) := (\nabla_{\xi,\eta}^2 \Phi)(\xi, \eta) [(\nabla_\xi^\perp \Phi)(\xi, \eta), (\nabla_\eta^\perp \Phi)(\xi, \eta)], \quad (4.35)$$

and in particular by its non-degeneracy close to the bad frequencies γ_0 and γ_1 , and to the resonant hypersurface $\Phi(\xi, \eta) = 0$. The properties of Υ that we are going to use are described in subsection 7.2, and in particular in Lemmas 7.1, 7.2, and 7.3.

(ii) We can insert S^∞ symbols and bounded factors that depend only on ξ or only on η in the integral in (4.33), without changing the conclusion. We will often use this lemma in the form

$$\begin{aligned} &\left| \iint_{\mathbb{R}^2 \times \mathbb{R}^2} e^{is\Lambda(\xi-\eta)} \chi(2^{-p}\Phi(\xi,\eta)) \mu(\xi,\eta) a(\xi,\eta) \widehat{P_{k_1} F_1}(\eta) \widehat{P_k F_2}(-\xi) d\xi d\eta \right| \\ &\lesssim 2^{30\delta m} (2^{(3/2)(p-k/2)} + 2^{p-k/2-m/3}) \|P_{k_1} F_1\|_{L^2} \|P_k F_2\|_{L^2}, \end{aligned} \quad (4.36)$$

provided that $k, k_1 \geq -80$, (4.32) and (4.34) hold, and $\|\mu\|_{S^\infty} \lesssim 1$. This follows by writing

$$\mu(\xi, \eta) = \iint_{\mathbb{R}^2 \times \mathbb{R}^2} P(x, y) e^{-ix \cdot \xi} e^{-iy \cdot \eta} d\xi d\eta,$$

with $\|P\|_{L^1(\mathbb{R}^2 \times \mathbb{R}^2)} \lesssim 1$, and then combining the oscillatory factors with the functions F_1, F_2 .

5. ENERGY ESTIMATES, III: PROOF OF PROPOSITION 4.1

In this section we prove Proposition 4.1, thus completing the proof of Proposition 2.2. Recall the definitions (4.8)-(4.11). For $G \in \mathcal{G}'$ and $W_1, W_2 \in \mathcal{W}'$ let

$$\mathcal{A}_{Y,m}^l[G, W_1, W_2] := \int_{\mathbb{R}} q_m(s) \iint_{\mathbb{R}^2 \times \mathbb{R}^2} \mu_l(\xi, \eta) \chi_Y(\xi - \eta) \widehat{G}(\xi - \eta, s) \widehat{W}_1(\eta, s) \widehat{W}_2(-\xi, s) d\xi d\eta ds, \quad (5.1)$$

where $l \in \{0, 1\}$, $m \in [\mathcal{D}^2, L]$, $Y \in \{go, ba\}$, and the symbols μ_l are as in (4.7). The conclusion of Proposition 4.1 is equivalent to the uniform bound

$$|\mathcal{A}_{Y,m}^l[G, W_1, W_2]| \lesssim \varepsilon_1^3 2^{2\delta^2 m}. \quad (5.2)$$

In proving this bound we further decompose the functions W_1 and W_2 dyadically and consider several cases. We remark that the most difficult case (which is treated in Lemma 5.1) is when the “bad” frequencies of G interact with the high frequencies of the functions W_1 and W_2 .

5.1. The main interactions. We prove the following lemma.

Lemma 5.1. *For $l \in \{0, 1\}$, $m \in [\mathcal{D}^2, L]$, $G \in \mathcal{G}'$, and $W_1, W_2 \in \mathcal{W}'$ we have*

$$\sum_{\min(k_1, k_2) \geq -40} |\mathcal{A}_{ba,m}^l[G, P_{k_1} W_1, P_{k_2} W_2]| \lesssim \varepsilon_1^3. \quad (5.3)$$

The rest of the subsection is concerned with the proof of this lemma. We need to further decompose our operators based on the size of the modulation. Assuming that $\bar{W}_2 \in \mathcal{W}'_\sigma$, $W_1 \in \mathcal{W}'_\nu$, $G \in \mathcal{G}'_\mu$, $\sigma, \nu, \mu \in \{+, -\}$, see (4.10)–(4.11), we define the associated phase

$$\Phi(\xi, \eta) = \Phi_{\sigma\mu\nu}(\xi, \eta) = \Lambda_\sigma(\xi) - \Lambda_\mu(\xi - \eta) - \Lambda_\nu(\eta). \quad (5.4)$$

Notice that in proving (5.3) we may assume that $\sigma = +$ (otherwise take complex conjugates) and that the sum is over $|k_1 - k_2| \leq 50$ (due to localization in $\xi - \eta$). For $k \geq -30$ let

$$\begin{aligned} \rho_k(s) &:= \sum_{i \in \{1, 2\}} \|P_{[k-40, k+40]} W_i(s)\|_{L^2} + 2^{5m/6 - \delta m} 2^{-k/2} \sum_{i \in \{1, 2\}} \|P_{[k-40, k+40]} \mathcal{E}_{W_i}(s)\|_{L^2}, \\ \rho_{k,m}^2 &:= \int_{\mathbb{R}} \rho_k(s)^2 [2^{-m} q_m(s) + |q'_m(s)|] ds, \end{aligned} \quad (5.5)$$

where $\mathcal{E}_{W_{1,2}}$ are the “semilinear” nonlinearities defined in (4.15). In view of (4.14) and (4.16),

$$\sum_{k \geq -30} \rho_{k,m}^2 \lesssim \varepsilon_1^2 2^{2\delta^2 m}. \quad (5.6)$$

Given $k \geq -30$, let $\underline{p} = \lfloor k/2 - 7m/9 \rfloor$ (the largest integer $\leq k/2 - 7m/9$). We define

$$\mathcal{A}_{ba}^{l,p}[F, H_1, H_2] := \iint_{\mathbb{R}^2 \times \mathbb{R}^2} \mu_l(\xi, \eta) \varphi_p^{[\underline{p}, \infty)}(\Phi(\xi, \eta)) \chi_{ba}(\xi - \eta) \widehat{F}(\xi - \eta) \widehat{H}_1(\eta) \widehat{H}_2(-\xi) d\xi d\eta, \quad (5.7)$$

where $p \in [\underline{p}, \infty)$ (here $\varphi_p^{[\underline{p}, \infty)} = \varphi_p$ if $p \geq \underline{p} + 1$ and $\varphi_p^{[\underline{p}, \infty)} = \varphi_{\leq \underline{p}}$ if $p = \underline{p}$). Assuming that $|k_1 - k| \leq 30$, $|k_2 - k| \leq 30$, let

$$\mathcal{A}_{ba,m}^{l,p}[G, P_{k_1} W_1, P_{k_2} W_2] := \int_{\mathbb{R}} q_m(s) \mathcal{A}_{ba}^{l,p}[G(s), P_{k_1} W_1(s), P_{k_2} W_2(s)] ds. \quad (5.8)$$

This gives a decomposition $\mathcal{A}_{ba,m}^l = \sum_{p \geq \underline{p}} \mathcal{A}_{ba,m}^{l,p}$ as a sum of operators localized in modulation. Notice that the sum is either over $p \in [\underline{p}, k/2 + \mathcal{D}]$ (if $\nu = +$ or if $\nu = -$ and $k \leq \mathcal{D}/2$) or over $|p - 3k/2| \leq \mathcal{D}$ (if $\nu = -$ and $k \geq \mathcal{D}/2$). For (5.3) it remains to prove that

$$|\mathcal{A}_{ba,m}^{l,p}[G, P_{k_1} W_1, P_{k_2} W_2]| \lesssim \varepsilon_1 2^{-\delta m} \rho_{k,m}^2, \quad (5.9)$$

for any $k \geq -30$, $p \geq \underline{p}$, and $k_1, k_2 \in \mathbb{Z}$ satisfying $|k_1 - k| \leq 30$, $|k_2 - k| \leq 30$.

Using Lemma 4.5 (see (4.26)), we have

$$|\mathcal{A}_{ba}^{l,p}[G(s), P_{k_1} W_1(s), P_{k_2} W_2(s)]| \lesssim \varepsilon_1 2^{2p+k} 2^{-5m/6 + \delta m} \|P_{k_1} W_1(s)\|_{L^2} \|P_{k_2} W_2(s)\|_{L^2},$$

for any $p \geq \underline{p}$, due to the L^∞ bound in (4.19). The desired bound (5.9) follows if $2p_+ - k \leq -m/5 + \mathcal{D}$. Also, using Lemma 4.6, we have

$$|\mathcal{A}_{ba}^{l,p}[G, P_{k_1}W_1, P_{k_2}W_2](s)| \lesssim \varepsilon_1 2^{-m-\delta m} \|P_{k_1}W_1(s)\|_{L^2} \|P_{k_2}W_2(s)\|_{L^2},$$

using (4.36), since $2^{p-k/2} \lesssim 2^{-7m/9}$ and $\|e^{is\Lambda_\mu}G(s)\|_{Z_1 \cap H_\Omega^{N_1/3,0}} \lesssim \varepsilon_1 2^{\delta m}$ (see (4.12)). Therefore (5.9) follows if $p = \underline{p}$. It remains to prove (5.9) when

$$p \geq \underline{p} + 1 \quad \text{and} \quad k \in [-30, 2p_+ + m/5], \quad |k_1 - k| \leq 30, \quad |k_2 - k| \leq 30. \quad (5.10)$$

In the remaining range in (5.10) we integrate by parts in s . We define

$$\tilde{\mathcal{A}}_{ba}^{l,p}[F, H_1, H_2] := \iint_{\mathbb{R}^2 \times \mathbb{R}^2} \mu(\xi, \eta) \tilde{\varphi}_p(\Phi(\xi, \eta)) \chi_{ba}(\xi - \eta) \widehat{F}(\xi - \eta) \widehat{H}_1(\eta) \widehat{H}_2(-\xi) d\xi d\eta, \quad (5.11)$$

where $\tilde{\varphi}_p(x) := 2^p x^{-1} \varphi_p(x)$. This is similar to the definition in (5.7), but with φ_p replaced by $\tilde{\varphi}_p$. Then we let $W_{k_1} := P_{k_1}W_1$, $W_{k_2} := P_{k_2}W_2$ and write

$$\begin{aligned} 0 &= \int_{\mathbb{R}} \frac{d}{ds} \left\{ q_m(s) \tilde{\mathcal{A}}_{ba}^{l,p}[G(s), W_{k_1}(s), W_{k_2}(s)] \right\} ds \\ &= \int_{\mathbb{R}} q'_m(s) \tilde{\mathcal{A}}_{ba}^{l,p}[G(s), W_{k_1}(s), W_{k_2}(s)] ds + \mathcal{J}_0^{l,p}(k_1, k_2) + \mathcal{J}_1^{l,p}(k_1, k_2) + \mathcal{J}_2^{l,p}(k_1, k_2) \\ &\quad + i2^p \int_{\mathbb{R}} q_m(s) \mathcal{A}_{ba}^{l,p}[G(s), W_{k_1}(s), W_{k_2}(s)] ds, \end{aligned} \quad (5.12)$$

where

$$\begin{aligned} \mathcal{J}_{ba,0}^{l,p}(k_1, k_2) &:= \int_{\mathbb{R}} q_m(s) \tilde{\mathcal{A}}_{ba}^{l,p}[(\partial_s + i\Lambda_\mu)G(s), W_{k_1}(s), W_{k_2}(s)] ds, \\ \mathcal{J}_{ba,1}^{l,p}(k_1, k_2) &:= \int_{\mathbb{R}} q_m(s) \tilde{\mathcal{A}}_{ba}^{l,p}[G(s), (\partial_s + i\Lambda_\nu)W_{k_1}(s), W_{k_2}(s)] ds, \\ \mathcal{J}_{ba,2}^{l,p}(k_1, k_2) &:= \int_{\mathbb{R}} q_m(s) \tilde{\mathcal{A}}_{ba}^{l,p}[G(s), W_{k_1}(s), (\partial_s + i\Lambda_{-\sigma})W_{k_2}(s)] ds. \end{aligned} \quad (5.13)$$

The integral in the last line of (5.12) is the one we have to estimate. Notice that

$$2^{-p} |\tilde{\mathcal{A}}_{ba}^{l,p}[G(s), W_{k_1}(s), W_{k_2}(s)]| \lesssim 2^{-p} 2^{-5m/6 + \delta m} \|W_{k_1}(s)\|_{L^2} \|W_{k_2}(s)\|_{L^2}$$

as a consequence of Lemma 4.5 and (4.19). It remains to prove that if (5.10) holds then

$$2^{-p} |\mathcal{J}_{ba,0}^{l,p}(k_1, k_2) + \mathcal{J}_{ba,1}^{l,p}(k_1, k_2) + \mathcal{J}_{ba,2}^{l,p}(k_1, k_2)| \lesssim \varepsilon_1 2^{-\delta m} \rho_{k,m}^2. \quad (5.14)$$

This bound will be proved in several steps, in Lemmas 5.2, 5.3, and 5.4 below.

5.1.1. *Quasilinear terms.* We consider first the quasilinear terms appearing in (5.14), which are those where $(\partial_t + i\Lambda)$ hits the high frequency inputs W_{k_1} and W_{k_2} . We start with the case when the frequencies k_1, k_2 are not too large relative to p_+ .

Lemma 5.2. *Assume that (5.10) holds and, in addition, $k \leq 2p_+/3 + m/4$. Then*

$$2^{-p} [|\mathcal{J}_{ba,1}^{l,p}(k_1, k_2)| + |\mathcal{J}_{ba,2}^{l,p}(k_1, k_2)|] \lesssim \varepsilon_1 2^{-\delta m} \rho_{k,m}^2. \quad (5.15)$$

Proof. It suffices to bound the contributions of $|\mathcal{J}_{ba,1}^{l,p}(k_1, k_2)|$ in (5.15), since the contributions of $|\mathcal{J}_{ba,2}^{l,p}(k_1, k_2)|$ are similar. We estimate, for $s \in [2^{m-1}, 2^{m+1}]$,

$$\|(\partial_s + i\Lambda_\nu)W_{k_1}(s)\|_{L^2} \lesssim \varepsilon_1 2^{-5m/6+\delta m} (2^{k_1} + 2^{3k_1/2} 2^{-5m/6}) \rho_k(s), \quad (5.16)$$

using (4.15)–(4.17). As before, we use Lemma 4.5 and the pointwise bound (4.19) to estimate

$$\begin{aligned} & |\tilde{\mathcal{A}}_{ba}^{l,p}[G(s), (\partial_s + i\Lambda_\nu)W_{k_1}(s), W_{k_2}(s)]| \\ & \lesssim \min(1, 2^{2p+k}) \varepsilon_1 2^{-5m/6+\delta m} \|(\partial_s + i\Lambda_\nu)W_{k_1}(s)\|_{L^2} \|W_{k_2}(s)\|_{L^2}. \end{aligned} \quad (5.17)$$

The bounds (5.16)–(5.17) suffice to prove (5.15) when $p \geq 0$ or when $-m/2 + k/2 \leq p \leq 0$.

It remains to prove (5.15) when

$$p + 1 \leq p \leq -m/2 + k/2, \quad k \leq m/5. \quad (5.18)$$

For this we would like to apply Lemma 4.6. We claim that, for $s \in [2^{m-1}, 2^{m+1}]$,

$$\begin{aligned} & |\tilde{\mathcal{A}}_{ba}^{l,p}[G(s), (\partial_s + i\Lambda_\nu)W_{k_1}(s), W_{k_2}(s)]| \\ & \lesssim 2^{-k} \varepsilon_1 2^{31\delta m} (2^{(3/2)(p-k/2)} + 2^{p-k/2-m/3}) \|(\partial_s + i\Lambda_\nu)W_{k_1}(s)\|_{L^2} \|W_{k_2}(s)\|_{L^2}. \end{aligned} \quad (5.19)$$

Assuming this and using also (5.16), it follows that

$$\begin{aligned} 2^{-p} |\mathcal{J}_{ba,1}^{l,p}(k_1, k_2)| & \lesssim 2^{-p} 2^m \cdot \varepsilon_1 \rho_{k,m}^2 2^{-5m/6+40\delta m} (2^{(3/2)(p-k/2)} + 2^{p-k/2-m/3}) \\ & \lesssim \varepsilon_1 \rho_{k,m}^2 2^{m/6+40\delta m} (2^{p/2-3k/4} + 2^{-k/2-m/3}), \end{aligned}$$

and the desired conclusion follows using also (5.18).

On the other hand, to prove the bound (5.19), we use (4.36). Clearly, with $g = e^{is\Lambda_\mu} G$, we have $\|g\|_{Z_1 \cap H_Q^{N_1/3,0}} \lesssim \varepsilon_1 2^{\delta^2 m}$, see (4.18). The factor 2^{-k} in the right-hand side of (5.19) is due to the symbols μ_0 and μ_1 . This is clear for the symbols μ_1 , which already contain a factor of 2^{-k} (see (4.7)). For the symbols μ_0 , we notice that we can take

$$A(\xi, \eta) := 2^k \mathfrak{d}(\xi, \eta) \varphi_{\leq 4}(\Phi(\xi, \eta)) \varphi_{[k_2-2, k_2+2]}(\xi) \varphi_{[-10, 10]}(\xi - \eta).$$

This satisfies the bounds required in (4.34), since $k \leq m/5$. This completes the proof. \square

We now look at the remaining cases for the quasilinear terms and prove the following:

Lemma 5.3. *Assume that (5.10) holds and, in addition,*

$$p \geq 0, \quad k \in [2p/3 + m/4, 2p + m/5]. \quad (5.20)$$

Then

$$2^{-p} |\mathcal{J}_{ba,1}^{l,p}(k_1, k_2) + \mathcal{J}_{ba,2}^{l,p}(k_1, k_2)| \lesssim \varepsilon_1 2^{-\delta m} \rho_{k,m}^2. \quad (5.21)$$

Proof. The main issue here is to deal with the case of large frequencies, relative to the time variable, and avoid the loss of derivatives coming from the terms $(\partial_t \pm i\Lambda)W_{1,2}$. For this we use ideas related to the local existence theory, such as symmetrization. Notice that in Lemma 5.3 we estimate the absolute value of the sum $\mathcal{J}_{ba,1}^{l,p} + \mathcal{J}_{ba,2}^{l,p}$, and not each term separately.

Notice first that we may assume that $\nu = + = \sigma$, since otherwise $\mathcal{J}_{ba,n}^{l,p}(k_1, k_2) = 0$, $n \in \{1, 2\}$, when $k \geq 2p/3 + m/4$. In particular $2^p \lesssim 2^{k/2}$. We deal first with the semilinear part of the nonlinearity, which is \mathcal{E}_{W_1} in equation (4.15). Using Lemma 4.5 and the definition (5.5),

$$\begin{aligned} |\tilde{\mathcal{A}}_{ba}^{l,p}[G(s), P_{k_1} \mathcal{E}_{W_1}(s), W_{k_2}(s)]| & \lesssim \varepsilon_1 2^{-5m/6+\delta m} \|P_{k_1} \mathcal{E}_{W_1}(s)\|_{L^2} \|W_{k_2}(s)\|_{L^2} \\ & \lesssim \varepsilon_1 2^{-5m/3+2\delta m} 2^{k/2} \rho_k(s)^2. \end{aligned}$$

Therefore

$$2^{-p} \int_{\mathbb{R}} q_m(s) |\tilde{\mathcal{A}}_{ba}^{l,p}[G(s), P_{k_1} \mathcal{E}_{W_1}(s), W_{k_2}(s)]| ds \lesssim \varepsilon_1 2^{-m/4} \rho_{k,m}^2.$$

It remains to bound the contributions of \mathcal{Q}_{W_1} and \mathcal{Q}_{W_2} . Using again Lemma 4.5, we can easily prove the estimate when $k \leq 6m/5$ or when $l = 1$. It remains to show that

$$2^{-p} \int_{\mathbb{R}} q_m(s) |\tilde{\mathcal{A}}_{ba}^{0,p}[G(s), P_{k_1} \mathcal{Q}_{W_1}(s), W_{k_2}(s)] + \tilde{\mathcal{A}}_{ba}^{0,p}[G(s), W_{k_1}(s), P_{k_2} \mathcal{Q}_{W_2}(s)]| ds \quad (5.22)$$

$$\lesssim \varepsilon_1 2^{-\delta m} \rho_{k,m}^2,$$

provided that

$$\nu = \sigma = +, \quad k \in [2p - \mathcal{D}, 2p + m/5], \quad k \geq 6m/5. \quad (5.23)$$

In this case we consider the full expression and apply a symmetrization procedure to recover the loss of derivatives. Since $W_1 \in \mathcal{W}'_+$ and $W_2 \in \mathcal{W}'_-$, recall from (4.16) that

$$\mathcal{Q}_{W_1} = -iT_{\Sigma_{\geq 2}} W_1 - iT_{V \cdot \zeta} W_1, \quad \mathcal{Q}_{W_2} = \overline{iT_{\Sigma_{\geq 2}} W_2} + \overline{iT_{V \cdot \zeta} W_2}.$$

Therefore, using the definition (5.11),

$$\begin{aligned} \tilde{\mathcal{A}}_{ba}^{0,p}[G, P_{k_1} \mathcal{Q}_{W_1}, W_{k_2}] &= \sum_{\sigma \in \{\Sigma_{\geq 2}, V \cdot \zeta\}} \iint_{\mathbb{R}^2 \times \mathbb{R}^2} \mu_0(\xi, \eta) \\ &\times \tilde{\varphi}_p(\Phi(\xi, \eta)) \chi_{ba}(\xi - \eta) \widehat{G}(\xi - \eta) \cdot \varphi_{k_1}(\eta) (-i) \widehat{T_{\sigma} W_1}(\eta) \cdot \varphi_{k_2}(\xi) \widehat{W_2}(-\xi) d\xi d\eta, \end{aligned}$$

and

$$\begin{aligned} \tilde{\mathcal{A}}_{ba}^{0,p}[G, W_{k_1}, P_{k_2} \mathcal{Q}_{W_2}] &= \sum_{\sigma \in \{\Sigma_{\geq 2}, V \cdot \zeta\}} \iint_{\mathbb{R}^2 \times \mathbb{R}^2} \mu_0(\xi, \eta) \\ &\times \tilde{\varphi}_p(\Phi(\xi, \eta)) \chi_{ba}(\xi - \eta) \widehat{G}(\xi - \eta) \cdot \varphi_{k_1}(\eta) \widehat{W_1}(\eta) \cdot \varphi_{k_2}(\xi) \overline{iT_{\sigma} W_2}(-\xi) d\xi d\eta. \end{aligned}$$

We use the definition (2.32) and make suitable changes of variables to write

$$\begin{aligned} \tilde{\mathcal{A}}_{ba}^{0,p}[G, P_{k_1} \mathcal{Q}_{W_1}, W_{k_2}] + \tilde{\mathcal{A}}_{ba}^{0,p}[G, W_{k_1}, P_{k_2} \mathcal{Q}_{W_2}] &= \\ = \sum_{\sigma \in \{\Sigma_{\geq 2}, V \cdot \zeta\}} \frac{-i}{4\pi^2} \iiint_{(\mathbb{R}^2)^3} \widehat{W_1}(\eta) \widehat{W_2}(-\xi) \widehat{G_{ba}}(\xi - \eta - \alpha) (\delta M)(\xi, \eta, \alpha) d\xi d\eta d\alpha, \end{aligned}$$

where $\widehat{G_{ba}} := \chi_{ba} \cdot \widehat{G}$ and

$$\begin{aligned} (\delta M)(\xi, \eta, \alpha) &= \mu_0(\xi, \eta + \alpha) \tilde{\varphi}_p(\Phi(\xi, \eta + \alpha)) \tilde{\sigma}(\alpha, \frac{2\eta + \alpha}{2}) \chi(\frac{|\alpha|}{|2\eta + \alpha|}) \varphi_{k_1}(\eta + \alpha) \varphi_{k_2}(\xi) \\ &\quad - \mu_0(\xi - \alpha, \eta) \tilde{\varphi}_p(\Phi(\xi - \alpha, \eta)) \tilde{\sigma}(\alpha, \frac{2\xi - \alpha}{2}) \chi(\frac{|\alpha|}{|2\xi - \alpha|}) \varphi_{k_1}(\eta) \varphi_{k_2}(\xi - \alpha). \end{aligned}$$

For (5.22) it suffices to prove that for any $s \in [2^{m-1}, 2^{m+1}]$ and $\sigma \in \{\Sigma_{\geq 2}, V \cdot \zeta\}$

$$2^{-p} \left| \iiint_{(\mathbb{R}^2)^3} \widehat{W_1}(\eta, s) \widehat{W_2}(-\xi, s) \widehat{G_{ba}}(\xi - \eta - \alpha, s) (\delta M)(\xi, \eta, \alpha, s) d\xi d\eta d\alpha \right| \quad (5.24)$$

$$\lesssim \varepsilon_1 \rho_k(s)^2 2^{-m-\delta m}.$$

Let

$$\begin{aligned} M(\xi, \eta, \alpha; \theta_1, \theta_2) &:= \mu_0(\xi - \theta_1, \eta + \alpha - \theta_1) \tilde{\varphi}_p(\Phi(\xi - \theta_1, \eta + \alpha - \theta_1)) \\ &\quad \times \varphi_{k_2}(\xi - \theta_1) \varphi_{k_1}(\eta + \alpha - \theta_1) \tilde{\sigma}(\alpha, \eta + \frac{\alpha}{2} + \theta_2) \chi\left(\frac{|\alpha|}{|2\eta + \alpha + 2\theta_2|}\right), \end{aligned} \quad (5.25)$$

therefore

$$\begin{aligned} (\delta M)(\xi, \eta, \alpha) &= M(\xi, \eta, \alpha; 0, 0) - M(\xi, \eta, \alpha; \alpha, \xi - \eta - \alpha) \\ &= \varphi_{\leq k-D}(\alpha) [\alpha \cdot \nabla_{\theta_1} M(\xi, \eta, \alpha; 0, 0) + (\xi - \eta - \alpha) \cdot \nabla_{\theta_2} M(\xi, \eta, \alpha; 0, 0)] + (eM)(\xi, \eta, \alpha). \end{aligned}$$

Using the formula for μ_0 in (4.7) and recalling that $\sigma \in \varepsilon_1 \mathcal{M}_{N_3-2}^{3/2,1}$ (see Definition 8.6), it follows that, in the support of the integral,

$$|(eM)(\xi, \eta, \alpha)| \lesssim (1 + |\alpha|^2) P(\alpha) 2^{-2k} 2^{3k/2}, \quad \|(1 + |\alpha|)^8 P\|_{L^2} \lesssim 2^{\delta m}.$$

The contribution of (eM) in (5.24) can then be estimated by $2^{-p} 2^{\delta m} 2^{-k/2} \varepsilon_1 \rho_k(s)^2$ which suffices due to the assumptions (5.23).

We are thus left with estimating the integrals

$$\begin{aligned} I &:= \iiint_{(\mathbb{R}^2)^3} \widehat{G}_{ba}(\xi - \eta - \alpha) \varphi_{\leq k-D}(\alpha) [(\xi - \eta - \alpha) \cdot \nabla_{\theta_2} M(\xi, \eta, \alpha; 0, 0)] \widehat{W}_1(\eta) \widehat{W}_2(-\xi) d\alpha d\eta d\xi, \\ II &:= \iiint_{(\mathbb{R}^2)^3} \widehat{G}_{ba}(\xi - \eta - \alpha) \varphi_{\leq k-D}(\alpha) [\alpha \cdot \nabla_{\theta_1} M(\xi, \eta, \alpha; 0, 0)] \widehat{W}_1(\eta) \widehat{W}_2(-\xi) d\alpha d\eta d\xi. \end{aligned}$$

If $|\alpha| \ll 2^k$ we have

$$\begin{aligned} (\xi - \eta - \alpha) \cdot \nabla_{\theta_2} M(\xi, \eta, \alpha; 0, 0) &= \mu_0(\xi, \eta + \alpha) \tilde{\varphi}_p(\Phi(\xi, \eta + \alpha)) \varphi_{k_2}(\xi) \varphi_{k_1}(\eta + \alpha) \\ &\quad \times (\xi - \eta - \alpha) \cdot (\nabla_{\zeta} \tilde{\sigma})(\alpha, \eta + \frac{\alpha}{2}). \end{aligned}$$

We make the change of variable $\alpha = \beta - \eta$ to rewrite

$$I = c \iiint_{(\mathbb{R}^2)^3} \widehat{G}_{ba}(\xi - \beta) \mu_0(\xi, \beta) \tilde{\varphi}_p(\Phi(\xi, \beta)) (\xi - \beta) \cdot \mathcal{F}\{P_{k_1} T_{P_{\leq k-D} \nabla_{\zeta} \sigma} W_1\}(\beta) \widehat{P}_{k_2} \widehat{W}_2(-\xi) d\beta d\xi.$$

Then we use Lemma 4.5, (4.19), and (8.22) (recall $\sigma \in \varepsilon_1 \mathcal{M}_{N_3-2}^{3/2,1}$) to estimate

$$\begin{aligned} 2^{-p} |I(s)| &\lesssim 2^{-p} 2^{2p-k} \varepsilon_1 2^{-5m/6+\delta m} \|P_{k_1} T_{P_{\leq k-D} \nabla_{\zeta} \sigma} W_1(s)\|_{L^2} \|P_{k_2} W_2(s)\|_{L^2} \\ &\lesssim \varepsilon_1 2^{-3m/2} 2^{p-k/2} \rho_k(s)^2. \end{aligned}$$

This is better than the desired bound (5.24). One can estimate $2^{-p} |II(s)|$ in a similar way, using the flexibility in Lemma 4.5 due to the fact that the symbol μ_0 is allowed to contain additional S^∞ symbols. This completes the proof of the bound (5.24) and the lemma. \square

5.1.2. Semilinear terms. The only term in (5.12) that remains to be estimated is $\mathcal{J}_0^{l,p}(k_1, k_2)$. This is a semilinear term, since the ∂_t derivative hits the low-frequency component, for which we will show the following:

Lemma 5.4. *Assume that (5.10) holds. Then*

$$2^{-p} |\mathcal{J}_{ba,0}^{l,p}(k_1, k_2)| \lesssim \varepsilon_1 2^{-\delta m} \rho_{k,m}^2. \quad (5.26)$$

Proof. Assume first that $p \geq -m/4$. Using integration by parts we can see that, for $\rho \in \mathbb{R}$,

$$\|\mathcal{F}^{-1}\{e^{i\rho\Lambda(\xi)}\varphi_{[-20,20]}(\xi)\}\|_{L_x^1} \lesssim 1 + |\rho|. \quad (5.27)$$

Combining this and the bounds in the second line of (4.18) we get

$$\sum_{|\rho| \leq 2^{-p+\delta m}} \|e^{i\rho\Lambda}[(\partial_s + i\Lambda_\mu)P_{[-10,10]}G(s)]\|_{L^\infty} \lesssim (2^{-p} + 1)2^{-5m/3+2\delta m}.$$

Using this in combination with Lemma 4.5 we get

$$|\tilde{\mathcal{A}}_{ba}^{l,p}[(\partial_s + i\Lambda_\mu)G(s), W_{k_1}(s), W_{k_2}(s)]| \lesssim (2^{-p} + 1)2^{-5m/3+2\delta m} \rho_k(s)^2, \quad (5.28)$$

which leads to an acceptable contribution.

Assume now that

$$\underline{p} + 1 \leq p \leq -m/4$$

Even though there is no loss of derivatives here, the information that we have so far is not sufficient to obtain the bound in this range. The main reason is that some components of $(\partial_s + i\Lambda_\mu)G(s)$ undergo oscillations which are not linear. To deal with this term we are going to use the following decomposition of $(\partial_s + i\Lambda_\mu)G$, which follows from Lemma 4.3 in [32],

$$\chi'_{ba}(\xi) \cdot \mathcal{F}\{(\partial_s + i\Lambda_\mu)G(s)\}(\xi) = g_d(\xi) + g_\infty(\xi) + g_2(\xi) \quad (5.29)$$

for any $s \in [2^{m-1}, 2^{m+1}]$, where $\chi'_{ba}(x) = \varphi_{\leq 4}(2^{\mathcal{D}}(|x| - \gamma_0)) + \varphi_{\leq 4}(2^{\mathcal{D}}(|x| - \gamma_1))$ and

$$\begin{aligned} \|g_2\|_{L^2} &\lesssim \varepsilon_1^2 2^{-3m/2+20\delta m}, & \|g_\infty\|_{L^\infty} &\lesssim \varepsilon_1^2 2^{-m-4\delta m}, \\ \sup_{|\rho| \leq 2^{7m/9+4\delta m}} \|\mathcal{F}^{-1}\{e^{i\rho\Lambda}g_d\}\|_{L^\infty} &\lesssim \varepsilon_1^2 2^{-16m/9-4\delta m}. \end{aligned} \quad (5.30)$$

Clearly, the contribution of g_d can be estimated as in (5.28), using Lemma 4.5. On the other hand, we estimate the contributions of g_2 and g_∞ in the Fourier space, using Schur's lemma. For this we need to use the volume bound in Proposition 7.4 (i) in [32]. We have

$$\sup_{\xi} \|\tilde{\varphi}_p(\Phi(\xi, \eta))\chi_{ba}(\xi - \eta)g_\infty(\xi - \eta)\|_{L_\eta^1} \lesssim 2^{(1-\delta)p} \|g_\infty\|_{L^\infty} \lesssim 2^{(1-\delta)p} 2^{-(1+4\delta)m} \varepsilon_1^2,$$

and also a similar bound for the ξ integral (keeping η fixed). Therefore, using Schur's lemma

$$|\tilde{\mathcal{A}}_{ba}^{l,p}[\mathcal{F}^{-1}g_\infty(s), W_{k_1}(s), W_{k_2}(s)]| \lesssim 2^{(1-\delta)p} 2^{-(1+4\delta)m} \varepsilon_1^2 \rho_k(s)^2,$$

and the corresponding contribution is bounded as claimed in (5.26). The contribution of g_2 can be bounded in a similar way, using Schur's lemma and the Cauchy-Schwarz inequality. This completes the proof of the lemma. \square

5.2. The other interactions. In this subsection we show how to bound all the remaining contributions to the energy increment in (5.1). We remark that we do not use the main L^2 lemma in the estimates in this subsection.

5.2.1. *Small frequencies.* We consider now the small frequencies and prove the following:

Lemma 5.5. *For $l \in \{0, 1\}$, $m \in [\mathcal{D}^2, L]$, $G \in \mathcal{G}'$, and $W_1, W_2 \in \mathcal{W}'$ we have*

$$\sum_{\min(k_1, k_2) \leq -40} |\mathcal{A}_{ba,m}^l[G, P_{k_1} W_1, P_{k_2} W_2]| \lesssim \varepsilon_1^3. \quad (5.31)$$

Proof. Let $\underline{k} := \min\{k_1, k_2\}$. Notice that we may assume that $\underline{k} \leq -40$, $\max(k_1, k_2) \in [-10, 10]$, and $l = 1$. We can easily estimate

$$|\mathcal{A}_{ba,m}^1[G, P_{k_1} W_1, P_{k_2} W_2]| \lesssim \sup_{s \in [2^{m-1}, 2^{m+1}]} 2^m 2^{\underline{k}} \|G(s)\|_{L^2} \|P_{k_1} W_1(s)\|_{L^2} \|P_{k_2} W_2(s)\|_{L^2}.$$

In view of (4.12) and (4.14), this suffices to estimate the sum corresponding to $\underline{k} \leq -m - 3\delta m$. Therefore, it suffices to show that if $-(1 + 3\delta)m \leq \bar{k} \leq -40$ then

$$|\mathcal{A}_{ba,m}^1[G, P_{k_1} W_1, P_{k_2} W_2]| \lesssim \varepsilon_1^3 2^{-\delta m}. \quad (5.32)$$

As in the proof of Lemma 5.1, assume that $W_2 \in \mathcal{W}'_{-\sigma}$, $W_1 \in \mathcal{W}'_{\nu}$, $G \in \mathcal{G}'_{\mu}$, $\sigma, \nu, \mu \in \{+, -\}$, and define the associated phase $\Phi = \Phi_{\sigma\mu\nu}$ as in (5.4). The important observation is that

$$|\Phi(\xi, \eta)| \approx 2^{\underline{k}/2} \quad (5.33)$$

in the support of the integral. We define $\mathcal{A}_{ba}^{1,p}$ and $\tilde{\mathcal{A}}_{ba,m}^{1,p}$ as in (5.7)–(5.8), by introducing the the cutoff function $\varphi_p(\Phi(\xi, \eta))$. In view of (5.33) we may assume that $|p - \underline{k}/2| \lesssim 1$. Then we integrate by parts as in (5.12) and similarly obtain

$$\begin{aligned} |\mathcal{A}_{ba,m}^{1,p}[G, P_{k_1} W_1, P_{k_2} W_2]| &\lesssim 2^{-p} \left| \int_{\mathbb{R}} q'_m(s) \tilde{\mathcal{A}}_{ba}^{1,p}[G(s), W_{k_1}(s), W_{k_2}(s)] ds \right| \\ &+ 2^{-p} |\mathcal{J}_{ba,0}^{1,p}(k_1, k_2)| + 2^{-p} |\mathcal{J}_{ba,1}^{1,p}(k_1, k_2)| + 2^{-p} |\mathcal{J}_{ba,2}^{1,p}(k_1, k_2)|, \end{aligned} \quad (5.34)$$

see (5.11) and (5.13) for definitions.

We apply Lemma 4.5 (see (4.26)) to control the terms in the right-hand side of (5.34). Using (4.14) and (4.19) (recall that $2^{-p} \leq 2^{-\underline{k}/2 + \delta m} \leq 2^{m/2 + 3\delta m}$), the first term is dominated by

$$C \varepsilon_1^3 2^{-p} 2^{\delta m} 2^{-5m/6 + \delta m} \lesssim \varepsilon_1^3 2^{-m/4}.$$

Similarly,

$$2^{-p} |\mathcal{J}_{ba,1}^{1,p}(k_1, k_2)| + 2^{-p} |\mathcal{J}_{ba,2}^{1,p}(k_1, k_2)| \lesssim \varepsilon_1^3 2^m 2^{-p} 2^{-5m/6 + \delta m} 2^{-5m/6 + 2\delta m} \lesssim \varepsilon_1^3 2^{-m/10}.$$

For $|\mathcal{J}_{ba,0}^{1,p}(k_1, k_2)|$ we estimate first, using also (5.27) and (4.18)

$$2^{-p} |\mathcal{J}_{ba,0}^{1,p}(k_1, k_2)| \lesssim \varepsilon_1^3 2^m 2^{-p} (2^{-p} 2^{-5m/3 + \delta m}) 2^{\delta m} \lesssim \varepsilon_1^3 2^{-2p} 2^{-2m/3 + 2\delta m}.$$

We can also estimate directly in the Fourier space (placing the factor at low frequency in L^1 and the other two factors in L^2),

$$2^{-p} |\mathcal{J}_{ba,0}^{1,p}(k_1, k_2)| \lesssim \varepsilon_1^3 2^m 2^{-p} 2^{\underline{k}} 2^{-5m/6 + 3\delta m} \lesssim \varepsilon_1^3 2^p 2^{m/6 + 3\delta m}.$$

These last two bounds show that $2^{-p} |\mathcal{J}_{ba,0}^{1,p}(k_1, k_2)| \lesssim \varepsilon_1^3 2^{-m/10}$. The desired conclusion (5.32) follows using (5.34). \square

5.2.2. *The “good” frequencies.* We estimate now the contribution of the terms in (5.1), corresponding to the cutoff χ_{go} . One should keep in mind that these terms are similar, but easier than the ones we have already estimated. We often use the sharp decay in (4.21) to bound the contribution of small modulations.

We may assume that $\overline{W}_2 \in \mathcal{W}'_\sigma$, $W_1 \in \mathcal{W}'_\nu$, and $G \in \mathcal{G}'_+$. For (5.2) it suffices to prove that

$$\sum_{k, k_1, k_2 \in \mathbb{Z}} |\mathcal{A}_{go, m}^l[P_k G, P_{k_1} W_1, P_{k_2} W_2]| \lesssim \varepsilon_1^3 2^{2\delta^2 m}. \quad (5.35)$$

Recalling the assumptions (4.7) on the symbols μ_l , we have the simple bound

$$|\mathcal{A}_{go, m}^l[P_k G, P_{k_1} W_1, P_{k_2} W_2]| \lesssim 2^m 2^{\min(k, k_1, k_2)} 2^{2k^+} \sup_{s \in I_m} \|P_k G(s)\|_{L^2} \|P_{k_1} W_1(s)\|_{L^2} \|P_{k_2} W_2(s)\|_{L^2}.$$

Using now (4.12) and (4.14), it follows that the sum over $k \geq 2\delta m$ or $k \leq -m - \delta m$ in (5.35) is dominated as claimed. Using also the L^∞ bounds (4.20) and Lemma 8.1, we have

$$\begin{aligned} |\mathcal{A}_{go, m}^l[P_k G, P_{k_1} W_1, P_{k_2} W_2]| &\lesssim 2^m 2^{2k^+} \sup_{s \in I_m} \|P_k G(s)\|_{L^\infty} \|P_{k_1} W_1(s)\|_{L^2} \|P_{k_2} W_2(s)\|_{L^2} \\ &\lesssim 2^m 2^{2k^+} \sup_{s \in I_m} \varepsilon_1 2^{k-m+50\delta m} \|P_{k_1} W_1(s)\|_{L^2} \|P_{k_2} W_2(s)\|_{L^2} \end{aligned}$$

if $|k| \geq 10$. This suffices to control the part of the sum over $k \leq -52\delta m$. Moreover

$$\sum_{\min(k_1, k_2) \leq -\mathcal{D} - |k|} |\mathcal{A}_{go, m}^l[P_k G, P_{k_1} W_1, P_{k_2} W_2]| \lesssim \varepsilon_1^3 2^{-\delta m}$$

if $k \in [-52\delta m, 2\delta m]$. This follows as in the proof of Lemma 5.5, once we notice that $\Phi(\xi, \eta) \approx 2^{\min(k_1, k_2)/2}$ in the support of the integral, so we can integrate by parts in s . After these reductions, for (5.35) it suffices to prove that, for any $k \in [-52\delta m, 2\delta m]$,

$$\sum_{k_1, k_2 \in [-\mathcal{D} - |k|, \infty)} |\mathcal{A}_{go, m}^l[P_k G, P_{k_1} W_1, P_{k_2} W_2]| \lesssim \varepsilon_1^3 2^{2\delta^2 m} 2^{-\delta|k|}. \quad (5.36)$$

To prove (5.36) we further decompose in modulation. Let $\bar{k} := \max(k, k_1, k_2)$ and $\underline{p} := \lfloor \bar{k}^+ / 2 - 110\delta m \rfloor$. We define, as in (5.7)–(5.8),

$$\mathcal{A}_{go}^{l, p}[F, H_1, H_2] := \iint_{\mathbb{R}^2 \times \mathbb{R}^2} \mu_l(\xi, \eta) \varphi_p^{[\underline{p}, \infty)}(\Phi(\xi, \eta)) \chi_{go}(\xi - \eta) \widehat{F}(\xi - \eta) \widehat{H}_1(\eta) \widehat{H}_2(-\xi) d\xi d\eta, \quad (5.37)$$

and

$$\mathcal{A}_{go, m}^{l, p}[P_k G, P_{k_1} W_1, P_{k_2} W_2] := \int_{\mathbb{R}} q_m(s) \mathcal{A}_{go}^{l, p}[P_k G(s), P_{k_1} W_1(s), P_{k_2} W_2(s)] ds. \quad (5.38)$$

For $p \geq \underline{p} + 1$ we integrate by parts in s . As in (5.11) and (5.13) let

$$\widetilde{\mathcal{A}}_{go}^{l, p}[F, H_1, H_2] := \iint_{\mathbb{R}^2 \times \mathbb{R}^2} \mu_l(\xi, \eta) \widetilde{\varphi}_p(\Phi(\xi, \eta)) \chi_{go}(\xi - \eta) \widehat{F}(\xi - \eta) \widehat{H}_1(\eta) \widehat{H}_2(-\xi) d\xi d\eta, \quad (5.39)$$

where $\tilde{\varphi}_p(x) := 2^p x^{-1} \varphi_p(x)$. Let $W_{k_1} = P_{k_1} W_1$, $W_{k_2} = P_{k_2} W_2$, and

$$\begin{aligned}\mathcal{J}_{go,0}^{l,p}(k_1, k_2) &:= \int_{\mathbb{R}} q_m(s) \tilde{\mathcal{A}}_{go}^{l,p}[P_k(\partial_s + i\Lambda_\mu)G(s), W_{k_1}(s), W_{k_2}(s)] ds, \\ \mathcal{J}_{go,1}^{l,p}(k_1, k_2) &:= \int_{\mathbb{R}} q_m(s) \tilde{\mathcal{A}}_{go}^{l,p}[P_k G(s), (\partial_s + i\Lambda_\nu)W_{k_1}(s), W_{k_2}(s)] ds, \\ \mathcal{J}_{go,2}^{l,p}(k_1, k_2) &:= \int_{\mathbb{R}} q_m(s) \tilde{\mathcal{A}}_{go}^{l,p}[P_k G(s), W_{k_1}(s), (\partial_s + i\Lambda_{-\sigma})W_{k_2}(s)] ds.\end{aligned}$$

As in (5.12), we have

$$\begin{aligned}|\mathcal{A}_{go,m}^{l,p}[P_k G, P_{k_1} W_1, P_{k_2} W_2]| &\lesssim 2^{-p} \left| \int_{\mathbb{R}} q'_m(s) \tilde{\mathcal{A}}_{go}^{l,p}[P_k G(s), W_{k_1}(s), W_{k_2}(s)] ds \right| \\ &+ 2^{-p} |\mathcal{J}_{go,0}^{l,p}(k_1, k_2) + \mathcal{J}_{go,1}^{l,p}(k_1, k_2) + \mathcal{J}_{go,2}^{l,p}(k_1, k_2)|.\end{aligned}\quad (5.40)$$

Using Lemma 4.5, (4.14), and (4.19), it is easy to see that

$$\sum_{k_1, k_2 \in [-\mathcal{D}-|k|, \infty)} \sum_{p \geq p+1} 2^{-p} \left| \int_{\mathbb{R}} q'_m(s) \tilde{\mathcal{A}}_{go}^{l,p}[P_k G(s), W_{k_1}(s), W_{k_2}(s)] ds \right| \lesssim \varepsilon_1^3 2^{-\delta m}.\quad (5.41)$$

Using also (5.27) and (4.18), as in the first part of the proof of Lemma 5.4, we have

$$\sum_{k_1, k_2 \in [-\mathcal{D}-|k|, \infty)} \sum_{p \geq p+1} 2^{-p} |\mathcal{J}_{go,0}^{l,p}(k_1, k_2)| \lesssim \varepsilon_1^3 2^{-\delta m}.\quad (5.42)$$

Using Lemma 4.5, (4.19), and (5.16), it follows that

$$\sum_{k_1, k_2 \in [-\mathcal{D}-|k|, 6m/5]} \sum_{p \geq p+1} 2^{-p} [|\mathcal{J}_{go,1}^{l,p}(k_1, k_2)| + |\mathcal{J}_{go,2}^{l,p}(k_1, k_2)|] \lesssim \varepsilon_1^3 2^{-\delta m}.\quad (5.43)$$

Finally, a symmetrization argument as in the proof of Lemma 5.3 shows that

$$\sum_{k_1, k_2 \in [6m/5-10, \infty)} \sum_{p \geq p+1} 2^{-p} |\mathcal{J}_{go,1}^{l,p}(k_1, k_2) + \mathcal{J}_{go,2}^{l,p}(k_1, k_2)| \lesssim \varepsilon_1^3 2^{-\delta m}.\quad (5.44)$$

In view of (5.40)–(5.44), to complete the proof of (5.36) it remains to bound the contribution of small modulations. In the case of “bad” frequencies, this is done using the main L^2 lemma. Here we need a different argument.

Lemma 5.6. *Assume that $k \in [-52\delta m, 2\delta m]$ and $p = \lfloor \bar{k}^+ / 2 - 110\delta m \rfloor$. Then*

$$\sum_{\min(k_1, k_2) \geq -\mathcal{D}-|k|} |\mathcal{A}_{go,m}^{l,p}[P_k G, P_{k_1} W_1, P_{k_2} W_2]| \lesssim \varepsilon_1^3 2^{2\delta^2 m} 2^{-\delta|k|}.\quad (5.45)$$

Proof. We need to further decompose the function G . Recall that $G \in \mathcal{G}'_+$ and let, for $(k, j) \in \mathcal{J}$

$$f(s) = e^{is\Lambda} G(s), \quad f_{j,k} = P_{[k-2, k+2]} Q_{jk} f, \quad g_{j,k} := A_{\leq 2\mathcal{D}, \gamma_0} A_{\leq 2\mathcal{D}, \gamma_1} f_{j,k}.\quad (5.46)$$

Compare with Lemma 2.6. The functions $g_{j,k}$ are supported away from the bad frequencies γ_0 and γ_1 and $\sum_j g_{j,k}(s) = e^{is\Lambda} G(s)$ away from these frequencies. This induces a decomposition

$$\mathcal{A}_{go,m}^{l,p}[P_k G, P_{k_1} W_1, P_{k_2} W_2] = \sum_{j \geq \max(-k, 0)} \mathcal{A}_{go,m}^{l,p}[e^{-is\Lambda} g_{j,k}, P_{k_1} W_1, P_{k_2} W_2].$$

Notice that for $j \leq m - \delta m$ we have the stronger estimate (4.21) on $\|e^{-is\Lambda} g_{j,k}\|_{L^\infty}$. Therefore, using Lemma 4.5, if $j \leq m - \delta m$ then

$$|\mathcal{A}_{g_{\bar{o},m}^{l,p}}[e^{-is\Lambda} g_{j,k}, P_{k_1} W_1, P_{k_2} W_2]| \lesssim \varepsilon_1 2^{k_2} 2^{-2k^+} 2^{-j/4} \sup_{s \in I_m} \|P_{k_1} W_1(s)\|_{L^2} \|P_{k_2} W_2(s)\|_{L^2}.$$

Therefore⁵

$$\sum_{j \leq m - \delta m} \sum_{\min(k_1, k_2) \geq -\mathcal{D} - |k|} |\mathcal{A}_{g_{\bar{o},m}^{l,p}}[e^{-is\Lambda} g_{j,k}, P_{k_1} W_1, P_{k_2} W_2]| \lesssim \varepsilon_1^3 2^{2\delta^2 m} 2^{-\delta|k|}.$$

Similarly, if $j \geq m + 60\delta m$ then we also have a stronger bound on $\|e^{-is\Lambda} g_{j,k}\|_{L^\infty}$ in the first line of (4.20), and the corresponding contributions are controlled in the same way.

It remains to show, for any $j \in [m - \delta m, m + 60\delta m]$,

$$\sum_{\min(k_1, k_2) \geq -\mathcal{D} - |k|} |\mathcal{A}_{g_{\bar{o},m}^{l,p}}[e^{-is\Lambda} g_{j,k}, P_{k_1} W_1, P_{k_2} W_2]| \lesssim \varepsilon_1^3 2^{-\delta m}. \quad (5.47)$$

For this we use Schur's test. Since $\min(k, k_1, k_2) \geq -53\delta m$ it follows from Proposition 7.4 (i) in [32] and the bound $\|\widehat{g_{j,k}}\|_{L^2} \lesssim \varepsilon_1 2^{-8k^+} 2^{-j+50\delta j}$ that

$$\int_{\mathbb{R}^2} |\mu_l(\xi, \eta)| \varphi_{\leq \underline{p}}(\Phi(\xi, \eta)) |\widehat{g_{j,k}}(\xi - \eta)| \varphi_{[k_1-2, k_1+2]}(\eta) d\eta \lesssim \varepsilon_1 2^{(p-\bar{k}^+)/2 + \delta m} 2^{-j+50\delta j}$$

for any $\xi \in \mathbb{R}^2$ fixed with $|\xi| \in [2^{k_2-4}, 2^{k_2+4}]$. The integral in ξ (for η fixed) can be estimated in the same way. Given the choice of \underline{p} , the desired bound (5.47) follows using Schur's lemma. \square

6. PROOF OF THE MAIN L^2 LEMMA

In this section we prove Lemma 4.6. We divide the proof into several cases. Let

$$\chi_{\gamma_l}(x) := \varphi(2^{\mathcal{D}}(|x| - \gamma_l)), \quad l \in \{0, 1\}.$$

We start the most difficult case when $|\xi - \eta|$ is close to γ_0 and $2^k \gg 1$. In this case $\widehat{\Upsilon}$ can vanish up to order 1 (so we can have $2^q \ll 1$ in the notation of the Lemma 6.1 below).

Lemma 6.1. *The conclusion of Lemma 4.6 holds if $k \geq 3\mathcal{D}_1/2$ and \widehat{g} is supported in the set $\{||\xi| - \gamma_0| \leq 2^{-100}\}$.*

Proof. We will often use the results in Lemma 7.1 below. We may assume that $\nu = +$ in the definition of Φ , since otherwise the operator is trivial. We may also assume that $\mu = +$, in view of the formula (7.30).

In view of Lemma 7.1 (ii) we may assume that either $(\xi - \eta) \cdot \xi^\perp \approx 2^k$ or $-(\xi - \eta) \cdot \xi^\perp \approx 2^k$ in the support of the integral, due to the factor $\chi(2^{-p}\Phi(\xi, \eta))$. Thus we may define

$$a^\pm(\xi, \eta) = a(\xi, \eta) \mathbf{1}_\pm((\xi - \eta) \cdot \xi^\perp), \quad (6.1)$$

and decompose the operator $L_{p,k} = L_{p,k}^+ + L_{p,k}^-$ accordingly. The two operators can be treated in similar ways, so we will concentrate on the operator $L_{p,k}^+$.

⁵This is the only place in the proof of the bound (5.2) where one needs the $2^{2\delta^2 m}$ factor in the right-hand side.

To apply the main TT^* argument we need to first decompose the operators $L_{p,k}$. For $\kappa := 2^{-\mathcal{D}^{3/2}}$ (a small parameter) and $\psi \in C_0^\infty(-2, 2)$ satisfying $\sum_{v \in \mathbb{Z}} \psi(\cdot + v) \equiv 1$, we write

$$\begin{aligned} L_{p,k}^+ &= \sum_{q,r \in \mathbb{Z}} \sum_{j \geq 0} L_{p,k,q}^{r,j}, \\ L_{p,k,q}^{r,j} f(x) &:= \int_{\mathbb{R}^2} e^{is\Phi(x,y)} \chi(2^{-p}\Phi(x,y)) \varphi_q(\widehat{\Upsilon}(x,y)) \psi(\kappa^{-1}2^{-q}\widehat{\Upsilon}(x,y) - r) \varphi_k(y) a_j^+(x,y) f(y) dy, \\ a_j^+(x,y) &:= A(x,y) \chi_{\gamma_0}(x-y) \mathbf{1}_+((x-y) \cdot x^\perp) \widehat{g}_j(x-y), \quad g_j := A_{\geq 0, \gamma_0}[\varphi_j^{[0,\infty)} \cdot g]. \end{aligned} \quad (6.2)$$

In other words, we insert the decompositions

$$g = \sum_{j \geq 0} g_j, \quad 1 = \sum_{q,r \in \mathbb{Z}} \varphi_q(\widehat{\Upsilon}(x,y)) \psi(\kappa^{-1}2^{-q}\widehat{\Upsilon}(x,y) - r)$$

in the formula (4.33) defining the operators $L_{p,k}$. The parameters j and r play a somewhat minor role in the proof (one can focus on the main case $j = 0$) but the parameter q is important. Notice that $q \leq -\mathcal{D}/2$, in view of (7.15). The hypothesis $\|g\|_{Z_1 \cap H_\Omega^{N_1/3,0}} \lesssim 1$ and Lemma 2.6 show that

$$\|\widehat{g}_j\|_{L^\infty} \lesssim 2^{-j(1/2-55\delta)}, \quad \left\| \sup_{\theta \in \mathbb{S}^1} |\widehat{g}_j(r\theta)| \right\|_{L^2(rdr)} \lesssim 2^{-j(1-55\delta)}. \quad (6.3)$$

Note that, for fixed x (respectively y) the support of integration is included in $\mathcal{S}_{p,q,r}^{1,-}(x)$ (respectively $\mathcal{S}_{p,q,r}^{2,-}(y)$), see (7.18)–(7.19). We can use this to estimate the Schur norm of the kernel. It follows from (7.20) and the first bound in (6.3) that

$$\sup_x \int_{\mathbb{R}^2} |\chi(2^{-p}\Phi(x,y)) \varphi_q(\widehat{\Upsilon}(x,y)) \varphi_k(y) a_j^+(x,y)| dy \lesssim \|a_j^+\|_{L^\infty} |\mathcal{S}_{p,q,r}^{1,-}(x)| \lesssim 2^{q+p-k/2} 2^{-j/3}. \quad (6.4)$$

A similar estimate holds for the x integral (keeping y fixed). Moreover, using (7.21) and the second bound in (6.3) to estimate the left-hand side of (6.4) by $C2^{-j+55\delta j} 2^{p-k/2}$. In view of Schur's lemma, we have

$$\|L_{p,k,q}^{r,j}\|_{L^2 \rightarrow L^2} \lesssim \min(2^{q+p-k/2} 2^{-j/3}, 2^{-j+55\delta j} 2^{q/2+p-k/2})$$

The desired conclusion follows, unless

$$q \geq \mathcal{D} + \max\left\{\frac{1}{2}(p - \frac{k}{2}), -\frac{m}{3}\right\} \quad \text{and} \quad 0 \leq j \leq \min\left\{\frac{4m}{9}, -\frac{2}{3}(p - \frac{k}{2})\right\}. \quad (6.5)$$

Therefore, in the rest of the proof we may assume that (6.5) holds, so $\kappa 2^q \gg 2^{p-\frac{k}{2}}$. We use the TT^* argument and Schur's test. It suffices to show that

$$\sup_x \int_{\mathbb{R}^2} |K(x, \xi)| d\xi + \sup_\xi \int_{\mathbb{R}^2} |K(x, \xi)| dx \lesssim 2^{6\delta^2 m} (2^{3(p-\frac{k}{2})} + 2^{2(p-\frac{k}{2})} 2^{-\frac{2}{3}m}) \quad (6.6)$$

for p, k, q, r, j fixed (satisfying (4.32) and (6.5)), where

$$\begin{aligned} K(x, \xi) &:= \int_{\mathbb{R}^2} e^{is\Theta(x,\xi,y)} \chi(2^{-p}\Phi(x,y)) \chi(2^{-p}\Phi(\xi,y)) \psi_{q,r}(x, \xi, y) a_j^+(x,y) \overline{a_j^+(\xi,y)} dy, \\ \Theta(x, \xi, y) &:= \Phi(x,y) - \Phi(\xi,y) = \Lambda(x) - \Lambda(\xi) - \Lambda(x-y) + \Lambda(\xi-y), \\ \psi_{q,r}(x, \xi, y) &:= \varphi_q(\widehat{\Upsilon}(x,y)) \varphi_q(\widehat{\Upsilon}(\xi,y)) \psi(\kappa^{-1}2^{-q}\widehat{\Upsilon}(x,y) - r) \psi(\kappa^{-1}2^{-q}\widehat{\Upsilon}(\xi,y) - r) \varphi_k(y)^2. \end{aligned} \quad (6.7)$$

Since $K(x, \xi) = \overline{K(\xi, x)}$, it suffices to prove the bound on the first term in the left-hand side of (6.6). The main idea of the proof is to show that K is essentially supported in the set where $\omega := x - \xi$ is small. Note first that, in view of (7.20), we may assume that

$$|\omega| = |x - \xi| \lesssim \kappa 2^q \ll 1. \quad (6.8)$$

Step 1: We will show in **Step 2** below that if

$$\text{if } |\omega| \geq L := 2^{2\delta^2 m} [2^{p-k/2} 2^{-q} + 2^{j-q-m} + 2^{-2m/3-q}] \quad \text{then } |K(x, \xi)| \lesssim 2^{-4m}. \quad (6.9)$$

Assuming this, we show now how to prove the bound on the first term in (6.6). Notice that $L \ll 1$, in view of (4.32) and (6.5). We decompose, for fixed x ,

$$\int_{\mathbb{R}^2} |K(x, \xi)| d\xi \lesssim \int_{\{|\omega| \leq L\}} |K(x, x - \omega)| d\omega + \int_{\{|\omega| \geq L\}} |K(x, x - \omega)| d\omega.$$

Combining (6.8) and (6.9), we obtain a suitable bound for the second integral. We now turn to the first integral, which we bound using Fubini and the formula (6.7) by

$$C \|a_j^+\|_{L^\infty} \int_{\mathbb{R}^2} |a_j^+(x, y)| |\chi(2^{-p}\Phi(x, y)) \varphi_q(\widehat{\Upsilon}(x, y)) \varphi_k(y)|^2 \left(\int_{\{|\omega| \leq L\}} |\chi(2^{-p}\Phi(x - \omega, y))| d\omega \right) dy. \quad (6.10)$$

We observe that, for fixed x, y satisfying $||x - y| - \gamma_0| \ll 1$, $|x| \approx 2^k \gg 1$, we have

$$\int_{\{|\omega| \leq L\}} |\chi(2^{-p}\Phi(x - \omega, y))| d\omega \lesssim 2^{p-k/2} L. \quad (6.11)$$

Indeed, it follows from (7.16) that if $z = (x - y - \omega) = (\rho \cos \theta, \rho \sin \theta)$, $|\omega| \leq L$, and $|\Phi(y + z, y)| \leq 2^p$, then $|\rho - |x - y|| \lesssim L$ and θ belongs to a union of two intervals of length $\lesssim 2^{p-k/2}$. The desired bound (6.11) follows.

Using also (6.4) and $\|a_j\|_{L^\infty} \lesssim 2^{-j/3}$, it follows that the expression in (6.10) is bounded by $C 2^{2(p-k/2)} 2^{-2j/3} 2^q L$. The desired bound (6.6) follows, using also the restrictions (6.5).

Step 2: We prove now (6.9). We define orthonormal frames (e_1, e_2) and (V_1, V_2) ,

$$\begin{aligned} e_1 &:= \frac{\nabla_x \Phi(x, y)}{|\nabla_x \Phi(x, y)|}, & e_2 &= e_1^\perp, & V_1 &:= \frac{\nabla_y \Phi(x, y)}{|\nabla_y \Phi(x, y)|}, & V_2 &= V_1^\perp, \\ \omega &= x - \xi = \omega_1 e_1 + \omega_2 e_2. \end{aligned} \quad (6.12)$$

Note that ω_1, ω_2 are functions of (x, y, ξ) . We first make a useful observation: if $|\Theta(x, \xi, y)| \lesssim 2^p$, and $|\omega| \ll 1$ then

$$|\omega_1| \lesssim 2^{-k/2} (2^p + |\omega|^2). \quad (6.13)$$

This follows from a simple Taylor expansion, since

$$|\Phi(x, y) - \Phi(\xi, y) - \omega \cdot \nabla_x \Phi(x, y)| \lesssim |\omega|^2.$$

We turn now to the proof of (6.9). Assuming that x, ξ are fixed with $|x - \xi| \geq L$ and using (6.13), we see that, on the support of integration, $|\omega_2| \approx |\omega|$ and

$$\begin{aligned} V_2 \cdot \nabla_y \Theta(x, \xi, y) &= V_2 \cdot \nabla_y \{-\Lambda(x - y) + \Lambda(\xi - y)\} \\ &= V_2 \cdot \nabla_{x, y}^2 \Phi(x, y) \cdot (x - \xi) + O(|\omega|^2) \\ &= \omega_2 \widehat{\Upsilon}(x, y) + O(|\omega_1| + |\omega|^2). \end{aligned} \quad (6.14)$$

Using (6.5), (6.9), (6.13) and (6.8) (this is where we need $\kappa \ll 1$), we obtain that

$$|V_2 \cdot \nabla_y \Theta(x, \xi, y)| \approx 2^q |\omega_2| \approx 2^q |\omega|$$

in the support of the integral. Using that

$$e^{is\Theta} = \frac{-i}{sV_2 \cdot \nabla_y \Theta} V_2 \cdot \nabla_y e^{is\Theta}, \quad |D_y^\alpha \Theta| \lesssim |\omega|,$$

and letting $\Theta_{(1)} := V_2 \cdot \nabla_y \Theta$, after integration by parts we have

$$K(x, \xi) = i \int_{\mathbb{R}^2} e^{is\Theta} \partial_l \left\{ V_2^l \frac{1}{s\Theta_{(1)}} \chi(2^{-p}\Phi(x, y)) \chi(2^{-p}\Phi(\xi, y)) \psi_{q,r}(x, \xi, y) a_j^+(x, y) \overline{a_j^+(\xi, y)} \right\} dy.$$

We observe that

$$V_2^l \partial_l [\chi(2^{-p}\Phi(x, y)) \chi(2^{-p}\Phi(\xi, y))] = -2^{-p} \Theta_{(1)} \cdot [\chi(2^{-p}\Phi(x, y)) \chi'(2^{-p}\Phi(\xi, y))].$$

This identity is the main reason for choosing V_2 as in (6.12), and this justifies the definition of the function Υ (intuitively, we can only integrate by parts in y along the level sets of the function Φ , due to the very large 2^{-p} factor). Moreover

$$\|D_y^\alpha \psi_{q,r}(x, \xi, y)\| \lesssim 2^{-q|\alpha|}, \quad |D_y^\alpha a_j^+(v, y)| \lesssim_\alpha 2^{|\alpha|j} + 2^{|\alpha|m/3}, \quad v \in \{x, \xi\},$$

in the support of the integral defining $K(x, \xi)$. We integrate by parts many times in y as above. At every step we gain a factor of $2^{m2^q}|\omega|$ and lose a factor of $2^{-p2^q}|\omega| + 2^{-q} + 2^j + 2^{m/3}$. The desired bound in (6.9) follows. This completes the proof. \square

We consider now the (easier) case when $|\xi - \eta|$ is close to γ_1 and k is large.

Lemma 6.2. *The conclusion of Lemma 4.6 holds if $k \geq 3\mathcal{D}_1/2$ and \widehat{g} is supported in the set $\{||\xi| - \gamma_1| \leq 2^{-100}\}$.*

Proof. Using (7.15), we see that on the support of integration we have $|\widehat{\Upsilon}(\xi, \eta)| \approx 1$. The proof is similar to the proof of Lemma 6.1 in the case $2^q \approx 1$. The new difficulties come from the less favorable decay in j close to γ_1 and from the fact that the conclusions in Lemma 7.1 (iii) do not apply. We define a_j^\pm as in (6.2) (with γ_1 replacing γ_0 and $g_j := A_{\geq 4, \gamma_1}[\varphi_j^{[0, \infty)} \cdot g]$), and

$$L_{p,k}^{x_0,j} f(x) := \varphi_{\leq -\mathcal{D}}(x - x_0) \int_{\mathbb{R}^2} e^{is\Phi(x,y)} \chi(2^{-p}\Phi(x, y)) \varphi_k(y) a_j^+(x, y) f(y) dy, \quad (6.15)$$

for any $x_0 \in \mathbb{R}^2$. We have

$$\|\widehat{g}_j\|_{L^\infty} \lesssim 2^{6\delta j}, \quad \left\| \sup_{\theta \in \mathbb{S}^1} \widehat{A_{n, \gamma_1} g_j}(r\theta) \right\|_{L^2(rdr)} \lesssim 2^{(1/2 - 49\delta)n - j(1 - 55\delta)}, \quad (6.16)$$

for $n \geq 1$, as a consequence of Lemma 2.6 (i). Notice that these bounds are slightly weaker than the bounds in (6.3). However, we can still estimate (compare with (6.4))

$$\sup_x \int_{\mathbb{R}^2} |\chi(2^{-p}\Phi(x, y)) \varphi_k(y) a_j^+(x, y)| dy \lesssim 2^{p-k/2} \cdot 2^{-(1-55\delta)j}. \quad (6.17)$$

Indeed, we use only the second bound in (6.16), decompose the integral as a sum of integrals over the dyadic sets $||x - y| - \gamma_1| \approx 2^{-n}$, $n \geq 1$, and use (7.16) and the Cauchy–Schwarz in each dyadic set. As a consequence of (6.17), it remains to consider the sum over $j \leq 4m/9$.

We can then proceed as in the proof of Lemma 6.1. Using the TT^* argument for the operators $L_{p,k}^{x_0,j}$ and Schur's lemma, it suffices to prove bounds similar to those in (6.6). Let $\omega = x - \xi$, and notice that $|\omega| \leq 2^{-\mathcal{D}+10}$. This replaces the diameter bound (6.8) and is the main reason for adding the localization factors $\varphi_{\leq -\mathcal{D}}(x - x_0)$ in (6.15). The main claim is that

$$\text{if } |\omega| \geq L := 2^{2\delta^2 m} (2^{p-k/2} + 2^{j-m} + 2^{-2m/3}) \quad \text{then } |K(x, \xi)| \lesssim 2^{-4m}. \quad (6.18)$$

The same argument as in **Step 1** in the proof of Lemma 6.1 shows that this claim suffices. Moreover, this claim can be proved using integration by parts, as in **Step 2** in the proof of Lemma 6.1. The conclusion of the lemma follows. \square

Finally, we now consider the case of low frequencies.

Lemma 6.3. *The conclusion of Lemma 4.6 holds if $k \in [-100, 7\mathcal{D}_1/4]$.*

Proof. For small frequencies, the harder case is when $|\xi - \eta|$ is close to γ_1 , since the conclusions of Lemma 7.3 are weaker than the conclusions of Lemma 7.2, and the decay in j is less favorable. So we will concentrate on this case.

We need to first decompose our operator. For $j \geq 0$ and $l \in \mathbb{Z}$ we define

$$a_{j,l}^\pm(x, y) := A(x, y)\chi_{\gamma_1}(x - y)\varphi_l^\pm((x - y) \cdot x^\perp)\widehat{g}_j(x - y), \quad g_j := A_{\geq 4, \gamma_1}[\varphi_j^{[0, \infty)}] \cdot P_{[-8, 8]}g, \quad (6.19)$$

where $\varphi_l^\pm(v) := \mathbf{1}_\pm(v)\varphi_l(v)$. This is similar to (6.2), but with the additional dyadic decomposition in terms of the angle $|(x - y) \cdot x^\perp| \approx 2^l$. Then we decompose, as in (6.2),

$$L_{p,k} = \sum_{q,r \in \mathbb{Z}} \sum_{j \geq 0} \sum_{l \in \mathbb{Z}} \sum_{\iota \in \pm} L_{p,k,q}^{r,j,l,\iota}, \quad (6.20)$$

where, with $\kappa = 2^{-\mathcal{D}^{3/2}}$ and $\psi \in C_0^\infty(-2, 2)$ satisfying $\sum_{v \in \mathbb{Z}} \psi(\cdot - v) \equiv 1$ as before,

$$L_{p,k,q}^{r,j,l,\iota} f(x) := \varphi_{\geq -100}(x) \int_{\mathbb{R}^2} e^{is\Phi(x,y)} \chi(2^{-p}\Phi(x,y)) \times \varphi_q(\Upsilon(x,y)) \psi(\kappa^{-1}2^{-q}\Upsilon(x,y) - r) \varphi_k(y) a_{j,l}^\iota(x,y) f(y) dy. \quad (6.21)$$

We consider two main cases, depending on the size of q .

Case 1. $q \leq -\mathcal{D}_1$. As a consequence of (7.32), the operators $L_{p,k,q}^{r,j,l,\iota}$ are nontrivial only if $2^k \approx 1$ and $2^l \approx 1$. Using also (7.31) it follows that

$$\begin{aligned} |\nabla_x \Phi| &\approx 1, & |\nabla_x \Upsilon \cdot \nabla_x^\perp \Phi| &\approx 1, \\ |\nabla_y \Phi| &\approx 1, & |\nabla_y \Upsilon \cdot \nabla_y^\perp \Phi| &\approx 1, \end{aligned} \quad (6.22)$$

in the support of the integrals defining the operators $L_{p,k,q}^{r,j,l,\iota}$.

Step 1. The proof proceeds as in Lemma 6.1. For simplicity, we assume that $\iota = +$. Let

$$\begin{aligned} \mathcal{S}_{p,q,r,l}^1(x) &:= \{z : |z| - \gamma_1 \leq 2^{-\mathcal{D}+1}, |\Phi(x, x - z)| \leq 2^{p+1}, |\Upsilon(x, x - z)| \leq 2^{q+2}, \\ &|\Upsilon(x, x - z) - r\kappa 2^q| \leq 10\kappa 2^q, z \cdot x^\perp \in [2^{l-2}, 2^{l+2}]\}. \end{aligned} \quad (6.23)$$

Recall that, if $z = (\rho \cos \theta, \rho \sin \theta)$ and $x = (|x| \cos \alpha, |x| \sin \alpha)$ then

$$\Phi(x, x - z) = \lambda(|x|) - \mu\lambda(\rho) - \nu\lambda(\sqrt{|x|^2 + \rho^2 - 2\rho|x|\cos(\theta - \alpha)}). \quad (6.24)$$

It follows from (6.22) and the change of variables argument in the proof of Lemma 7.1 (iii) that

$$|\mathcal{S}_{p,q,r,l}^1(x)| \lesssim 2^{p+q}, \quad \text{diam}(\mathcal{S}_{p,q,r,l}^1(x)) \lesssim 2^p + \kappa 2^q \quad (6.25)$$

if $|x| \approx 1$ and $2^l \approx 1$. Moreover, using (6.24), for any x and ρ ,

$$|\{\theta : z = (\rho \cos \theta, \rho \sin \theta) \in \mathcal{S}_{p,q,r,l}^1(x)\}| \lesssim 2^p. \quad (6.26)$$

Therefore, using (6.16) and these last two bounds, if $|x| \approx 1$ then

$$\int_{\mathbb{R}^2} |\chi(2^{-p}\Phi(x,y))\varphi_q(\Upsilon(x,y))\varphi_k(y)a_{j,l}^+(x,y)| dy \lesssim \min(2^{p+q}2^{6\delta j}, 2^{p}2^{-j+55\delta j}). \quad (6.27)$$

One can prove a similar bound for the x integral, keeping y fixed. The desired conclusion follows from Schur's lemma unless

$$q \geq \mathcal{D} + \max \left\{ \frac{p}{2}, -\frac{m}{3} \right\} \quad \text{and} \quad 0 \leq j \leq \min \left\{ \frac{4m}{9}, -\frac{2p}{3} \right\}. \quad (6.28)$$

Step 2. Assuming (6.28), we use the TT^* argument and Schur's test. It suffices to show that

$$\sup_x \int_{\mathbb{R}^2} |K(x, \xi)| d\xi + \sup_{\xi} \int_{\mathbb{R}^2} |K(x, \xi)| dx \lesssim 2^{6\delta m} (2^{3p} + 2^{2p-2m/3}) \quad (6.29)$$

for p, k, q, r, j, l fixed, where

$$\begin{aligned} K(x, \xi) &:= \varphi_{\geq -100}(x) \varphi_{\geq -100}(\xi) \int_{\mathbb{R}^2} e^{is\Theta(x, \xi, y)} \\ &\quad \times \chi(2^{-p}\Phi(x, y)) \chi(2^{-p}\Phi(\xi, y)) \psi_{q,r}(x, \xi, y) a_{j,l}^+(x, y) \overline{a_{j,l}^+(\xi, y)} dy, \end{aligned} \quad (6.30)$$

and, as in (6.7),

$$\begin{aligned} \Theta(x, \xi, y) &:= \Phi(x, y) - \Phi(\xi, y) = \Lambda(x) - \Lambda(\xi) - \Lambda_{\mu}(x - y) + \Lambda_{\mu}(\xi - y), \\ \psi_{q,r}(x, \xi, y) &:= \varphi_q(\Upsilon(x, y)) \varphi_q(\Upsilon(\xi, y)) \psi(\kappa^{-1}2^{-q}\Upsilon(x, y) - r) \psi(\kappa^{-1}2^{-q}\Upsilon(\xi, y) - r) \varphi_k(y)^2. \end{aligned}$$

Let $\omega := x - \xi$. As in the proof of Lemma 6.1 the main claim is that

$$\text{if } |\omega| \geq L := 2^{\delta^2 m} (2^{p-q} + 2^{j-q-m} + 2^{-q-2m/3}) \quad \text{then} \quad |K(x, \xi)| \lesssim 2^{-4m}. \quad (6.31)$$

The same argument as in **Step 1** in the proof of Lemma 6.1, using (6.27), shows that this claim suffices. Moreover, this claim can be proved using integration by parts, as in **Step 2** in the proof of Lemma 6.1. The desired bound (6.29) follows.

Case 2. $q \geq -\mathcal{D}_1$. There are several new issues in this case, mostly when the angular parameter 2^l is very small (and bounds like (6.26) fail). As in the proof of Lemma 6.2, we also need to modify the main decomposition (6.20). Let

$$L_{p,k,q}^{x_0, j, l} f(x) := \varphi_{\leq -\mathcal{D}}(x - x_0) \int_{\mathbb{R}^2} e^{is\Phi(x, y)} \chi(2^{-p}\Phi(x, y)) \varphi_q(\Upsilon(x, y)) \varphi_k(y) a_{j,l}^+(x, y) f(y) dy. \quad (6.32)$$

Here $x_0 \in \mathbb{R}^2$, $|x_0| \geq 2^{-110}$, and the localization factor on $x - x_0$ leads to a good upper bound on $|x - \xi|$ in the TT^* argument below. It remains to prove that if $q \geq -\mathcal{D}_1$ then

$$\|L_{p,k,q}^{x_0, j, l}\|_{L^2 \rightarrow L^2} \lesssim 2^{\delta^2 l} 2^{-\delta^2 j} 2^{30\delta m} (2^{(3/2)p} + 2^{p-m/3}). \quad (6.33)$$

Step 1. We start with a Schur bound. For $x \in \mathbb{R}^2$ with $|x| \in [2^{-120}, 2^{\mathcal{D}_1+10}]$ let

$$\begin{aligned} \mathcal{S}_{p,q,l}^1(x) &:= \{z : ||z| - \gamma_1| \leq 2^{-\mathcal{D}+1}, |\Phi(x, x-z)| \leq 2^{p+1}, \\ &\quad |\Upsilon(x, x-z)| \in [2^{q-2}, 2^{q+2}], z \cdot x^\perp \in [2^{l-2}, 2^{l+2}]\}. \end{aligned} \quad (6.34)$$

The condition $|\Upsilon(x, x-z)| \geq 2^{-\mathcal{D}_1-4}$ shows that $|\nabla_z[\Phi(x, x-z)]| \in [2^{-4\mathcal{D}_1}, 2^{\mathcal{D}_1}]$ for $z \in \mathcal{S}_{p,q,l}^1(x)$. The formula (6.24) shows that

$$|\{\theta : z = (\rho \cos \theta, \rho \sin \theta) \in \mathcal{S}_{p,q,l}^1(x)\}| \lesssim 2^{p-l}. \quad (6.35)$$

Moreover, we claim that for any x ,

$$|\mathcal{S}_{p,q,l}^1(x)| \lesssim 2^{p+l}. \quad (6.36)$$

Indeed, this follows from (6.35) if $l \geq -\mathcal{D}$. On the other hand, if $l \leq -\mathcal{D}$ then $\partial_\theta[\Phi(x, x-z)] \leq 2^{-\mathcal{D}/2}$ (due to (6.24)), so $\partial_\rho[\Phi(x, x-z)] \geq 2^{-5\mathcal{D}_1}$ (due to the inequality $|\nabla_z[\Phi(x, x-z)]| \in [2^{-4\mathcal{D}_1}, 2^{\mathcal{D}_1}]$). Recalling also (6.16), it follows from these last two bounds that

$$\int_{\mathbb{R}^2} |\chi(2^{-p}\Phi(x, y))\varphi_q(\Upsilon(x, y))\varphi_k(y)a_{j,l}^+(x, y)|dy \lesssim \min(2^{6\delta j}2^{p+l}, 2^{-j+55\delta j}2^{p-l}), \quad (6.37)$$

if $|x| \in [2^{-120}, 2^{\mathcal{D}_1+10}]$. In particular, the integral is also bounded by $C2^p2^{-j/2+31\delta j}$. The integral in x , keeping y fixed, can be estimated in a similar way. The desired bound (6.33) follows unless

$$j \leq \min(2m/3, -p) - \mathcal{D}, \quad l \geq \max(p/2, -m/3) + \mathcal{D}. \quad (6.38)$$

Step 2. Assuming (6.38), we use the TT^* argument and Schur's test. It suffices to show that

$$\sup_x \int_{\mathbb{R}^2} |K(x, \xi)| d\xi \lesssim 2^{55\delta m} (2^{3p} + 2^{2p-2m/3}) \quad (6.39)$$

for p, k, q, x_0, j, l fixed, where $\Theta(x, \xi, y) = \Phi(x, y) - \Phi(\xi, y)$ and

$$\begin{aligned} K(x, \xi) := & \varphi_{\leq -\mathcal{D}}(x - x_0)\varphi_{\leq -\mathcal{D}}(\xi - x_0) \int_{\mathbb{R}^2} e^{is\Theta(x, \xi, y)} \chi(2^{-p}\Phi(x, y))\chi(2^{-p}\Phi(\xi, y)) \\ & \times \varphi_q(\Upsilon(x, y))\varphi_q(\Upsilon(\xi, y))\varphi_k(y)^2 a_{j,l}^+(x, y)\overline{a_{j,l}^+(\xi, y)} dy. \end{aligned} \quad (6.40)$$

Let $\omega = x - \xi$. As before, the main claim is that

$$\text{if } |\omega| \geq L := 2^{\delta m}(2^p + 2^{j-m} + 2^{-2m/3}) \quad \text{then } |K(x, \xi)| \lesssim 2^{-4m}. \quad (6.41)$$

To see that this claim suffices, we use an argument similar to the one in **Step 1** in the proof of Lemma 6.1. Indeed, up to acceptable errors, the left-hand side of (6.39) is bounded by

$$\begin{aligned} C\|a_{j,l}^+\|_{L^\infty} \sup_{|x-x_0| \leq 2^{-\mathcal{D}+2}} \int_{\mathbb{R}^2} |a_{j,l}^+(x, y)|\chi(2^{-p}\Phi(x, y))\varphi_q(\Upsilon(x, y)) \\ \times \left(\int_{\{|\omega| \leq L\}} |\chi(2^{-p}\Phi(x - \omega, y))| d\omega \right) dy. \end{aligned} \quad (6.42)$$

Notice that if $|\Upsilon(x, y)| \geq 2^{-\mathcal{D}_1-2}$ then $|(\nabla_x \Phi)(x, y)| \geq 2^{-4\mathcal{D}_1}$, thus $|(\nabla_y \Phi)(x - \omega, y)| \geq 2^{-4\mathcal{D}_1-1}$ if $|\omega| \leq L \leq 2^{-\mathcal{D}}$. Therefore, the integral in ω in the expression above is bounded by $C2^p L$. Using also (6.37), the expression in (6.42) is bounded by

$$C2^{6\delta j}2^p L \cdot 2^p 2^{-j/2+32\delta j} \lesssim 2^{\delta m}2^{3p} + 2^{40\delta m}2^{2p+j/2-m} + 2^{\delta m}2^{2p-2m/3}$$

The desired bound (6.39) follows using also that $j \leq 2m/3$, see (6.38).

The claim (6.41) follows by the same integration by parts argument as in **Step 2** in the proof of Lemma 6.1, once we recall that $|(\nabla_x \Phi)(x, y)| \geq 2^{-4\mathcal{D}_1}$ and $|(\nabla_y \Phi)(x, y)| \geq 2^{-4\mathcal{D}_1}$ in the support of the integral, while $|\omega| \leq 2^{-\mathcal{D}+4}$. This completes the proof of the lemma. \square

7. THE FUNCTION Υ

In this section we collect and prove some important facts about the functions Φ and Υ .

7.1. Basic properties. Recall that

$$\begin{aligned}\Phi(\xi, \eta) &= \Phi_{\sigma\mu\nu}(\xi, \eta) = \Lambda_\sigma(\xi) - \Lambda_\mu(\xi - \eta) - \Lambda_\nu(\eta), \quad \sigma, \mu, \nu \in \{+, -\}, \\ \Lambda_\kappa(\xi) &= \lambda_\kappa(|\xi|) = \kappa\lambda(|\xi|) = \kappa\sqrt{|\xi| + |\xi|^3}.\end{aligned}\tag{7.1}$$

We have

$$\lambda'(x) = \frac{1 + 3x^2}{2\sqrt{x + x^3}}, \quad \lambda''(x) = \frac{3x^4 + 6x^2 - 1}{4(x + x^3)^{3/2}}, \quad \lambda'''(x) = \frac{3(1 + 5x^2 - 5x^4 - x^6)}{8(x + x^3)^{5/2}}.\tag{7.2}$$

Therefore

$$\lambda''(x) \geq 0 \text{ if } x \geq \gamma_0, \quad \lambda''(x) \leq 0 \text{ if } x \in [0, \gamma_0], \quad \gamma_0 := \sqrt{\frac{2\sqrt{3} - 3}{3}} \approx 0.393.\tag{7.3}$$

It follows that

$$\lambda(\gamma_0) \approx 0.674, \quad \lambda'(\gamma_0) \approx 1.086, \quad \lambda'''(\gamma_0) \approx 4.452, \quad \lambda''''(\gamma_0) \approx -28.701.\tag{7.4}$$

Let $\gamma_1 := \sqrt{2} \approx 1.414$ denote the radius of the space-time resonant sphere, and notice that

$$\lambda(\gamma_1) = \sqrt{3\sqrt{2}} \approx 2.060, \quad \lambda'(\gamma_1) = \frac{7}{2\sqrt{3\sqrt{2}}} \approx 1.699, \quad \lambda''(\gamma_1) = \frac{23}{4\sqrt{54\sqrt{2}}} \approx 0.658.\tag{7.5}$$

The following simple observation will be used many times: if $U_2 \geq 1$, $\xi, \eta \in \mathbb{R}^2$, $\max(|\xi|, |\eta|, |\xi - \eta|) \leq U_2$, $\min(|\xi|, |\eta|, |\xi - \eta|) = a \leq 2^{-10}U_2^{-1}$, then

$$|\Phi(\xi, \eta)| \geq \lambda(a) - \sup_{b \in [a, U_2]} (\lambda(a + b) - \lambda(b)) \geq \lambda(a) - a \max\{\lambda'(a), \lambda'(U_2 + 1)\} \geq \lambda(a)/4.\tag{7.6}$$

7.2. The function Υ . The analysis in the proofs of the crucial L^2 lemmas in section 6 depends on understanding the properties of the function $\Upsilon : \mathbb{R}^2 \times \mathbb{R}^2 \rightarrow \mathbb{R}$,

$$\Upsilon(\xi, \eta) := (\nabla_{\xi, \eta}^2 \Phi)(\xi, \eta) [(\nabla_\xi^\perp \Phi)(\xi, \eta), (\nabla_\eta^\perp \Phi)(\xi, \eta)].\tag{7.7}$$

We calculate

$$\begin{aligned}(\nabla_\eta \Phi)(\xi, \eta) &= -\lambda'_\nu(|\eta|) \frac{\eta}{|\eta|} + \lambda'_\mu(|\xi - \eta|) \frac{\xi - \eta}{|\xi - \eta|}, \\ (\nabla_\xi \Phi)(\xi, \eta) &= \lambda'_\sigma(|\xi|) \frac{\xi}{|\xi|} - \lambda'_\mu(|\xi - \eta|) \frac{\xi - \eta}{|\xi - \eta|},\end{aligned}\tag{7.8}$$

and

$$\begin{aligned}(\nabla_{\xi, \eta}^2 \Phi)(\xi, \eta) [\partial_i, \partial_j] &= \lambda''_\mu(|\xi - \eta|) \frac{(\xi_i - \eta_i)(\xi_j - \eta_j)}{|\xi - \eta|^2} \\ &\quad + \lambda'_\mu(|\xi - \eta|) \frac{\delta_{ij}|\xi - \eta|^2 - (\xi_i - \eta_i)(\xi_j - \eta_j)}{|\xi - \eta|^3}.\end{aligned}\tag{7.9}$$

Using these formulas we calculate

$$\begin{aligned}\Upsilon(\xi, \eta) &= \frac{\lambda'_\mu(|\xi - \eta|) - |\xi - \eta| \lambda''_\mu(|\xi - \eta|)}{|\xi - \eta|^3} \frac{\lambda'_\sigma(|\xi|)}{|\xi|} \frac{\lambda'_\nu(|\eta|)}{|\eta|} (\eta \cdot \xi^\perp)^2 \\ &\quad + \frac{\lambda'_\mu(|\xi - \eta|)}{|\xi - \eta|} (\nabla_\xi \Phi)(\xi, \eta) \cdot (\nabla_\eta \Phi)(\xi, \eta).\end{aligned}\tag{7.10}$$

Using (7.8) and the identity $(v \cdot w^\perp)^2 + (v \cdot w)^2 = |v|^2|w|^2$, this becomes, with $z := \xi - \eta$,

$$\begin{aligned} -\Upsilon(\xi, \eta) &= \frac{\lambda''_\mu(|z|)}{|z|^2} \frac{\lambda'_\sigma(|\xi|)}{|\xi|} \frac{\lambda'_\nu(|\eta|)}{|\eta|} (\eta \cdot \xi^\perp)^2 \\ &\quad + \frac{\lambda'_\mu(|z|)}{|z|^3} \left\{ \lambda'_\mu(|z|)|z| - \frac{\lambda'_\sigma(|\xi|)}{|\xi|} \xi \cdot z \right\} \left\{ \lambda'_\mu(|z|)|z| - \frac{\lambda'_\nu(|\eta|)}{|\eta|} \eta \cdot z \right\}. \end{aligned} \quad (7.11)$$

We define also the normalized function $\widehat{\Upsilon}$,

$$\widehat{\Upsilon}(\xi, \eta) := \frac{\Upsilon(\xi, \eta)}{|(\nabla_\xi \Phi)(\xi, \eta)| \cdot |(\nabla_\eta \Phi)(\xi, \eta)|}. \quad (7.12)$$

We consider first the case of large frequencies:

Lemma 7.1. *Assume that $\sigma = \nu = +$, $k \geq \mathcal{D}_1$, and $p - k/2 \leq -\mathcal{D}_1$.*

(i) *Assume that*

$$|\Phi(\xi, \eta)| \leq 2^p, \quad |\xi|, |\eta| \in [2^{k-2}, 2^{k+2}], \quad 2^{-20} \leq |\xi - \eta| \leq 2^{20}. \quad (7.13)$$

Let $z := \xi - \eta$. Then, with $p^+ = \max(p, 0)$,

$$\frac{|\xi \cdot \eta^\perp|}{|\xi||\eta|} \approx 2^{-k}, \quad \frac{|\xi \cdot z|}{|\xi||z|} + \frac{|\eta \cdot z|}{|\eta||z|} \lesssim 2^{p^+ - k/2}. \quad (7.14)$$

Moreover, we can write

$$\begin{aligned} -\mu \Upsilon(\xi, \eta) &= \lambda''(|z|)A(\xi, \eta) + B(\xi, z)B(\eta, z), \\ |A(\xi, \eta)| &\gtrsim 2^k, \quad \|D^\alpha A\|_{L^\infty} \lesssim_\alpha 2^k, \quad \|B\|_{L^\infty} \lesssim 2^{p^+}, \quad \|D^\alpha B\|_{L^\infty} \lesssim_\alpha 2^{k/2}. \end{aligned} \quad (7.15)$$

(ii) *Assume that $z = (\rho \cos \theta, \rho \sin \theta)$, $|\rho| \in [2^{-20}, 2^{20}]$. There exists functions $\theta^1 = \theta^1_{|\xi|, \mu}$ and $\theta^2 = \theta^2_{|\eta|, \mu}$ such that,*

$$\begin{aligned} \text{if } 2^{k-2} \leq |\xi| \leq 2^{k+2} \text{ and } |\Phi(\xi, \xi - z)| \leq 2^p \text{ then } \min_{\mp} |\theta - \arg(\xi) \mp \theta^1(\rho)| &\lesssim 2^{p-k/2}, \\ \text{if } 2^{k-2} \leq |\eta| \leq 2^{k+2} \text{ and } |\Phi(\eta + z, \eta)| \leq 2^p \text{ then } \min_{\mp} |\theta - \arg(\eta) \mp \theta^2(\rho)| &\lesssim 2^{p-k/2}. \end{aligned} \quad (7.16)$$

Moreover

$$|\theta^1(\rho) - \pi/2| + |\theta^2(\rho) - \pi/2| \lesssim 2^{-k/2}, \quad |\partial_\rho \theta^1| + |\partial_\rho \theta^2| \lesssim 2^{-k/2}. \quad (7.17)$$

(iii) *Assume that $|\xi|, |\eta| \in [2^{k-2}, 2^{k+2}]$. For $0 < \kappa \leq 2^{-\mathcal{D}_1}$ and integers r, q such that $q \leq -\mathcal{D}_1$, $|\kappa r| \in [1/4, 4]$, define*

$$\begin{aligned} \mathcal{S}_{p,q,r}^{1,\mp}(\xi) &:= \{z : |z| \in [2^{-15}, 2^{15}], |\Phi(\xi, \xi - z)| \leq 2^p, \\ &\quad |\arg(z) - \arg(\xi) \mp \theta^1(\rho)| \leq 2^{-\mathcal{D}_1/2}, |\widehat{\Upsilon}(\xi, \xi - z) - \kappa r 2^q| \leq 10\kappa 2^q\}, \end{aligned} \quad (7.18)$$

and

$$\begin{aligned} \mathcal{S}_{p,q,r}^{2,\mp}(\eta) &:= \{z : |z| \in [2^{-15}, 2^{15}], |\Phi(\eta + z, \eta)| \leq 2^p, \\ &\quad |\arg(z) - \arg(\eta) \mp \theta^2(\rho)| \leq 2^{-\mathcal{D}_1/2}, |\widehat{\Upsilon}(\eta + z, \eta) - \kappa r 2^q| \leq 10\kappa 2^q\}. \end{aligned} \quad (7.19)$$

Then, for any $\iota \in \{+, -\}$,

$$|\mathcal{S}_{p,q,r}^{1,\iota}(\xi)| + |\mathcal{S}_{p,q,r}^{2,\iota}(\eta)| \lesssim 2^{q+p-k/2}, \quad \text{diam}(\mathcal{S}_{p,q,r}^{1,\iota}(\xi)) + \text{diam}(\mathcal{S}_{p,q,r}^{2,\iota}(\eta)) \lesssim 2^{p-k/2} + \kappa 2^q. \quad (7.20)$$

Moreover, if $2^{p-k/2} \ll \kappa 2^q$ then there exist intervals $I_{p,q,r}^1$ and $I_{p,q,r}^2$ such that

$$\begin{aligned} \mathcal{S}_{p,q,r}^{1,\mp}(\xi) &\subseteq \{(\rho \cos \theta, \rho \sin \theta) : \rho \in I_{p,q,r}^1, |\theta - \arg(\xi) \mp \theta^1(\rho)| \lesssim 2^{p-k/2}\}, & |I_{p,q,r}^1| &\lesssim \kappa 2^q, \\ \mathcal{S}_{p,q,r}^{2,\mp}(\eta) &\subseteq \{(\rho \cos \theta, \rho \sin \theta) : \rho \in I_{p,q,r}^2, |\theta - \arg(\eta) \mp \theta^2(\rho)| \lesssim 2^{p-k/2}\}, & |I_{p,q,r}^2| &\lesssim \kappa 2^q. \end{aligned} \quad (7.21)$$

Proof. (i) Notice that if $|\xi| = s$, $|\eta| = r$, and $z = \xi - \eta = (\rho \cos \theta, \rho \sin \theta)$ then

$$\begin{aligned} 2\xi \cdot \eta &= r^2 + s^2 - \rho^2, & 2z \cdot \xi &= \rho^2 + s^2 - r^2, & 2z \cdot \eta &= s^2 - r^2 - \rho^2, \\ (2\eta \cdot \xi^\perp)^2 &= 4r^2 s^2 - (r^2 + s^2 - \rho^2)^2. \end{aligned} \quad (7.22)$$

Under the assumptions (7.13), we see that $|\lambda(r) - \lambda(s)| \lesssim 2^{p^+}$, therefore $|r - s| \lesssim 2^{-k/2} 2^{p^+}$. The bounds (7.14) follow using also (7.22). The decomposition (7.15) follows from (7.11), with

$$A(x, y) := \frac{\lambda'(|x|)}{|x|} \frac{\lambda'(|y|)}{|y|} \frac{(x \cdot y^\perp)^2}{|x - y|^2}, \quad B(w, z) := \frac{\sqrt{\lambda'(|z|)}}{|z|^{3/2}} \left\{ |z| \lambda'(|z|) - \frac{\lambda'(|w|)}{|w|} (w \cdot z) \right\}.$$

The bounds in the second line of (7.15) follow from this definition and (7.14).

(ii) We will show the estimates for fixed ξ , since the estimates for fixed η are similar. We may assume that $\xi = (s, 0)$, so

$$\Phi(\xi, \xi - z) = \lambda(s) - \lambda_\mu(\rho) - \lambda(\sqrt{s^2 + \rho^2 - 2s\rho \cos \theta}). \quad (7.23)$$

Let $f(\theta) := -\lambda(s) + \lambda_\mu(\rho) + \lambda(\sqrt{s^2 + \rho^2 - 2s\rho \cos \theta})$. We notice that $-f(0) \gtrsim 2^{k/2}$, $f(\pi) \gtrsim 2^{k/2}$, and $f'(\theta) \approx 2^{k/2} \sin \theta$ for $\theta \in [0, \pi]$. Therefore f is increasing on the interval $[0, \pi]$ and vanishes at a unique point $\theta^1(\rho) = \theta_{s,\mu}^1(\rho)$. Moreover, it is easy to see that $|\cos(\theta^1(\rho))| \lesssim 2^{-k/2}$, therefore $|\theta^1(\rho) - \pi/2| \lesssim 2^{-k/2}$. The remaining conclusions in (7.16)–(7.17) follow easily.

(iii) We will only prove the estimates for the sets $\mathcal{S}_{p,q,r}^{1,-}(\xi)$, since the others are similar. With $z = (\rho \cos \theta, \rho \sin \theta)$ and $\xi = (s, 0)$, we define $F(\rho, \theta) := \Phi(\xi, \xi - z)$ and $G(\rho, \theta) := \widehat{\Upsilon}(\xi, \xi - z)$. The condition $|\widehat{\Upsilon}(\xi, \xi - z)| \lesssim 2^{-\mathcal{D}_1}$ shows that $|\Upsilon(\xi, \xi - z)| \lesssim 2^{k-\mathcal{D}_1}$, thus $|\rho - \gamma_0| \leq 2^{-\mathcal{D}_1/2}$ (see (7.15)). Moreover, $|\theta - \pi/2| \lesssim 2^{-\mathcal{D}_1/2}$ in view of (7.16)–(7.17). Using (7.23),

$$|\partial_\theta F(\rho, \theta)| \approx 2^{k/2}, \quad |\partial_\rho F(\rho, \theta)| \lesssim 2^{k/2-\mathcal{D}_1/2}$$

in the set $\{(\rho, \theta) : |\rho - \gamma_0| \leq 2^{-\mathcal{D}_1/2}, |\theta - \pi/2| \lesssim 2^{-\mathcal{D}_1/2}\}$. In addition, using (7.15) we have

$$\begin{aligned} -\mu \partial_\rho G(\rho, \theta) &= \lambda'''(\rho) \frac{A(\xi, \xi - z)}{|\Lambda'(\xi) - \Lambda'_\mu(z)| |\Lambda'(\xi - z) - \Lambda'_\mu(z)|} + O(2^{-\mathcal{D}_1} + |\lambda''(\rho)|), \\ |\partial_\theta G(\rho, \theta)| &= O(2^{-\mathcal{D}_1} + |\lambda''(\rho)|). \end{aligned}$$

Therefore, the mapping $(\rho, \theta) \mapsto [2^{-k/2} F(\rho, \theta), G(\rho, \theta)]$ is a regular change of variables for ρ, θ satisfying $|\rho - \gamma_0| \leq 2^{-\mathcal{D}_1/2}$, $|\theta - \pi/2| \lesssim 2^{-\mathcal{D}_1/2}$. The desired conclusions follow. \square

It follows from (7.11) and (7.22) that if $|\xi| = s$, $|\eta| = r$, $|\xi - \eta| = \rho$ then

$$\begin{aligned} -4\Upsilon(\xi, \eta) \frac{\rho^3}{\lambda'_\mu(\rho)} \frac{s}{\lambda'_\sigma(s)} \frac{r}{\lambda'_\nu(r)} &= \frac{\rho \lambda''_\mu(\rho)}{\lambda'_\mu(\rho)} [4r^2 s^2 - (r^2 + s^2 - \rho^2)^2] \\ &+ \left[2\rho s \frac{\lambda'_\mu(\rho)}{\lambda'_\sigma(s)} - (\rho^2 + s^2 - r^2) \right] \left[2\rho r \frac{\lambda'_\mu(\rho)}{\lambda'_\nu(r)} + (\rho^2 + r^2 - s^2) \right]. \end{aligned} \quad (7.24)$$

We assume now that $|\xi - \eta|$ is close to γ_0 and consider the case of bounded frequencies.

Lemma 7.2. *If $|\xi| = s$, $|\eta| = r$, $|\xi - \eta| = \rho$, $|\rho - \gamma_0| \leq 2^{-8\mathcal{D}_1}$, and $2^{-20} \leq r, s \leq 2^{2\mathcal{D}_1}$ then*

$$|\Phi(\xi, \eta)| + |\Upsilon(\xi, \eta)| \gtrsim 1. \quad (7.25)$$

Proof. Case 1: $(\sigma, \mu, \nu) = (+, +, +)$. Notice first that the function $f(r) := \lambda(r) + \lambda(\gamma_0) - \lambda(r + \gamma_0)$ is concave down for $r \in [0, \gamma_0]$ (in view of (7.3)) and satisfies $f(0) = 0$, $f(\gamma_0) \geq 0.1$. Therefore $f(r) \gtrsim 1$ if $r \in [2^{-20}, \gamma_0]$, so

$$|\Phi(\xi, \eta)| \gtrsim 1 \quad \text{if} \quad r \leq \gamma_0 \quad \text{or} \quad s \leq 2\gamma_0. \quad (7.26)$$

Assume, for contradiction, that (7.25) fails. In view of (7.24), $|\Phi(\xi, \eta)| \ll 1$ and

$$\left| \left[2\rho r \frac{\lambda'(\rho)}{\lambda'(r)} + (\rho^2 + r^2 - s^2) \right] \left[2\rho s \frac{\lambda'(\rho)}{\lambda'(s)} - (\rho^2 + s^2 - r^2) \right] \right| \ll 1 + s + r. \quad (7.27)$$

It is easy to see that if $|\Phi(\xi, \eta)| = |\lambda(s) - \lambda(\rho) - \lambda(r)| \ll 1$, $r \geq 100$, and $|\rho - \gamma_0| \leq 2^{-8\mathcal{D}_1}$ then

$$r \leq s - \frac{\lambda(\rho) - 0.1}{\lambda'(s)} \quad \text{and} \quad s \geq r + \frac{\lambda(\rho) - 0.1}{\lambda'(r)}.$$

Therefore, using (7.2)–(7.4), if $r \geq 100$ then

$$\begin{aligned} -2\rho s \frac{\lambda'(\rho)}{\lambda'(s)} + \rho^2 + s^2 - r^2 &\geq \frac{2s}{\lambda'(s)} (\lambda(\rho) - 0.1 - \rho\lambda'(\rho)) \gtrsim \sqrt{s} \\ -2\rho r \frac{\lambda'(\rho)}{\lambda'(r)} - \rho^2 - r^2 + s^2 &\geq \frac{2r}{\lambda'(r)} (\lambda(\rho) - 0.1 - \rho\lambda'(\rho)) - \rho^2 \gtrsim \sqrt{r}. \end{aligned}$$

In particular, (7.27) cannot hold if $r \geq 100$.

For $y \in [0, \infty)$, the equation $\lambda(x) = y$ admits a unique solution $x \in [0, \infty)$,

$$x = -\frac{1}{Y(y)} + \frac{Y(y)}{3}, \quad Y(y) := \left(\frac{27y^2 + \sqrt{27}\sqrt{27y^4 + 4}}{2} \right)^{1/3}. \quad (7.28)$$

Assuming $|\rho - \gamma_0| \leq 2^{-8\mathcal{D}_1}$, $2\gamma_0 \leq s \leq 110$, and $|\lambda(s) - \lambda(r) - \lambda(\rho)| \ll 1$, we show now that

$$-2\rho r \frac{\lambda'(\rho)}{\lambda'(r)} - \rho^2 - r^2 + s^2 \gtrsim 1, \quad -2\rho s \frac{\lambda'(\rho)}{\lambda'(s)} + \rho^2 + s^2 - r^2 \gtrsim 1. \quad (7.29)$$

Indeed, solving the equation $\lambda(r(s)) = \lambda(s) - \lambda(\gamma_0)$ according to (7.28), we define the functions

$$F_1(s) := s^2 - \gamma_0^2 - r(s)^2 - 2\gamma_0 r(s) \frac{\lambda'(\gamma_0)}{\lambda'(r(s))}, \quad F_2(s) := -2\gamma_0 s \frac{\lambda'(\gamma_0)}{\lambda'(s)} + \gamma_0^2 + s^2 - r(s)^2.$$

A simple *Mathematica* program shows that $F_1(s) \gtrsim 1$ and $F_2(s) \gtrsim 1$ if $2\gamma_0 \leq s \lesssim 1$. The bound (7.29) follows. This completes the proof of (7.25) when $(\sigma, \mu, \nu) = (+, +, +)$.

Case 2: the other triplets. Notice that if $(\sigma, \mu, \nu) = (+, -, +)$ then

$$\Phi_{+-+}(\xi, \eta) = -\Phi_{+++}(\eta, \xi), \quad \Upsilon_{+-+}(\xi, \eta) = -\Upsilon_{+++}(\eta, \xi). \quad (7.30)$$

The desired bound in this case follows from the case $(\sigma, \mu, \nu) = (+, +, +)$ analyzed earlier.

On the other hand, if $(\sigma, \mu, \nu) = (+, -, -)$ then $\Phi(\xi, \eta) = \lambda(s) + \lambda(r) + \lambda(\rho) \gtrsim 1$, so (7.25) is clearly verified. Finally, if $(\sigma, \mu, \nu) = (+, +, -)$ then $\Phi(\xi, \eta) = \lambda(s) + \lambda(r) - \lambda(\rho)$ and we estimate, assuming $10^{-4} \leq r \leq \rho/2$,

$$\lambda(s) + \lambda(r) - \lambda(\rho) \geq \lambda(r) + \lambda(\rho - r) - \lambda(\rho) = \int_0^r [\lambda'(x) - \lambda'(x + \rho - r)] dx \gtrsim 1.$$

A similar estimate holds if $10^{-4} \leq s \leq \rho/2$ or if $s, r \geq \rho/2$. Therefore $\Phi(\xi, \eta) \gtrsim 1$ in this case.

The cases corresponding to $\sigma = -$ are similar by replacing Φ with $-\Phi$ and Υ with $-\Upsilon$. This completes the proof of the lemma. \square

Finally, we consider the case when $|\xi - \eta|$ is close to γ_1 .

Lemma 7.3. *If $|\xi| = s$, $|\eta| = r$, $|\xi - \eta| = \rho$, $|\rho - \gamma_1| \leq 2^{-\mathcal{D}_1}$, and $2^{-20} \leq r, s$ then*

$$\begin{aligned} |\Phi(\xi, \eta)| + \frac{|\Upsilon(\xi, \eta)|}{|\xi| + |\eta|} + \frac{|(\nabla_\eta \Upsilon)(\xi, \eta) \cdot (\nabla_\eta^\perp \Phi)(\xi, \eta)|}{(|\xi| + |\eta|)^6} &\gtrsim 1, \\ |\Phi(\xi, \eta)| + \frac{|\Upsilon(\xi, \eta)|}{|\xi| + |\eta|} + \frac{|(\nabla_\xi \Upsilon)(\xi, \eta) \cdot (\nabla_\xi^\perp \Phi)(\xi, \eta)|}{(|\xi| + |\eta|)^6} &\gtrsim 1, \end{aligned} \quad (7.31)$$

and

$$\begin{aligned} |\Phi(\xi, \eta)| + \frac{|\Upsilon(\xi, \eta)|}{|\xi| + |\eta|} + \frac{|(\xi - \eta) \cdot (\nabla_\eta^\perp \Phi)(\xi, \eta)|}{(|\xi| + |\eta|)^6} &\gtrsim 1, \\ |\Phi(\xi, \eta)| + \frac{|\Upsilon(\xi, \eta)|}{|\xi| + |\eta|} + \frac{|(\xi - \eta) \cdot (\nabla_\xi^\perp \Phi)(\xi, \eta)|}{(|\xi| + |\eta|)^6} &\gtrsim 1. \end{aligned} \quad (7.32)$$

Proof. Case 1: $(\sigma, \mu, \nu) = (+, +, +)$. Notice first that the function $f(r) := \lambda(r) + \lambda(\gamma_1) - \lambda(r + \gamma_1)$ is concave down for $r \in [0, 0.3]$ (in view of (7.3)) and satisfies $f(0) = 0$, $f(0.3) \geq 0.02$. Therefore $f(r) \gtrsim 1$ if $r \in [2^{-20}, 0.3]$, so

$$|\Phi(\xi, \eta)| \gtrsim 1 \quad \text{if} \quad r \leq 0.3 \quad \text{or} \quad s \leq \gamma_1 + 0.3. \quad (7.33)$$

On the other hand, if $|\Phi(\xi, \eta)| \ll 1$, $r \geq 100$, and $|\rho - \gamma_1| \leq 2^{-\mathcal{D}_1}$ then

$$s \leq r + \frac{\lambda(\rho) + 0.4}{\lambda'(r)} \quad \text{and} \quad r \geq s - \frac{\lambda(\rho) + 0.4}{\lambda'(s)}.$$

Therefore, using also (7.5), if $r \geq 100$ then

$$\begin{aligned} 2\rho r \frac{\lambda'(\rho)}{\lambda'(r)} + \rho^2 + r^2 - s^2 &\geq \frac{2r}{\lambda'(r)} (\rho\lambda'(\rho) - \lambda(\rho) - 0.4) \gtrsim \sqrt{r}, \\ 2\rho s \frac{\lambda'(\rho)}{\lambda'(s)} - \rho^2 - s^2 + r^2 &\geq \frac{2s}{\lambda'(s)} (\rho\lambda'(\rho) - \lambda(\rho) - 0.4) - \rho^2 \gtrsim \sqrt{s}, \\ \frac{\rho\lambda''(\rho)}{\lambda'(\rho)} [4r^2s^2 - (r^2 + s^2 - \rho^2)^2] &\gtrsim r^2. \end{aligned}$$

Using the formula (7.24) and assuming $|\rho - \gamma_1| \leq 2^{-\mathcal{D}_1}$, it follows that

$$\text{if } |\Phi(\xi, \eta)| \ll 1 \text{ and } r \geq 100 \text{ then } -\Upsilon(\xi, \eta) \gtrsim r. \quad (7.34)$$

Therefore both (7.31) and (7.32) follow if $r \geq 100$.

It remains to consider the case $\gamma_1 + 0.3 \leq s \leq 110$. We show first that

$$\text{if } 3 \leq s \leq 110 \text{ and } |\lambda(s) - \lambda(r) - \lambda(\rho)| \ll 1 \text{ then } -\Upsilon(\xi, \eta) \gtrsim 1. \quad (7.35)$$

Indeed, we solve the equation $\lambda(r(s)) = \lambda(s) - \lambda(\gamma_1)$ according to (7.28), and define the function $G_1(s) := G(s, r(s), \gamma_1)$, where

$$\begin{aligned} G(s, r, \rho) &:= \frac{\rho\lambda''(\rho)}{\lambda'(\rho)} [4r^2s^2 - (r^2 + s^2 - \rho^2)^2] \\ &\quad + \left[2\rho s \frac{\lambda'(\rho)}{\lambda'(s)} - \rho^2 - s^2 + r^2 \right] \left[2\rho r \frac{\lambda'(\rho)}{\lambda'(r)} + \rho^2 + r^2 - s^2 \right], \end{aligned} \quad (7.36)$$

compare with (7.24). A simple *Mathematica* program shows that $G_1(s) \gtrsim 1$ if $3 \leq s \leq 110$. The bound (7.35) follows, so both (7.31) and (7.32) follow if $3 \leq s \leq 110$.

On the other hand the function $G_1(s)$ does vanish for some $s \in [\gamma_1 + 0.3, 3]$ (more precisely at $s \approx 1.94$). In this range we can only prove the weaker estimates in the lemma. Notice that

$$\Upsilon(\xi, \eta) = \tilde{\Upsilon}(|\xi|, |\eta|, |\xi - \eta|), \quad \tilde{\Upsilon}(s, r, \rho) := -\frac{1}{4}G(s, r, \rho) \frac{\lambda'(\rho)}{\rho^3} \frac{\lambda'(s)}{s} \frac{\lambda'(r)}{r}.$$

Then, using also (7.8), we have

$$\begin{aligned} (\nabla_\eta \Upsilon)(\xi, \eta) \cdot (\nabla_\eta^\perp \Phi)(\xi, \eta) &= (r\rho)^{-1}(\eta \cdot \xi^\perp) [(\partial_r \tilde{\Upsilon})(s, r, \rho)\lambda'(\rho) - (\partial_\rho \tilde{\Upsilon})(s, r, \rho)\lambda'(r)], \\ (\nabla_\xi \Upsilon)(\xi, \eta) \cdot (\nabla_\xi^\perp \Phi)(\xi, \eta) &= (s\rho)^{-1}(\xi \cdot \eta^\perp) [(\partial_s \tilde{\Upsilon})(s, r, \rho)\lambda'(\rho) + (\partial_\rho \tilde{\Upsilon})(s, r, \rho)\lambda'(s)]. \end{aligned} \quad (7.37)$$

It is easy to see, using the formulas (7.22) and (7.24), that

$$|\Phi(\xi, \eta)| + |\Upsilon(\xi, \eta)| + |\xi \cdot \eta^\perp| \gtrsim 1 \quad (7.38)$$

if $s \in [\gamma_1 + 0.3, 3]$. Moreover, let

$$\begin{aligned} G_{11}(s) &:= (\partial_r \tilde{\Upsilon})(s, r(s), \gamma_1)\lambda'(\gamma_1) - (\partial_\rho \tilde{\Upsilon})(s, r(s), \gamma_1)\lambda'(r(s)), \\ G_{12}(s) &:= (\partial_s \tilde{\Upsilon})(s, r(s), \gamma_1)\lambda'(\gamma_1) + (\partial_\rho \tilde{\Upsilon})(s, r(s), \gamma_1)\lambda'(s), \end{aligned}$$

where, as before, $r(s)$ is the unique solution of the equation $\lambda(r(s)) = \lambda(s) - \lambda(\gamma_1)$, according to (7.28). A simple *Mathematica* program shows that $G_1(s) + G_{11}(s) \gtrsim 1$ and $G_1(s) + G_{12}(s) \gtrsim 1$ if $s \in [\gamma_1 + 0.3, 3]$. Using also (7.37) and (7.38) it follows that

$$\begin{aligned} |\Upsilon(\xi, \eta)| + |(\nabla_\eta \Upsilon)(\xi, \eta) \cdot (\nabla_\eta^\perp \Phi)(\xi, \eta)| &\gtrsim 1, \\ |\Upsilon(\xi, \eta)| + |(\nabla_\xi \Upsilon)(\xi, \eta) \cdot (\nabla_\xi^\perp \Phi)(\xi, \eta)| &\gtrsim 1, \end{aligned} \quad (7.39)$$

if $s \in [\gamma_1 + 0.3, 3]$, $|\Phi(\xi, \eta)| \ll 1$, and $|\rho - \gamma_0| \leq 2^{-\mathcal{D}_1}$. The bounds (7.31) follow from (7.33)–(7.35) and (7.39). The bounds (7.32) follow from (7.33)–(7.35), and (7.38).

Case 2: the other triplets. The desired bounds in the case $(\sigma, \mu, \nu) = (+, -, +)$ follow from the corresponding bounds the case $(\sigma, \mu, \nu) = (+, +, +)$ and (7.30). Moreover, if $(\sigma, \mu, \nu) = (+, -, -)$ then $\Phi(\xi, \eta) = \lambda(s) + \lambda(r) + \lambda(\rho) \gtrsim 1$, so (7.31)–(7.32) are clearly verified.

Finally, if $(\sigma, \mu, \nu) = (+, +, -)$ then $\Phi(\xi, \eta) = \lambda(s) + \lambda(r) - \lambda(\rho)$. We may assume that $s, r \in [2^{-20}, \gamma_1]$. In this case we prove the stronger bound

$$|\Phi(\xi, \eta)| + |\Upsilon(\xi, \eta)| \gtrsim 1. \quad (7.40)$$

Indeed, for this suffices to notice that the function $x \rightarrow \lambda(x) + \lambda(\gamma_1 - x) - \lambda(\gamma_1)$ is nonnegative for $x \in [0, \gamma_1]$ and vanishes only when $x \in \{0, \gamma_1/2, \gamma_1\}$. Moreover $\Upsilon((\gamma_1/2)e, (\gamma_1/2)e) \neq 0$ if $|e| = 1$ (using (7.10)), and the lower bound (7.40) follows.

The cases corresponding to $\sigma = -$ are similar by replacing Φ with $-\Phi$ and Υ with $-\Upsilon$. This completes the proof of the lemma. \square

8. PARADIFFERENTIAL CALCULUS

The paradifferential calculus allows us to understand the high frequency structure of our system. In this section we record the definitions, and state and prove several useful lemmas.

8.1. Operators bounds. In this subsection we define our main objects, and prove several basic nonlinear bounds.

8.1.1. *Fourier multipliers.* We will mostly work with bilinear and trilinear multipliers. Many of the simpler estimates follow from the following basic result (see [46, Lemma 5.2] for the proof).

Lemma 8.1. (i) *Assume $l \geq 2$, $f_1, \dots, f_l, f_{l+1} \in L^2(\mathbb{R}^2)$, and $m : (\mathbb{R}^2)^l \rightarrow \mathbb{C}$ is a continuous compactly supported function. Then*

$$\left| \int_{(\mathbb{R}^2)^l} m(\xi_1, \dots, \xi_l) \widehat{f}_1(\xi_1) \cdots \widehat{f}_l(\xi_l) \cdot \widehat{f_{l+1}}(-\xi_1 - \dots - \xi_l) d\xi_1 \dots d\xi_l \right| \lesssim \|\mathcal{F}^{-1}(m)\|_{L^1} \|f_1\|_{L^{p_1}} \cdots \|f_{l+1}\|_{L^{p_{l+1}}}, \quad (8.1)$$

for any exponents $p_1, \dots, p_{l+1} \in [1, \infty]$ satisfying $\frac{1}{p_1} + \dots + \frac{1}{p_{l+1}} = 1$.

(ii) *Assume $l \geq 2$ and L_m is the multilinear operator defined by*

$$\mathcal{F}\{L_m[f_1, \dots, f_l]\}(\xi) = \int_{(\mathbb{R}^2)^{l-1}} m(\xi, \eta_2, \dots, \eta_l) \widehat{f}_1(\xi - \eta_2) \cdots \widehat{f_{l-1}}(\eta_{l-1} - \eta_l) \widehat{f}_l(\eta_l) d\eta_2 \dots d\eta_l.$$

Then, for any exponents $p, q_1, \dots, q_l \in [1, \infty]$ satisfying $\frac{1}{q_1} + \dots + \frac{1}{q_l} = \frac{1}{p}$, we have

$$\|L_m[f_1, \dots, f_l]\|_{L^p} \lesssim \|\mathcal{F}^{-1}(m)\|_{S^\infty} \|f_1\|_{L^{q_1}} \cdots \|f_l\|_{L^{q_l}}. \quad (8.2)$$

Given a multiplier $m : (\mathbb{R}^2)^2 \rightarrow \mathbb{C}$, we define the bilinear operator M by the formula

$$\mathcal{F}[M[f, g]](\xi) = \frac{1}{4\pi^2} \int_{\mathbb{R}^2} m(\xi, \eta) \widehat{f}(\xi - \eta) \widehat{g}(\eta) d\eta. \quad (8.3)$$

With $\Omega = x_1 \partial_2 - x_2 \partial_1$, we notice the formula

$$\Omega M[f, g] = M[\Omega f, g] + M[f, \Omega g] + \widetilde{M}[f, g], \quad (8.4)$$

where \widetilde{M} is the bilinear operator defined by the multiplier $\widetilde{m}(\xi, \eta) = (\Omega_\xi + \Omega_\eta)m(\xi, \eta)$.

For simplicity of notation, we define the following classes of bilinear multipliers:

$$\begin{aligned} S^\infty &:= \{m : (\mathbb{R}^2)^n \rightarrow \mathbb{C} : m \text{ continuous and } \|m\|_{S^\infty} := \|\mathcal{F}^{-1}m\|_{L^1} < \infty\}, \\ S_\Omega^\infty &:= \{m : (\mathbb{R}^2)^2 \rightarrow \mathbb{C} : m \text{ continuous and } \|m\|_{S_\Omega^\infty} := \sup_{l \leq N_1} \|(\Omega_\xi + \Omega_\eta)^l m\|_{S^\infty} < \infty\}. \end{aligned} \quad (8.5)$$

We will often need to analyze bilinear operators more carefully, by localizing in the frequency space. We therefore define, for any symbol m ,

$$m^{k, k_1, k_2}(\xi, \eta) := \varphi_k(\xi) \varphi_{k_1}(\xi - \eta) \varphi_{k_2}(\eta) m(\xi, \eta). \quad (8.6)$$

For any $t \in [0, T]$, $p \geq -N_3$, and $m \geq 1$ let $\langle t \rangle = 1 + t$ and let $\mathcal{O}_{m,p} = \mathcal{O}_{m,p}(t)$ denote the Banach spaces of functions $f \in L^2$ defined by the norms

$$\|f\|_{\mathcal{O}_{m,p}} := \langle t \rangle^{(m-1)(5/6-20\delta^2)-\delta^2} [\|f\|_{H^{N_0+p}} + \|f\|_{H_\Omega^{N_1, N_3+p}} + \langle t \rangle^{5/6-2\delta^2} \|f\|_{\widetilde{W}_\Omega^{N_1/2, N_2+p}}]. \quad (8.7)$$

This is similar to the definition of the spaces $\mathcal{O}_{m,p}$ in Definition 2.4, except for the supremum over $t \in [0, T]$. We show first that these spaces are compatible with S_Ω^∞ multipliers.

Lemma 8.2. *Assume M is a bilinear operator with symbol m satisfying $\|m^{k, k_1, k_2}\|_{S_\Omega^\infty} \leq 1$, for any $k, k_1, k_2 \in \mathbb{Z}$. Then, if $p \in [-N_3, 10]$, $t \in [0, T]$, and $m, n \geq 1$,*

$$\langle t \rangle^{12\delta^2} \|M[f, g]\|_{\mathcal{O}_{m+n,p}} \lesssim \|f\|_{\mathcal{O}_{m,p}} \|g\|_{\mathcal{O}_{n,p}}. \quad (8.8)$$

Proof. In view of the definition we may assume that $m = n = 1$ and $\|f\|_{\mathcal{O}_{m,p}} = \|g\|_{\mathcal{O}_{n,p}} = 1$. Therefore, we may assume that

$$\|h\|_{H^{N_0+p}} + \sup_{j \leq N_1} \|\Omega^j h\|_{H^{N_3+p}} \leq \langle t \rangle^{\delta^2}, \quad \sup_{j \leq N_1/2} \|\Omega^j h\|_{\widetilde{W}^{N_2+p}} \leq \langle t \rangle^{3\delta^2-5/6}, \quad (8.9)$$

where $h \in \{f(t), g(t)\}$. With $F := M[f(t), g(t)]$, it suffices to prove that

$$\begin{aligned} \|F\|_{H^{N_0+p}} + \sup_{j \leq N_1} \|\Omega^j F\|_{H^{N_3+p}} &\lesssim \langle t \rangle^{6\delta^2-5/6}, \\ \sup_{j \leq N_1/2} \|\Omega^j P_k F\|_{\widetilde{W}^{N_2+p}} &\lesssim \langle t \rangle^{8\delta^2-5/3}. \end{aligned} \quad (8.10)$$

For $k, k_1, k_2 \in \mathbb{Z}$ let

$$F_k := P_k M[f(t), g(t)], \quad F_{k,k_1,k_2} := P_k M[P_{k_1} f(t), P_{k_2} g(t)].$$

For $k \in \mathbb{Z}$ let

$$\begin{aligned} \mathcal{X}_k^1 &:= \{(k_1, k_2) \in \mathbb{Z} \times \mathbb{Z} : k_1 \leq k - 8, |k_2 - k| \leq 4\}, \\ \mathcal{X}_k^2 &:= \{(k_1, k_2) \in \mathbb{Z} \times \mathbb{Z} : k_2 \leq k - 8, |k_1 - k| \leq 4\}, \\ \mathcal{X}_k^3 &:= \{(k_1, k_2) \in \mathbb{Z} \times \mathbb{Z} : \min(k_1, k_2) \geq k - 7, |k_1 - k_2| \leq 20\}, \end{aligned}$$

and let $\mathcal{X}_k := \mathcal{X}_k^1 \cup \mathcal{X}_k^2 \cup \mathcal{X}_k^3$. Let

$$\begin{aligned} a_k &:= \|P_k h\|_{H^{N_0+p}}, \quad b_k := \sup_{0 \leq j \leq N_1} \|\Omega^j P_k h\|_{H^{N_3+p}}, \quad c_k := \sup_{0 \leq j \leq N_1/2} \|\Omega^j P_k h\|_{\widetilde{W}^{N_2+p}}, \\ \tilde{a}_k &:= \sum_{m \in \mathbb{Z}} a_{k+m} 2^{-|m|/100}, \quad \tilde{b}_k := \sum_{m \in \mathbb{Z}} b_{k+m} 2^{-|m|/100}, \quad \tilde{c}_k := \sum_{m \in \mathbb{Z}} c_{k+m} 2^{-|m|/100}. \end{aligned} \quad (8.11)$$

We can prove now (8.10). Assuming $k \in \mathbb{Z}$ fixed we estimate, using Lemma 8.1 (ii),

$$\begin{aligned} \|F_{k,k_1,k_2}\|_{H^{N_0+p}} &\lesssim a_{k_1} (2^{-4 \max(k_2, 0)} c_{k_2}) && \text{if } (k_1, k_2) \in \mathcal{X}_k^2, \\ \|F_{k,k_1,k_2}\|_{H^{N_0+p}} &\lesssim a_{k_2} (2^{-4 \max(k_1, 0)} c_{k_1}) && \text{if } (k_1, k_2) \in \mathcal{X}_k^1 \cup \mathcal{X}_k^3. \end{aligned} \quad (8.12)$$

Since $\sum_l c_l \leq \langle t \rangle^{3\delta^2-5/6}$, it follows that

$$\sum_{(k_1, k_2) \in \mathcal{X}_k} \|F_{k,k_1,k_2}\|_{H^{N_0+p}} \lesssim \langle t \rangle^{3\delta^2-5/6} [\tilde{a}_k + \sum_{l \geq k} \tilde{a}_l 2^{-4l}]. \quad (8.13)$$

Therefore, since $\sum_{k \in \mathbb{Z}} \tilde{a}_k^2 \lesssim \langle t \rangle^{2\delta^2}$, it follows that

$$\left[\sum_{2^k \geq (1+t)^{-10}} \|F_k\|_{H^{N_0+p}}^2 \right]^{1/2} \lesssim \langle t \rangle^{6\delta^2-5/6}. \quad (8.14)$$

To bound the contribution of small frequencies, $2^k \leq \langle t \rangle^{-10}$, we also use the bound

$$\|F_{k,k_1,k_2}\|_{L^2} \lesssim 2^k \|F_{k,k_1,k_2}\|_{L^1} \lesssim 2^k a_{k_1} a_{k_2}. \quad (8.15)$$

when $(k_1, k_2) \in \mathcal{X}_k^3$, in addition to the bounds (8.12). Therefore

$$\sum_{(k_1, k_2) \in \mathcal{X}_k} \|F_{k,k_1,k_2}\|_{H^{N_0+p}} \lesssim \langle t \rangle^{3\delta^2-5/6} \tilde{a}_k + 2^k \sum_{l \in \mathbb{Z}} a_l^2, \quad (8.16)$$

if $2^k \leq \langle t \rangle^{-10}$. It follows that

$$\left[\sum_{2^k \leq \langle t \rangle^{-10}} \|F_k\|_{H^{N_0+p}}^2 \right]^{1/2} \lesssim \langle t \rangle^{6\delta^2-5/6}, \quad (8.17)$$

and the desired bound $\|F\|_{H^{N_0+p}} \lesssim (1+t)^{6\delta^2-5/6}$ in (8.10) follows.

The proof of the second bound in (8.10) is similar. We start by estimating, as in (8.12),

$$\|\Omega^j F_{k,k_1,k_2}\|_{H^{N_3+p}} \lesssim 2^{(N_3+p)k^+} [b_{k_1} 2^{-(N_3+p)k_1^+} c_{k_2} 2^{-(N_2+p)k_2^+} + b_{k_2} 2^{-(N_3+p)k_2^+} c_{k_1} 2^{-(N_2+p)k_1^+}]$$

for any $j \in [0, N_1]$. We remark that this is weaker than (8.12) since the Ω derivatives can distribute on either $P_{k_1}f(t)$ or $P_{k_2}(t)$, and we are forced to estimate the factor with more than $N_1/2$ Ω derivatives in L^2 . To bound the contributions of small frequencies we also estimate

$$\|\Omega^j F_{k,k_1,k_2}\|_{H^{N_3+p}} \lesssim 2^{\min(k,k_1,k_2)} b_{k_1} b_{k_2},$$

as in (8.15). Recall that $N_2 - N_3 \geq 5$. We combine these two bounds to estimate

$$\sum_{(k_1,k_2) \in \mathcal{X}_k} \|\Omega^j F_{k,k_1,k_2}\|_{H^{N_3+p}} \lesssim \langle t \rangle^{3\delta^2-5/6} [\tilde{b}_k + \sum_{l \geq k} \tilde{b}_l 2^{-4l^+}] + \langle t \rangle^{2\delta^2} 2^{-(N_2-N_3)k^+} \tilde{c}_k.$$

When $2^k \leq (1+t)^{-10}$ this does not suffice; we have instead the bound

$$\sum_{(k_1,k_2) \in \mathcal{X}_k} \|\Omega^j F_{k,k_1,k_2}\|_{H^{N_3+p}} \lesssim \langle t \rangle^{3\delta^2-5/6} \tilde{b}_k + 2^k \sum_{l \in \mathbb{Z}} b_l^2 + \langle t \rangle^{2\delta^2} 2^{-(N_2-N_3)k^+} \tilde{c}_k.$$

The desired estimate $\|\Omega^j F\|_{H^{N_3+p}} \lesssim \langle t \rangle^{6\delta^2-5/6}$ in (8.10) follows.

For the last bound in (8.10), we estimate as before for any $j \in [0, N_1/2]$,

$$\|\Omega^j F_{k,k_1,k_2}\|_{\widetilde{W}^{N_2+p}} \lesssim 2^{(N_2+p)k^+} c_{k_1} 2^{-(N_2+p)k_1^+} c_{k_2} 2^{-(N_2+p)k_2^+}, \quad \|\Omega^j F_{k,k_1,k_2}\|_{\widetilde{W}^{N_2+p}} \lesssim 2^{2k} b_{k_1} b_{k_2},$$

where the last estimate holds only for $k \leq 0$. The desired bound follows as before. \square

8.1.2. Paradifferential operators. We recall first the definition of paradifferential operators (see (2.32): given a symbol $a = a(x, \zeta) : \mathbb{R}^2 \times \mathbb{R}^2 \rightarrow \mathbb{C}$, we define the operator T_a by

$$\mathcal{F} \{T_a f\}(\xi) = \frac{1}{4\pi^2} \int_{\mathbb{R}^2} \chi \left(\frac{|\xi - \eta|}{|\xi + \eta|} \right) \tilde{a}(\xi - \eta, (\xi + \eta)/2) \hat{f}(\eta) d\eta, \quad (8.18)$$

where \tilde{a} denotes the partial Fourier transform of a in the first coordinate and $\chi = \varphi_{-20}$. We define the Poisson bracket between two symbols a and b by

$$\{a, b\} := \nabla_x a \nabla_\zeta b - \nabla_\zeta a \nabla_x b. \quad (8.19)$$

We will use several norms to estimate symbols of degree 0. For $q \in \{2, \infty\}$, $r \in \mathbb{Z}_+$, let

$$\|a\|_{\mathcal{M}_{r,q}} := \sup_{\zeta} \| |a|_r(\cdot, \zeta) \|_{L_x^q}, \quad \text{where} \quad |a|_r(x, \zeta) := \sum_{|\alpha|+|\beta| \leq r} |\zeta^{|\beta|} |\partial_\zeta^\beta \partial_x^\alpha a(x, \zeta)|. \quad (8.20)$$

At later stages we will use more complicated norms, which also keep track of multiplicity and degree. For now we record a few simple properties, which follow directly from definitions:

$$\begin{aligned} \|ab\|_{\mathcal{M}_{r,q}} + \|\zeta\{a, b\}\|_{\mathcal{M}_{r-2,q}} &\lesssim \|a\|_{\mathcal{M}_{r,q_1}} \|b\|_{\mathcal{M}_{r,q_2}}, & \{\infty, q\} &= \{q_1, q_2\}, \\ \|P_k a\|_{\mathcal{M}_{r,q}} &\lesssim 2^{-sk} \|P_k a\|_{\mathcal{M}_{r+s,q}}, & q &\in \{2, \infty\}, \quad k \in \mathbb{Z}, \quad s \in \mathbb{Z}_+. \end{aligned} \quad (8.21)$$

We start with some simple properties.

Lemma 8.3. (i) Let a be a symbol and $1 \leq q \leq \infty$, then

$$\|P_k T_a f\|_{L^q} \lesssim \|a\|_{\mathcal{M}_{8,\infty}} \|P_{[k-2,k+2]} f\|_{L^q} \quad (8.22)$$

and

$$\|P_k T_a f\|_{L^2} \lesssim \|a\|_{\mathcal{M}_{8,2}} \|P_{[k-2,k+2]} f\|_{L^\infty}. \quad (8.23)$$

(ii) If $a \in \mathcal{M}_{8,\infty}$ is real-valued then T_a is a bounded self-adjoint operator on L^2 .

(iii) We have

$$\overline{T_a f} = T_{a'} \overline{f}, \quad \text{where} \quad a'(y, \zeta) := \overline{a(y, -\zeta)} \quad (8.24)$$

and

$$\Omega(T_a f) = T_a(\Omega f) + T_{a''} f \quad \text{where} \quad a''(y, \zeta) = (\Omega_y a)(y, \zeta) + (\Omega_\zeta a)(y, \zeta). \quad (8.25)$$

Proof. (i) Inspecting the Fourier transform, we directly see that $P_k T_a f = P_k T_a P_{[k-2,k+2]} f$. By rescaling, we may assume that $k = 0$ and write

$$\begin{aligned} \langle P_0 T_a h, g \rangle &= C \int_{\mathbb{R}^4} \overline{g}(x) h(y) I(x, y) dx dy, \\ I(x, y) &= \int_{\mathbb{R}^6} a(z, (\xi + \eta)/2) e^{i\xi \cdot (x-z)} e^{i\eta \cdot (z-y)} \chi\left(\frac{|\xi - \eta|}{|\xi + \eta|}\right) \varphi_0(\xi) d\eta d\xi dz \\ &= \int_{\mathbb{R}^6} a(z, \xi + \theta/2) e^{i\theta \cdot (z-y)} e^{i\xi \cdot (x-y)} \chi\left(\frac{|\theta|}{|2\xi + \theta|}\right) \varphi_0(\xi) d\xi d\theta dz. \end{aligned}$$

We observe that

$$\begin{aligned} (1 + |x - y|^2)^2 I(x, y) &= \int_{\mathbb{R}^6} \frac{a(z, \xi + \theta/2)}{(1 + |z - y|^2)^2} \chi\left(\frac{|\theta|}{|2\xi + \theta|}\right) \varphi_0(\xi) \\ &\quad \times \left[(1 - \Delta_\theta)^2 (1 - \Delta_\xi)^2 \{ e^{i\theta \cdot (z-y)} e^{i\xi \cdot (x-y)} \} \right] d\xi d\theta dz. \end{aligned}$$

By integration by parts in ξ and θ it follows that

$$(1 + |x - y|^2)^2 |I(x, y)| \lesssim \int_{\mathbb{R}^6} \frac{|a|_8(z, \xi + \theta/2)}{(1 + |z - y|^2)^2} \varphi_{[-4,4]}(\xi) \varphi_{\leq -10}(\theta) d\xi d\theta dz, \quad (8.26)$$

where $|a|_8$ is defined as in (8.20).

The bounds (8.22) and (8.23) now follow easily. Indeed, it follows from (8.26) that

$$(1 + |x - y|^2)^2 |I(x, y)| \lesssim \|a\|_{\mathcal{M}_{8,\infty}}.$$

Therefore $|\langle P_0 T_a h, g \rangle| \lesssim \|a\|_{\mathcal{M}_{8,\infty}} \|h\|_{L^q} \|g\|_{L^{q'}}$. This gives (8.22), and (8.23) follows similarly.

Part (ii) and (8.24) follow directly from definitions. To prove (8.25) we start from the formula

$$\mathcal{F}\{\Omega T_a f\}(\xi) = \frac{1}{4\pi^2} \int_{\mathbb{R}^2} (\Omega_\xi + \Omega_\eta) \left[\chi\left(\frac{|\xi - \eta|}{|\xi + \eta|}\right) \tilde{a}(\xi - \eta, (\xi + \eta)/2) \widehat{f}(\eta) \right] d\eta,$$

and notice that $(\Omega_\xi + \Omega_\eta) \left[\chi\left(\frac{|\xi - \eta|}{|\xi + \eta|}\right) \right] \equiv 0$. The formula (8.25) follows. \square

The paradifferential calculus is useful to linearize products and compositions. More precisely:

Lemma 8.4. (i) If $f, g \in L^2$ then

$$fg = T_f g + T_g f + \mathcal{H}(f, g)$$

where \mathcal{H} is smoothing in the sense that

$$\|P_k \mathcal{H}(f, g)\|_{L^q} \lesssim \sum_{k', k'' \geq k-40, |k'-k''| \leq 40} \min(\|P_{k'} f\|_{L^q} \|P_{k''} g\|_{L^\infty}, \|P_{k'} f\|_{L^\infty} \|P_{k''} g\|_{L^q}).$$

As a consequence, if $f \in \mathcal{O}_{m,-5}$ and $g \in \mathcal{O}_{n,-5}$ then

$$\langle t \rangle^{12\delta^2} \|\mathcal{H}(f, g)\|_{\mathcal{O}_{m+n,5}} \lesssim \|f\|_{\mathcal{O}_{m,-5}} \|g\|_{\mathcal{O}_{n,-5}}. \quad (8.27)$$

(ii) Assume that $F(z) = z + h(z)$, where h is analytic for $|z| < 1/2$ and satisfies $|h(z)| \lesssim |z|^3$. If $\|u\|_{L^\infty} \leq 1/100$ and $N \geq 10$ then

$$F(u) = T_{F'(u)}u + E(u), \quad (8.28)$$

$$\langle t \rangle^{12\delta^2} \|E(u)\|_{\mathcal{O}_{3,5}} \lesssim \|u\|_{\mathcal{O}_{1,-5}}^3 \quad \text{if} \quad \|u\|_{\mathcal{O}_{1,-5}} \leq 1.$$

Proof. (i) This follows easily by defining $\mathcal{H}(f, g) = fg - T_f g - T_g f$ and observing that

$$P_k \mathcal{H}(P_{k'} f, P_{k''} g) \equiv 0 \quad \text{unless} \quad k', k'' \geq k - 40, |k' - k''| \leq 40.$$

The bound (8.27) follows as in the proof of Lemma 8.2 (the remaining bilinear interactions correspond essentially to the set \mathcal{X}_k^3)

(ii) Since F is analytic, it suffices to show this for $F(x) = x^n$, $n \geq 3$. This follows, however, as in part (i), using the Littlewood–Paley decomposition for u . \square

We show now that compositions of paradifferential operators can be approximated well by paradifferential operators with suitable symbols. More precisely:

Proposition 8.5. *Let $1 \leq q \leq \infty$. Given symbols a and b , we may decompose*

$$T_a T_b = T_{ab} + \frac{i}{2} T_{\{a,b\}} + E(a, b). \quad (8.29)$$

The error E obeys the following bounds: assuming $k \geq -100$,

$$\|P_k E(a, b) f\|_{L^q} \lesssim 2^{-2k} \|a\|_{\mathcal{M}_{16,\infty}} \|b\|_{\mathcal{M}_{16,\infty}} \|P_{[k-5,k+5]} f\|_{L^q}, \quad \text{for } q \in \{2, \infty\}, \quad (8.30)$$

$$\begin{aligned} \|P_k E(a, b) f\|_{L^2} &\lesssim 2^{-2k} \|a\|_{\mathcal{M}_{16,2}} \|b\|_{\mathcal{M}_{16,\infty}} \|P_{[k-5,k+5]} f\|_{L^\infty}, \\ \|P_k E(a, b) f\|_{L^2} &\lesssim 2^{-2k} \|a\|_{\mathcal{M}_{16,\infty}} \|b\|_{\mathcal{M}_{16,2}} \|P_{[k-5,k+5]} f\|_{L^\infty}. \end{aligned} \quad (8.31)$$

Moreover $E(a, b) = 0$ if both a and b are independent of x .

Proof. We may assume that $a = P_{\leq k-100} a$ and $b = P_{\leq k-100} b$, since the other contributions can also be estimated using Lemma 8.3 (i) and (8.21). In this case we write

$$\begin{aligned} (16\pi^4) \mathcal{F} \{P_k (T_a T_b - T_{ab}) f\}(\xi) &= \varphi_k(\xi) \int_{\mathbb{R}^4} \widehat{f}(\eta) \varphi_{\leq k-100}(\xi - \theta) \varphi_{\leq k-100}(\theta - \eta) \\ &\quad \times \left[\widetilde{a}(\xi - \theta, \frac{\xi + \theta}{2}) \widetilde{b}(\theta - \eta, \frac{\eta + \theta}{2}) - \widetilde{a}(\xi - \theta, \frac{\xi + \eta}{2}) \widetilde{b}(\theta - \eta, \frac{\xi + \eta}{2}) \right] d\eta d\theta. \end{aligned}$$

Moreover, using the definition,

$$\begin{aligned} (16\pi^4) \mathcal{F} \{P_k (i/2) T_{\{a,b\}} f\}(\xi) &= \varphi_k(\xi) \int_{\mathbb{R}^4} \widehat{f}(\eta) \varphi_{\leq k-100}(\xi - \theta) \varphi_{\leq k-100}(\theta - \eta) \\ &\quad \times \left[\frac{\theta - \eta}{2} (\nabla_\zeta \widetilde{a})(\xi - \theta, \frac{\xi + \eta}{2}) \widetilde{b}(\theta - \eta, \frac{\xi + \eta}{2}) - \widetilde{a}(\xi - \theta, \frac{\xi + \eta}{2}) \frac{\xi - \theta}{2} (\nabla_\zeta \widetilde{b})(\theta - \eta, \frac{\xi + \eta}{2}) \right] d\eta d\theta. \end{aligned}$$

Therefore

$$(16\pi^4)P_k E(a, b)f = U^1 f + U^2 f + U^3 f, \quad (8.32)$$

$$\mathcal{F}(U^j f)(\xi) = \varphi_k(\xi) \int_{\mathbb{R}^4} \widehat{f}(\eta) \varphi_{\leq k-100}(\xi - \theta) \varphi_{\leq k-100}(\theta - \eta) m^j(\xi, \eta, \theta) d\eta d\theta,$$

where

$$m^1(\xi, \eta, \theta) := \widetilde{a}(\xi - \theta, \frac{\xi + \eta}{2}) \widetilde{b}(\theta - \eta, \frac{\eta + \theta}{2}) - \widetilde{a}(\xi - \theta, \frac{\xi + \eta}{2}) \widetilde{b}(\theta - \eta, \frac{\xi + \eta}{2}) \\ - \widetilde{a}(\xi - \theta, \frac{\xi + \eta}{2}) \frac{\theta - \xi}{2} (\nabla_{\zeta} \widetilde{b})(\theta - \eta, \frac{\xi + \eta}{2}), \quad (8.33)$$

$$m^2(\xi, \eta, \theta) := \widetilde{a}(\xi - \theta, \frac{\xi + \theta}{2}) \widetilde{b}(\theta - \eta, \frac{\eta + \theta}{2}) - \widetilde{a}(\xi - \theta, \frac{\xi + \eta}{2}) \widetilde{b}(\theta - \eta, \frac{\eta + \theta}{2}) \\ - \frac{\theta - \eta}{2} (\nabla_{\zeta} \widetilde{a})(\xi - \theta, \frac{\xi + \eta}{2}) \widetilde{b}(\theta - \eta, \frac{\eta + \theta}{2}), \quad (8.34)$$

and

$$m^3(\xi, \eta, \theta) := \frac{\theta - \eta}{2} (\nabla_{\zeta} \widetilde{a})(\xi - \theta, \frac{\xi + \eta}{2}) \left[\widetilde{b}(\theta - \eta, \frac{\eta + \theta}{2}) - \widetilde{b}(\theta - \eta, \frac{\xi + \eta}{2}) \right]. \quad (8.35)$$

It remains to prove the bounds (8.30) and (8.31) for the operators U^j , $j \in \{1, 2, 3\}$. The operators U^j are similar, so we will only provide the details for the operator U^1 . We rewrite

$$m^1(\xi, \eta, \theta) = \int_0^1 \widetilde{a}(\xi - \theta, \frac{\xi + \eta}{2}) \frac{(\theta - \xi)_j (\theta - \xi)_k}{4} (\partial_{\zeta_j} \partial_{\zeta_k} \widetilde{b})(\theta - \eta, \frac{\xi + \eta}{2} + s \frac{\theta - \xi}{2}) (1-s) ds. \quad (8.36)$$

Therefore

$$U^1 f(x) = \int_{\mathbb{R}^2} f(y) K^1(x, y) \quad (8.37)$$

where

$$K^1(x, y) := C \int_{\mathbb{R}^6} e^{-iy \cdot \eta} e^{ix \cdot \xi} \varphi_k(\xi) \varphi_{\leq k-100}(\xi - \theta) \varphi_{\leq k-100}(\theta - \eta) m^1(\xi, \eta, \theta) d\eta d\theta d\xi.$$

We use the formula (8.36) and make changes to variables to rewrite

$$K^1(x, y) = C \int_0^1 ds (1-s) \int_{\mathbb{R}^{10}} e^{-iy \cdot (\xi + \mu + \nu)} e^{ix \cdot \xi} e^{iz \cdot \mu} e^{iw \cdot \nu} \varphi_k(\xi) \varphi_{\leq k-100}(\mu) \varphi_{\leq k-100}(\nu) \\ \times (\partial_{x_j} \partial_{x_k} a)(z, \xi + \mu/2 + \nu/2) (\partial_{\zeta_j} \partial_{\zeta_k} b)(w, \xi + \mu/2 + \nu/2 + s\mu/2) d\mu d\nu d\xi dz dw.$$

We integrate by parts in ξ, μ, ν , using the operators $(2^{-2k} - \Delta_{\xi})^2$, $(2^{-2k} - \Delta_{\mu})^2$ and $(2^{-2k} - \Delta_{\nu})^2$. It follows that

$$|K^1(x, y)| \lesssim \int_{\mathbb{R}^{10}} \frac{2^{-2k}}{(2^{-2k} + |x - y|^2)^2} \frac{2^{-2k}}{(2^{-2k} + |z - y|^2)^2} \frac{2^{-2k}}{(2^{-2k} + |w - y|^2)^2} F_{a,b}(z, w) dz dw, \quad (8.38)$$

where, with $\underline{\varphi}(X, Y, Z) := \varphi_0(X) \varphi_{\leq -100}(Y) \varphi_{\leq -100}(Z)$,

$$F_{a,b}(z, w) := 2^{6k} \int_0^1 ds \int_{\mathbb{R}^6} \left| [(2^{-2k} - \Delta_{\xi})^2 (2^{-2k} - \Delta_{\mu})^2 (2^{-2k} - \Delta_{\nu})^2] \{ \underline{\varphi}(2^{-k}\xi, 2^{-k}\mu, 2^{-k}\nu) \right. \\ \left. \times (\partial_{x_j} \partial_{x_k} a)(z, \xi + \mu/2 + \nu/2) (\partial_{\zeta_j} \partial_{\zeta_k} b)(w, \xi + \mu/2 + \nu/2 + s\mu/2) \right| d\xi d\mu d\nu.$$

With $|a|_{16}$ and $|b|_{16}$ defined as in (8.20), it follows that

$$\begin{aligned} |F_{a,b}(z, w)| &\lesssim 2^{-2k} \int_0^1 ds \int_{\mathbb{R}^6} |a|_{16}(z, \xi + \mu/2 + \nu/2) |b|_{16}(w, \xi + \mu/2 + \nu/2 + s\mu/2) \\ &\quad \times \varphi_{[-4,4]}(2^{-k}\xi) \varphi_{\leq -10}(2^{-k}\mu) \varphi_{\leq -10}(2^{-k}\nu) \frac{d\xi d\mu d\nu}{2^{6k}}. \end{aligned}$$

The desired bounds (8.30) and (8.31) for U^1 follow using also (8.37) and (8.38). \square

We also make the following observation: if $a = a(\zeta)$ is a Fourier multiplier, b is a symbol, and f is a function, then

$$\begin{aligned} \widehat{E(a, b)}f(\xi) &= \frac{1}{4\pi^2} \int_{\mathbb{R}^2} \chi\left(\frac{|\xi - \eta|}{|\xi + \eta|}\right) \left(a(\xi) - a\left(\frac{\xi + \eta}{2}\right) - \frac{\xi - \eta}{2} \cdot \nabla a\left(\frac{\xi + \eta}{2}\right) \right) \tilde{b}\left(\xi - \eta, \frac{\xi + \eta}{2}\right) \widehat{f}(\eta) d\eta, \\ \widehat{E(b, a)}f(\xi) &= \frac{1}{4\pi^2} \int_{\mathbb{R}^2} \chi\left(\frac{|\xi - \eta|}{|\xi + \eta|}\right) \left(a(\eta) - a\left(\frac{\xi + \eta}{2}\right) - \frac{\eta - \xi}{2} \cdot \nabla a\left(\frac{\xi + \eta}{2}\right) \right) \tilde{b}\left(\xi - \eta, \frac{\xi + \eta}{2}\right) \widehat{f}(\eta) d\eta. \end{aligned} \quad (8.39)$$

8.2. Decorated norms and estimates. In the previous subsection we proved bounds on paraproduct operators. In our study of the water wave problem, we need to keep track of several parameters, such as order, decay, and vector-fields. It is convenient to use two compatible hierarchies of bounds, one for functions and one for symbols of operators.

8.2.1. Decorated norms. Recall the spaces $\mathcal{O}_{m,p}$ defined in (8.7). We define now the norms we will use to measure symbols.

Definition 8.6. For $l \in [-10, 10]$, $r \in \mathbb{Z}_+$, $m \in \{1, 2, 3, 4\}$, $t \in [0, T]$, and $q \in \{2, \infty\}$, we define classes of symbols $\mathcal{M}_{r,q}^{l,m} = \mathcal{M}_{r,q}^{l,m}(t) \subseteq C(\mathbb{R}^2 \times \mathbb{R}^2 : \mathbb{C})$ by the norms

$$\|a\|_{\mathcal{M}_{r,\infty}^{l,m}} := \sup_{j \leq N_1/2} \sup_{|\alpha|+|\beta| \leq r} \sup_{\zeta \in \mathbb{R}^2} \langle t \rangle^{m(5/6-20\delta^2)+16\delta^2} \langle \zeta \rangle^{-l} \| |\zeta|^{|\beta|} \partial_\zeta^\beta \partial_x^\alpha \Omega_{x,\zeta}^j a \|_{L_x^\infty}, \quad (8.40)$$

$$\|a\|_{\mathcal{M}_{r,2}^{l,m}} := \sup_{j \leq N_1/2} \sup_{|\alpha|+|\beta| \leq r} \sup_{\zeta \in \mathbb{R}^2} \langle t \rangle^{(m-1)(5/6-20\delta^2)-2\delta^2} \langle \zeta \rangle^{-l} \| |\zeta|^{|\beta|} \partial_\zeta^\beta \partial_x^\alpha \Omega_{x,\zeta}^j a \|_{L_x^2}. \quad (8.41)$$

Here

$$\Omega_{x,\zeta} a := \Omega_x a + \Omega_\zeta a = (x_1 \partial_{x_2} - x_2 \partial_{x_1} + \zeta_1 \partial_{\zeta_2} - \zeta_2 \partial_{\zeta_1}) a,$$

see (8.25). We also define

$$\|a\|_{\mathcal{M}_r^{l,m}} := \|a\|_{\mathcal{M}_{r,\infty}^{l,m}} + \|a\|_{\mathcal{M}_{r,2}^{l,m}}, \quad m \geq 1. \quad (8.42)$$

Note that this hierarchy is naturally related to the hierarchy in terms of $\mathcal{O}_{m,p}$. In this definition the parameters m (the ‘‘multiplicity’’ of a , related to the decay rate) and l (the ‘‘order’’) will play an important role. Observe that for a function $f = f(x)$, and $m \in [1, 4]$,

$$\|f\|_{\mathcal{M}_{N_3+p}^{0,m}} \lesssim \|f\|_{\mathcal{O}_{m,p}}. \quad (8.43)$$

Note also that we have the simple linear rule

$$\|P_k a\|_{\mathcal{M}_{r,q}^{l,m}} \lesssim 2^{-sk} \|P_k a\|_{\mathcal{M}_{r+s,q}^{l,m}}, \quad k \in \mathbb{Z}, s \geq 0, q \in \{2, \infty\}, \quad (8.44)$$

and the basic multiplication rules

$$\langle t \rangle^{2\delta^2} [\|ab\|_{\mathcal{M}_r^{l_1+l_2, m_1+m_2}} + \|\zeta\{a, b\}\|_{\mathcal{M}_{r-2}^{l_1+l_2, m_1+m_2}}] \lesssim \|a\|_{\mathcal{M}_r^{l_1, m_1}} \|b\|_{\mathcal{M}_r^{l_2, m_2}}. \quad (8.45)$$

8.2.2. *Bounds on operators.* We may now pass the bounds proved in subsection 8.1 to decorated norms. We consider the action of paradifferential operators on the classes $\mathcal{O}_{k,p}$. We will often use the following simple facts: paradifferential operators preserve frequency localizations,

$$P_k T_a f = P_k T_a P_{[k-4, k+4]} f = P_k T_{a(x, \zeta) \varphi_{\leq k+4}(\zeta)} f; \quad (8.46)$$

the rotation vector-field Ω acts nicely on such operators, see (8.25),

$$\Omega(T_a f) = T_{\Omega_{x, \zeta} a} f + T_a(\Omega f); \quad (8.47)$$

the following relations between basic and decorated norms for symbols hold:

$$\begin{aligned} \|\Omega_{x, \zeta}^j a(x, \zeta) \varphi_{\leq k}(\zeta)\|_{\mathcal{M}_{r, \infty}} &\lesssim 2^{lk_+} \|a\|_{\mathcal{M}_{r, \infty}^{l, m}} \langle t \rangle^{-m(5/6-20\delta^2)-16\delta^2}, \quad 0 \leq j \leq N_1/2, \\ \|\Omega_{x, \zeta}^j a(x, \zeta) \varphi_{\leq k}(\zeta)\|_{\mathcal{M}_{r, 2}} &\lesssim 2^{lk_+} \|a\|_{\mathcal{M}_{r, 2}^{l, m}} \langle t \rangle^{-(m-1)(5/6-20\delta^2)+2\delta^2}, \quad 0 \leq j \leq N_1. \end{aligned} \quad (8.48)$$

A simple application of the above remarks and Lemma 8.3 (i) gives the bound

$$\|T_\sigma f\|_{H^s} \lesssim \langle t \rangle^{-m(5/6-20\delta^2)-16\delta^2} \|\sigma\|_{\mathcal{M}_8^{l, m}} \|f\|_{H^{s+l}}. \quad (8.49)$$

We prove now two useful lemmas:

Lemma 8.7. *If $q, q-l \in [-N_3, 10]$ and $m, m_1 \geq 1$ then*

$$\langle t \rangle^{12\delta^2} T_a \mathcal{O}_{m, q} \subseteq \mathcal{O}_{m+m_1, q-l}, \quad \text{for } a \in \mathcal{M}_{10}^{l, m_1}, \quad (8.50)$$

In particular, using also (8.43),

$$\langle t \rangle^{12\delta^2} T_{\mathcal{O}_{m_1, -10}} \mathcal{O}_{m, q} \subseteq \mathcal{O}_{m+m_1, q}. \quad (8.51)$$

Proof. The estimate (8.50) follows using the definitions and the linear estimates (8.22) and (8.23) in Lemma 8.3. We may assume $m = m_1 = 1$. Using (8.22) and (8.48) we estimate

$$\begin{aligned} 2^{(N_0+q-l)k_+} \|P_k T_a f\|_{L^2} &\lesssim \|a\|_{\mathcal{M}_{8, \infty}} 2^{(N_0+q-l)k_+} \|P_{[k-2, k+2]} f\|_{L^2} \\ &\lesssim \langle t \rangle^{-5/6+4\delta^2} \|a\|_{\mathcal{M}_{8, \infty}^{l, 1}} 2^{(N_0+q)k_+} \|P_{[k-2, k+2]} f\|_{L^2}, \end{aligned}$$

for any $f \in \mathcal{O}_{1, q}$. By orthogonality we deduce the desired bound on the H^{N_0} norm.

To estimate the weighted norm we use (8.22), (8.23), and (8.48) to estimate

$$\begin{aligned} 2^{(N_3+q-l)k_+} \|\Omega^j P_k T_a f\|_{L^2} &\lesssim \sum_{n \leq j/2} 2^{(N_3+q-l)k_+} [\|P_k T_{\Omega_{x, \zeta}^n a} \Omega^{j-n} f\|_{L^2} + \|P_k T_{\Omega_{x, \zeta}^{j-n} a} \Omega^n f\|_{L^2}] \\ &\lesssim \sum_{n \leq j/2} 2^{(N_3+q-l)k_+} [\|\Omega_{x, \zeta}^n a\|_{\mathcal{M}_{8, \infty}} \|P_{[k-2, k+2]} \Omega^{j-n} f\|_{L^2} + \|\Omega_{x, \zeta}^{j-n} a\|_{\mathcal{M}_{8, 2}} \|P_{[k-2, k+2]} \Omega^n f\|_{L^\infty}] \\ &\lesssim \sum_{n \leq j/2} 2^{(N_3+q)k_+} \|a\|_{\mathcal{M}_8^{l, 1}} [\langle t \rangle^{-5/6+4\delta^2} \|P_{[k-2, k+2]} \Omega^{j-n} f\|_{L^2} + \langle t \rangle^{2\delta^2} \|P_{[k-2, k+2]} \Omega^n f\|_{L^\infty}], \end{aligned}$$

for every $j \in [0, N_1]$. The desired weighted L^2 bound follows since

$$\begin{aligned} \left[\sum_{k \in \mathbb{Z}} 2^{2(N_3+q)k_+} \|P_{[k-2, k+2]} \Omega^{j-n} f\|_{L^2}^2 \right]^{1/2} + \langle t \rangle^{5/6-2\delta^2} \left[\sum_{k \in \mathbb{Z}} 2^{2(N_3+q)k_+} \|P_{[k-2, k+2]} \Omega^n f\|_{L^\infty}^2 \right]^{1/2} \\ \lesssim \langle t \rangle^{2\delta^2} \|f\|_{\mathcal{O}_{1, q}}. \end{aligned}$$

Finally, for the L^∞ bound we use (8.22) to estimate

$$\begin{aligned} 2^{(N_2+q-l)k_+} \|\Omega^j P_k T_a f\|_{L^\infty} &\lesssim \sum_{j_1, j_2 \leq N_1/2} 2^{(N_2+q-l)k_+} \|\Omega_{x,\zeta}^{j_1} a\|_{\mathcal{M}_{8,\infty}} \|P_{[k-2, k+2]} \Omega^{j_2} f\|_{L^\infty} \\ &\lesssim \langle t \rangle^{-5/6+4\delta^2} \|a\|_{\mathcal{M}_{8,\infty}^{l_1}} \sum_{j_2 \leq N_1/2} 2^{(N_2+q)k_+} \|P_{[k-2, k+2]} \Omega^{j_2} f\|_{L^\infty}, \end{aligned}$$

for any $j \in [0, N_1/2]$. The desired bound follows by summation over k . \square

Lemma 8.8. *Let E be defined as in Proposition 8.5. Assume that $m, m_1, m_2 \geq 1$, $q, q-l_1, q-l_2, q-l_1-l_2 \in [-N_3, 10]$ and consider $a \in \mathcal{M}_{20}^{l_1, m_1}$, $b \in \mathcal{M}_{20}^{l_2, m_2}$. Then*

$$\begin{aligned} \langle t \rangle^{12\delta^2} P_{\geq -100} E(a, b) \mathcal{O}_{m, q} &\subseteq \mathcal{O}_{m+m_1+m_2, q-l_1-l_2+2}, \\ \langle t \rangle^{12\delta^2} P_{\geq -100} (T_a T_b + T_b T_a - 2T_{ab}) \mathcal{O}_{m, q} &\subseteq \mathcal{O}_{m+m_1+m_2, q-l_1-l_2+2}. \end{aligned} \quad (8.52)$$

In addition,

$$\begin{aligned} \langle t \rangle^{12\delta^2} [T_a, T_b] \mathcal{O}_{m, q} &\subseteq \mathcal{O}_{m+m_1+m_2, q-l_1-l_2+1}, \\ \langle t \rangle^{12\delta^2} (T_a T_b - T_{ab}) \mathcal{O}_{m, q} &\subseteq \mathcal{O}_{m+m_1+m_2, q-l_1-l_2+1}. \end{aligned} \quad (8.53)$$

Moreover, if $a \in \mathcal{M}_{20}^{0, m_1}$, $b \in \mathcal{M}_{20}^{0, m_2}$ are functions then

$$\langle t \rangle^{12\delta^2} (T_a T_b - T_{ab}) \mathcal{O}_{m, -5} \subseteq \mathcal{O}_{m+m_1+m_2, 5}. \quad (8.54)$$

Proof. We record the formulas

$$\Omega_{x,\zeta}(ab) = (\Omega_{x,\zeta} a)b + a(\Omega_{x,\zeta} b), \quad \Omega_{x,\zeta}(\{a, b\}) = \{\Omega_{x,\zeta} a, b\} + \{a, \Omega_{x,\zeta} b\}. \quad (8.55)$$

Therefore, letting $U(a, b) := T_a T_b - T_{ab}$, we have

$$\begin{aligned} [T_a, T_b] &= U(a, b) - U(b, a), \quad E(a, b) = U(a, b) - (i/2)T_{\{a, b\}}, \\ T_a T_b + T_b T_a - 2T_{ab} &= E(a, b) + E(b, a), \end{aligned} \quad (8.56)$$

and

$$\begin{aligned} \Omega(U(a, b)f) &= U(\Omega_{x,\zeta} a, b)f + U(a, \Omega_{x,\zeta} b)f + U(a, b)\Omega f, \\ \Omega(T_{\{a, b\}}f) &= T_{\{\Omega_{x,\zeta} a, b\}}f + T_{\{a, \Omega_{x,\zeta} b\}}f + T_{\{a, b\}}\Omega f, \\ \Omega(E(a, b)f) &= E(\Omega_{x,\zeta} a, b)f + E(a, \Omega_{x,\zeta} b)f + E(a, b)\Omega f. \end{aligned} \quad (8.57)$$

The bound (8.54) follows as in the proof of Lemma 8.2, once we notice that

$$P_k[(T_a T_b - T_{ab})f] = \sum_{\max(k_1, k_2) \geq k-40} P_k[(T_{P_{k_1} a} T_{P_{k_2} b} - T_{P_{k_1} a P_{k_2} b})f].$$

The bounds (8.52) follow from (8.30)–(8.31) and (8.48), in the same way the bound (8.50) in Lemma 8.7 follows from (8.22)–(8.23). Moreover, using (8.45),

$$\langle t \rangle^{12\delta^2} \|\{a, b\}(x, \zeta) \varphi_{\geq -200}(\zeta)\|_{\mathcal{M}_{18}^{l_1+l_2-1, m_1+m_2}} \lesssim \|a\|_{\mathcal{M}_{20}^{l_1, m_1}} \|b\|_{\mathcal{M}_{20}^{l_2, m_2}}.$$

Therefore, using (8.50) and frequency localization,

$$\langle t \rangle^{12\delta^2} P_{\geq -100} T_{\{a, b\}} \mathcal{O}_{m, q} \subseteq \mathcal{O}_{m+m_1+m_2, q-l_1-l_2+1}. \quad (8.58)$$

Therefore, using (8.56) and (8.52),

$$\langle t \rangle^{12\delta^2} P_{\geq -100} U(a, b) \mathcal{O}_{m, q} \subseteq \mathcal{O}_{m+m_1+m_2, q-l_1-l_2+1}.$$

For (8.53) it remains to prove that

$$\langle t \rangle^{12\delta^2} P_{\leq 0} U(a, b) \mathcal{O}_{m,q} \subseteq \mathcal{O}_{m+m_1+m_2, q-l_1-l_2+1}. \quad (8.59)$$

However, using (8.50) and (8.45),

$$\langle t \rangle^{12\delta^2} T_a T_b \mathcal{O}_{m,q} \subseteq \mathcal{O}_{m+m_1+m_2, q-l_1-l_2}, \quad \langle t \rangle^{12\delta^2} T_{ab} \mathcal{O}_{m,q} \subseteq \mathcal{O}_{m+m_1+m_2, q-l_1-l_2},$$

and (8.59) follows. This completes the proof of (8.53). \square

9. THE DIRICHLET-NEUMANN OPERATOR

Assume (h, ϕ) are as in Proposition 2.2 and let $\Omega := \{(x, z) \in \mathbb{R}^3 : z \leq h(x)\}$. Let Φ denote the (unique in a suitable space, see Lemma 9.4) harmonic function in Ω satisfying $\Phi(x, h(x)) = \phi(x)$. We define the Dirichlet-Neumann⁶ map as

$$G(h)\phi = \sqrt{1 + |\nabla h|^2} (\nu \cdot \nabla \Phi) \quad (9.1)$$

where ν denotes the outward pointing unit normal to the domain Ω . The main result of this section is the following parilinearization of the Dirichlet-Neumann map.

Proposition 9.1. *Assume that $t \in [0, T]$ is fixed and (h, ϕ) satisfy*

$$\|\langle \nabla \rangle h\|_{\mathcal{O}_{1,0}} + \|\langle \nabla \rangle^{1/2} \phi\|_{\mathcal{O}_{1,0}} \lesssim \varepsilon_1. \quad (9.2)$$

Define

$$B := \frac{G(h)\phi + \nabla_x h \cdot \nabla_x \phi}{1 + |\nabla h|^2}, \quad V := \nabla_x \phi - B \nabla_x h, \quad \omega := \phi - T_B h. \quad (9.3)$$

Then we can parilinearize the Dirichlet-Neumann operator as

$$G(h)\phi = T_{\lambda_{DN}} \omega - \operatorname{div}(T_V h) + G_2 + \varepsilon_1^3 \mathcal{O}_{3,3/2}, \quad (9.4)$$

recall the definition (8.7), where

$$\begin{aligned} \lambda_{DN} &:= \lambda^{(1)} + \lambda^{(0)}, \\ \lambda^{(1)}(x, \zeta) &:= \sqrt{(1 + |\nabla h|^2) |\zeta|^2 - (\zeta \cdot \nabla h)^2}, \\ \lambda^{(0)}(x, \zeta) &:= \left(\frac{(1 + |\nabla h|^2)^2}{2\lambda^{(1)}} \left\{ \frac{\lambda^{(1)}}{1 + |\nabla h|^2}, \frac{\zeta \cdot \nabla h}{1 + |\nabla h|^2} \right\} + \frac{1}{2} \Delta h \right) \varphi_{\geq 0}(\zeta). \end{aligned} \quad (9.5)$$

The quadratic terms are given by

$$G_2 = G_2(h, |\nabla|^{1/2} \omega) \in \varepsilon_1^2 \mathcal{O}_{2,5/2}, \quad \widehat{G}_2(\xi) = \frac{1}{4\pi^2} \int_{\mathbb{R}^2} g_2(\xi, \eta) \widehat{h}(\xi - \eta) |\eta|^{1/2} \widehat{\omega}(\eta) d\eta, \quad (9.6)$$

where g_2 is a symbol satisfying (see the definition of the class S_Ω^∞ in (8.5))

$$\|g_2^{k, k_1, k_2}(\xi, \eta)\|_{S_\Omega^\infty} \lesssim 2^k 2^{\min\{k_1, k_2\}/2} \left(\frac{1 + 2^{\min\{k_1, k_2\}}}{1 + 2^{\max\{k_1, k_2\}}} \right)^{7/2}. \quad (9.7)$$

⁶To be precise this is $\sqrt{1 + |\nabla h|^2}$ times the standard Dirichlet-Neumann operator, but we will slightly abuse notation and call $G(h)\phi$ the Dirichlet-Neumann operator.

Remark 9.2. Using (9.5), Definition 8.6, and (8.43)–(8.45) we see that, for any $t \in [0, T]$,

$$\lambda^{(1)} = |\zeta|(1 + \varepsilon_1^2 \mathcal{M}_{N_3-1}^{0,2}), \quad \lambda^{(0)} \in \varepsilon_1 \mathcal{M}_{N_3-2}^{0,1}. \quad (9.8)$$

For later use we further decompose $\lambda^{(0)}$ into its linear and higher order parts:

$$\lambda^{(0)} = \lambda_1^{(0)} + \lambda_2^{(0)}, \quad \lambda_1^{(0)} := \left[\frac{1}{2} \Delta h - \frac{1}{2} \frac{\zeta_j \zeta_k \partial_j \partial_k h}{|\zeta|^2} \right] \varphi_{\geq 0}(\zeta), \quad \lambda_2^{(0)} \in \varepsilon_1^3 \mathcal{M}_{N_3-2}^{0,3}. \quad (9.9)$$

According to the formulas in (9.5) and (9.9) we have:

$$\lambda_{DN} - |\zeta| - \lambda_1^{(0)} \in \varepsilon_1^2 \mathcal{M}_{N_3-2}^{1,2}, \quad \lambda_{DN} - \lambda^{(1)} - \lambda_1^{(0)} \in \varepsilon_1^3 \mathcal{M}_{N_3-2}^{0,3}. \quad (9.10)$$

The proof of Proposition 9.1 relies on several results and is given at the end of the section.

9.1. Linearization. We start with a result that identifies the linear and quadratic part of the Dirichlet-Neumann operator.

We first use a change of variable to flatten the surface. We thus define

$$\begin{aligned} u(x, y) &:= \Phi(x, h(x) + y), & (x, y) &\in \mathbb{R}^2 \times (-\infty, 0], \\ \Phi(x, z) &= u(x, z - h(x)). \end{aligned} \quad (9.11)$$

In particular $u|_{y=0} = \phi$, $\partial_y u|_{y=0} = B$, and the Dirichlet-Neumann operator is given by

$$G(h)\phi = (1 + |\nabla h|^2) \partial_y u|_{y=0} - \nabla_x h \cdot \nabla_x u|_{y=0}. \quad (9.12)$$

A simple computation yields

$$0 = \Delta_{x,z} \Phi = (1 + |\nabla_x h|^2) \partial_y^2 u + \Delta_x u - 2 \partial_y \nabla_x u \cdot \nabla_x h - \partial_y u \Delta_x h. \quad (9.13)$$

Since we will also need to study the linearized operator, it is convenient to also allow for error terms and consider the equation

$$(1 + |\nabla_x h|^2) \partial_y^2 u + \Delta_x u - 2 \partial_y \nabla_x u \cdot \nabla_x h - \partial_y u \Delta_x h = \partial_y \mathbf{e}_a + |\nabla| \mathbf{e}_b. \quad (9.14)$$

With $\mathcal{R} := |\nabla|^{-1} \nabla$ (the Riesz transform), this can be rewritten in the form

$$\begin{aligned} (\partial_y^2 - |\nabla|^2) u &= \partial_y Q_a + |\nabla| Q_b, \\ Q_a &:= \nabla u \cdot \nabla h - |\nabla h|^2 \partial_y u + \mathbf{e}_a, & Q_b &:= \mathcal{R}(\partial_y u \nabla h) + \mathbf{e}_b. \end{aligned} \quad (9.15)$$

To study the solution y we will need an additional class of Banach spaces, to measure functions that depend on $y \in (-\infty, 0]$ and $x \in \mathbb{R}^2$. These spaces are only used in this section.

Definition 9.3. For $t \in [0, T]$, $p \geq -10$, and $m \geq 1$ let $\mathcal{L}_{m,p} = \mathcal{L}_{m,p}(t)$ denote the Banach space of functions $g \in C((-\infty, 0] : \dot{H}^{1/2, 1/2})$ defined by the norm

$$\|g\|_{\mathcal{L}_{m,p}} := \| |\nabla| g \|_{L_y^2 \mathcal{O}_{m,p}} + \|\partial_y g\|_{L_y^2 \mathcal{O}_{m,p}} + \| |\nabla|^{1/2} g \|_{L_y^\infty \mathcal{O}_{m,p}}. \quad (9.16)$$

The point of these spaces is to estimate solutions of equations of the form $(\partial_y - |\nabla|)u = \mathcal{N}$, in terms of the initial data $u(0) = \psi$. It is easy to see that if $|\nabla|^{1/2} \psi \in \mathcal{O}_{m,p}$ then

$$\|e^{y|\nabla|} \psi\|_{\mathcal{L}_{m,p}} \lesssim \| |\nabla|^{1/2} \psi \|_{\mathcal{O}_{m,p}}. \quad (9.17)$$

To see this estimate for the $L_y^2 \widetilde{W}_\Omega^{N_1/2, N_2+p}$ component we use the bound $\|c\|_{L_y^2 \ell_k^1} \lesssim \|c\|_{\ell_k^1 L_y^2}$ for any $c : \mathbb{Z} \times (-\infty, 0] \rightarrow \mathbb{C}$. Moreover, if $Q \in L_y^2 \mathcal{O}_{m,p}$ then

$$\left\| |\nabla|^{1/2} \int_{-\infty}^0 e^{-|y-s||\nabla|} \mathbf{1}_\pm(y-s) Q(s) ds \right\|_{L_y^\infty \mathcal{O}_{m,p}} \lesssim \langle t \rangle^{\delta^2/2} \|Q\|_{L_y^2 \mathcal{O}_{m,p}} \quad (9.18)$$

and

$$\left\| |\nabla| \int_{-\infty}^0 e^{-|y-s||\nabla|} \mathbf{1}_{\pm}(y-s) Q(s) ds \right\|_{L_y^2 \mathcal{O}_{m,p}} \lesssim \langle t \rangle^{\delta^2/2} \|Q\|_{L_y^2 \mathcal{O}_{m,p}}. \quad (9.19)$$

Indeed, these bounds follow directly from the definitions for the L^2 -based components of the space $\mathcal{O}_{m,p}$, which are H^{N_0+p} and $H_{\Omega}^{N_1, N_3+p}$. For the remaining component one can control uniformly the $\widetilde{W}_{\Omega}^{N_1/2, N_2+p}$ norm of the function localized at every single dyadic frequency, without the factor of $\langle t \rangle^{\delta^2/2}$ in the right-hand side. The full bounds follow once we notice that only the frequencies satisfying $2^k \in [\langle t \rangle^{-8}, \langle t \rangle^8]$ are relevant in the $\widetilde{W}_{\Omega}^{N_1/2, N_2+p}$ component of the space $\mathcal{O}_{1,p}$; the other frequencies are already accounted by the stronger Sobolev norms.

Our first result is the following:

Lemma 9.4. (i) Assume that $t \in [0, T]$ is fixed, $\|\langle \nabla \rangle h\|_{\mathcal{O}_{1,0}} \lesssim \varepsilon_1$, as in (9.2), and

$$\| |\nabla|^{1/2} \psi \|_{\mathcal{O}_{1,p}} \leq A < \infty, \quad \|\mathbf{e}_a\|_{L_y^2 \mathcal{O}_{1,p}} + \|\mathbf{e}_b\|_{L_y^2 \mathcal{O}_{1,p}} \leq A \varepsilon_1 \langle t \rangle^{-12\delta^2}, \quad (9.20)$$

for some $p \in [-10, 0]$. Then there is a unique solution $u \in \mathcal{L}_{1,p}$ of the equation

$$\begin{aligned} u(y) &= e^{y|\nabla|} \left(\psi - \frac{1}{2} \int_{-\infty}^0 e^{s|\nabla|} (Q_a(s) - Q_b(s)) ds \right) \\ &\quad + \frac{1}{2} \int_{-\infty}^0 e^{-|y-s||\nabla|} (\operatorname{sgn}(y-s) Q_a(s) - Q_b(s)) ds, \end{aligned} \quad (9.21)$$

where Q_a and Q_b are as in (9.15). Moreover, u is a solution of the equation $(\partial_y^2 - |\nabla|^2)u = \partial_y Q_a + |\nabla| Q_b$ in (9.15) (and therefore a solution of (9.14) in $\mathbb{R}^2 \times (-\infty, 0]$), and

$$\|u\|_{\mathcal{L}_{1,p}} = \| |\nabla| u \|_{L_y^2 \mathcal{O}_{1,p}} + \|\partial_y u\|_{L_y^2 \mathcal{O}_{1,p}} + \| |\nabla|^{1/2} u \|_{L_y^{\infty} \mathcal{O}_{1,p}} \lesssim A. \quad (9.22)$$

(ii) Assume that we make the stronger assumptions, compare with (9.20),

$$\| |\nabla|^{1/2} \psi \|_{\mathcal{O}_{1,p}} \leq A < 0, \quad \|\partial_y^j \mathbf{e}\|_{L_y^2 \mathcal{O}_{2,p-j}} + \|\partial_y^j \mathbf{e}\|_{L_y^{\infty} \mathcal{O}_{2,p-1/2-j}} \leq A \varepsilon_1 \langle t \rangle^{-12\delta^2}, \quad (9.23)$$

for $\mathbf{e} \in \{\mathbf{e}_a, \mathbf{e}_b\}$ and $j \in \{0, 1, 2\}$. Then

$$\|\partial_y^j (\partial_y u - |\nabla| u)\|_{L_y^2 \mathcal{O}_{2,p-j}} + \|\partial_y^j (\partial_y u - |\nabla| u)\|_{L_y^{\infty} \mathcal{O}_{2,p-1/2-j}} \lesssim A \varepsilon_1. \quad (9.24)$$

Proof. (i) We use a fixed point argument in a ball of radius $\approx A$ in $\mathcal{L}_{1,p}$, for the functional

$$\begin{aligned} \Phi(u) &:= e^{y|\nabla|} \left[\psi - \frac{1}{2} \int_{-\infty}^0 e^{s|\nabla|} (Q_a(s) - Q_b(s)) ds \right] \\ &\quad + \frac{1}{2} \int_{-\infty}^0 e^{-|y-s||\nabla|} (\operatorname{sign}(y-s) Q_a(s) - Q_b(s)) ds. \end{aligned} \quad (9.25)$$

Notice that, using Lemma 8.2 and (9.20), if $\|u\|_{\mathcal{L}_{1,p}} \lesssim 1$ then

$$\|Q_a\|_{L_y^2 \mathcal{O}_{1,p}} + \|Q_b\|_{L_y^2 \mathcal{O}_{1,p}} \lesssim A \varepsilon_1 \langle t \rangle^{-12\delta^2}.$$

Therefore, using (9.17)–(9.19), $\|\Phi(u) - e^{y|\nabla|} \psi\|_{\mathcal{L}_{1,p}} \lesssim A \varepsilon_1$. Similarly, one can show that $\|\Phi(u) - \Phi(v)\|_{\mathcal{L}_{1,p}} \lesssim \varepsilon_1 \|u - v\|_{\mathcal{L}_{1,p}}$, and the desired conclusion follows.

(ii) The identity (9.21) shows that

$$\partial_y u(y) - |\nabla| u(y) = Q_a(y) + \int_{-\infty}^y |\nabla| e^{-|s-y||\nabla|} (Q_b(s) - Q_a(s)) ds. \quad (9.26)$$

Given (9.22), the definition (9.15), and the stronger assumptions in (9.23), we have

$$\|Q\|_{L_y^2 \mathcal{O}_{2,p}} + \|Q\|_{L_y^\infty \mathcal{O}_{2,p-1/2}} \lesssim A\varepsilon_1 \langle t \rangle^{-12\delta^2}, \quad (9.27)$$

for $Q \in \{Q_a, Q_b\}$. Using estimates similar to (9.18) and (9.19) it follows that

$$\|\partial_y u - |\nabla|u\|_{L_y^2 \mathcal{O}_{2,p}} + \|\partial_y u - |\nabla|u\|_{L_y^\infty \mathcal{O}_{2,p-1/2}} \lesssim A\varepsilon_1. \quad (9.28)$$

To prove (9.24) for $j \in \{1, 2\}$, we observe that, as a consequence of (9.14),

$$\partial_y^2 u - |\nabla|^2 u = (1 + |\nabla_x h|^2)^{-1} (-|\nabla|^2 u |\nabla_x h|^2 + 2\partial_y \nabla_x u \cdot \nabla_x h + \partial_y u \Delta_x h + \partial_y \mathbf{e}_a + |\nabla| \mathbf{e}_b). \quad (9.29)$$

Using (9.22) and (9.28), together with Lemma 8.2, it follows that

$$\|\partial_y^2 u - |\nabla|^2 u\|_{L_y^2 \mathcal{O}_{2,p-1}} + \|\partial_y^2 u - |\nabla|^2 u\|_{L_y^\infty \mathcal{O}_{2,p-3/2}} \lesssim A\varepsilon_1.$$

The desired bound (9.24) for $j = 1$ follows using also (9.28). The bound for $j = 2$ then follows by differentiating (9.29) with respect to y . This completes the proof of the lemma. \square

9.2. Paralinearization. The previous analysis allows us to isolate the linear (and the higher order) components of the Dirichlet-Neumann operator. However, this is insufficient for our purpose because we also need to avoid losses of derivatives in the equation. To deal with this we follow the approach of Alazard-Metivier [5], Alazard-Burq-Zuily [1, 2] and Alazard-Delort [3] using paradifferential calculus. Our choice is to work with the (somewhat unusual) Weyl quantization, instead of the standard one used by the cited authors. We refer to section 8 for a review of the paraproduct calculus using the Weyl quantization.

For simplicity of notation, we set $\alpha = |\nabla h|^2$ and let

$$\omega := u - T_{\partial_y u} h. \quad (9.30)$$

Notice that ω is naturally extended to the fluid domain, compare with the definition (9.3). We will also assume (9.2) and use Lemma 9.4. Using (8.51) in Lemma 8.7 and (9.24), we see that

$$\|\omega - u\|_{L_y^2 \mathcal{O}_{2,1} \cap L_y^\infty \mathcal{O}_{2,1}} \lesssim \varepsilon_1^2. \quad (9.31)$$

Using Lemma 8.4 to paralinearize products, we may rewrite the equation as

$$T_{1+\alpha} \partial_y^2 \omega + \Delta \omega - 2T_{\nabla h} \nabla \partial_y \omega - T_{\Delta h} \partial_y \omega = \mathcal{Q} + \mathcal{C} \quad (9.32)$$

where

$$\begin{aligned} -\mathcal{Q} &= -2\mathcal{H}(\nabla h, \nabla \partial_y u) - \mathcal{H}(\Delta h, \partial_y u), \\ -\mathcal{C} &= \partial_y (T_{1+\alpha} T_{\partial_y^2 u} h + T_{\Delta u} - 2T_{\nabla h} T_{\nabla \partial_y u} - T_{\Delta h} T_{\partial_y u}) h + 2(T_{\partial_y^2 u} T_{\nabla h} - T_{\nabla h} T_{\partial_y^2 u}) \nabla h \\ &\quad + T_{\partial_y^2 u} \mathcal{H}(\nabla h, \nabla h) + \mathcal{H}(\alpha, \partial_y^2 u). \end{aligned} \quad (9.33)$$

Notice that the error terms are quadratic and cubic strongly semilinear. More precisely, using Lemma 8.4, Lemma 8.8, and the equation (9.13), we see that

$$\mathcal{Q} \in \varepsilon_1^2 [L_y^2 \mathcal{O}_{2,4} \cap L_y^\infty \mathcal{O}_{2,4}], \quad \mathcal{C} \in \varepsilon_1^3 \langle t \rangle^{-11\delta^2} [L_y^2 \mathcal{O}_{3,4} \cap L_y^\infty \mathcal{O}_{3,4}]. \quad (9.34)$$

We now look for a factorization of the main elliptic equation into

$$\begin{aligned} &T_{1+\alpha} \partial_y^2 + \Delta - 2T_{\nabla h} \nabla \partial_y - T_{\Delta h} \partial_y \\ &= (T_{\sqrt{1+\alpha}} \partial_y - A + B)(T_{\sqrt{1+\alpha}} \partial_y - A - B) + \mathcal{E} \\ &= T_{\sqrt{1+\alpha}}^2 \partial_y^2 - \{(AT_{\sqrt{1+\alpha}} + T_{\sqrt{1+\alpha}}A) + [T_{\sqrt{1+\alpha}}, B]\} \partial_y + A^2 - B^2 + [A, B] + \mathcal{E} \end{aligned}$$

where the error term is acceptable (in a suitable sense to be made precise later), and $[A, \partial_y] = 0$, $[B, \partial_y] = 0$. Identifying the terms, this leads to the system

$$\begin{aligned} T_{\sqrt{1+\alpha}}A + AT_{\sqrt{1+\alpha}} + [T_{\sqrt{1+\alpha}}, B] &= 2Ti_{\zeta} \cdot \nabla h + \mathcal{E}, \\ A^2 - B^2 + [A, B] &= \Delta + \mathcal{E}. \end{aligned}$$

We may now look for paraproduct solutions in the form

$$A = iT_a, \quad a = a^{(1)} + a^{(0)}, \quad B = T_b, \quad b = b^{(1)} + b^{(0)}$$

where both a and b are real and are a sum of a two symbols of order 1 and 0. Therefore A corresponds to the skew-symmetric part of the system, while B corresponds to the symmetric part. Using Proposition 8.5, and formally identifying the symbols, we obtain the system

$$\begin{aligned} 2ia\sqrt{1+\alpha} + i\{\sqrt{1+\alpha}, b\} &= 2i\zeta \cdot \nabla h + \varepsilon_1^2 \mathcal{M}_{N_3-2}^{-1,2}, \\ a^2 + b^2 + \{a, b\} &= |\zeta|^2 + \varepsilon_1^2 \mathcal{M}_{N_3-2}^{0,2}. \end{aligned}$$

We can solve this by letting

$$\begin{aligned} a^{(1)} &:= \frac{\zeta \cdot \nabla h}{\sqrt{1+\alpha}}, & a^{(0)} &:= -\frac{1}{2\sqrt{1+\alpha}} \{\sqrt{1+\alpha}, b^{(1)}\} \varphi_{\geq 0}(\zeta), \\ b^{(1)} &= \sqrt{|\zeta|^2 - (a^{(1)})^2}, & b^{(0)} &= \frac{1}{2b^{(1)}} (-2a^{(1)}a^{(0)} - \{a^{(1)}, b^{(1)}\} \varphi_{\geq 0}(\zeta)). \end{aligned}$$

This gives us the following formulas:

$$a^{(1)} = \frac{1}{\sqrt{1+|\nabla h|^2}} (\zeta \cdot \nabla h) = (\zeta \cdot \nabla h)(1 + \varepsilon_1^2 \mathcal{M}_{N_3}^{0,2}), \quad (9.35)$$

$$b^{(1)} = \sqrt{\frac{(1+|\nabla h|^2)|\zeta|^2 - (\zeta \cdot \nabla h)^2}{1+|\nabla h|^2}} = |\zeta|(1 + \varepsilon_1^2 \mathcal{M}_{N_3}^{0,2}), \quad (9.36)$$

$$a^{(0)} = -\frac{\{\sqrt{1+|\nabla h|^2}, b^{(1)}\}}{2\sqrt{1+|\nabla h|^2}} \varphi_{\geq 0}(\zeta) = \varphi_{\geq 0}(\zeta) \varepsilon_1^2 \mathcal{M}_{N_3-1}^{0,2}, \quad (9.37)$$

$$b^{(0)} = -\frac{\sqrt{1+|\nabla h|^2}}{2b^{(1)}} \left\{ \frac{\zeta \cdot \nabla h}{1+|\nabla h|^2}, b^{(1)} \right\} \varphi_{\geq 0}(\zeta) = \varphi_{\geq 0}(\zeta) \left[-\frac{\zeta_j \zeta_k \partial_j \partial_k h}{2|\zeta|^2} + \varepsilon_1^3 \mathcal{M}_{N_3-1}^{0,3} \right]. \quad (9.38)$$

We now verify that

$$\begin{aligned} (T_{\sqrt{1+\alpha}} \partial_y - iT_a + T_b)(T_{\sqrt{1+\alpha}} \partial_y - iT_a - T_b) \\ = T_{1+\alpha} \partial_y^2 - (2T_{a\sqrt{1+\alpha}} + T_{\{\sqrt{1+\alpha}, b^{(1)}\}}) i \partial_y - T_{a^2} - T_{b^2} - T_{\{a^{(1)}, b^{(1)}\} \varphi_{\geq 0}(\zeta)} + \mathcal{E}, \end{aligned} \quad (9.39)$$

where

$$\begin{aligned} \mathcal{E} &:= (T_{\sqrt{1+\alpha}} T_{\sqrt{1+\alpha}} - T_{1+\alpha}) \partial_y^2 - (T_a T_{\sqrt{1+\alpha}} + T_{\sqrt{1+\alpha}} T_a - 2T_{a\sqrt{1+\alpha}}) i \partial_y - [T_{\sqrt{1+\alpha}}, T_{b^{(0)}}] \partial_y \\ &\quad - ([T_{\sqrt{1+\alpha}}, T_{b^{(1)}}] - iT_{\{\sqrt{1+\alpha}, b^{(1)}\}}) \partial_y + (T_{a^2} - T_a^2) + (T_{b^2} - T_b^2) + i[T_a, T_b] + T_{\{a^{(1)}, b^{(1)}\} \varphi_{\geq 0}(\zeta)}. \end{aligned}$$

We also verify that

$$\begin{aligned} 2a\sqrt{1+\alpha} + \{\sqrt{1+\alpha}, b^{(1)}\} &= 2\zeta \cdot \nabla h + \{\sqrt{1+\alpha}, b^{(1)}\} \varphi_{\leq -1}(\zeta), \\ a^2 + b^2 + \{a^{(1)}, b^{(1)}\} \varphi_{\geq 0}(\zeta) &= |\zeta|^2 + (a^{(0)})^2 + (b^{(0)})^2. \end{aligned}$$

Lemma 9.5. *With the definitions above, we have*

$$(T_{\sqrt{1+\alpha}}\partial_y - iT_a + T_b)(T_{\sqrt{1+\alpha}}\partial_y - iT_a - T_b)\omega = Q_0 + \tilde{\mathcal{C}}, \quad (9.40)$$

where

$$\begin{aligned} \tilde{\mathcal{C}} &\in \varepsilon_1^3 \langle t \rangle^{-11\delta^2} [L_y^\infty \mathcal{O}_{3,1/2} \cap L_y^2 \mathcal{O}_{3,1}], \quad Q_0 \in \varepsilon_1^2 [L_y^\infty \mathcal{O}_{2,3/2} \cap L_y^2 \mathcal{O}_{2,2}], \\ \widehat{Q}_0(\xi, y) &= \frac{1}{4\pi^2} \int_{\mathbb{R}^2} q_0(\xi, \eta) \widehat{h}(\xi - \eta) \widehat{u}(\eta, y) d\eta, \end{aligned} \quad (9.41)$$

and

$$\begin{aligned} q_0(\xi, \eta) &:= \chi\left(\frac{|\xi - \eta|}{|\xi + \eta|}\right) \frac{(|\xi| - |\eta|)^2 (|\xi| + |\eta|)}{2} \left[\frac{2\xi \cdot \eta - 2|\xi||\eta|}{|\xi + \eta|^2} \varphi_{\geq 0}\left(\frac{\xi + \eta}{2}\right) + \varphi_{\leq -1}\left(\frac{\xi + \eta}{2}\right) \right] \\ &+ \left[1 - \chi\left(\frac{|\xi - \eta|}{|\xi + \eta|}\right) - \chi\left(\frac{|\eta|}{|2\xi - \eta|}\right) \right] (|\eta|^2 - |\xi|^2) |\eta|. \end{aligned} \quad (9.42)$$

Notice that (see (8.6) for the definition),

$$\|q_0^{k,k_1,k_2}\|_{S_\Omega^\infty} \lesssim 2^{k_2} 2^{2k_1} \left[2^{-(2k_2-2k_1)} \mathbf{1}_{[-40,\infty)}(k_2 - k_1) + \mathbf{1}_{(-\infty,4]}(k_2) \right], \quad (\Omega_\xi + \Omega_\eta)q_0 = 0. \quad (9.43)$$

Proof. Using (9.32) and (9.39) we have

$$\begin{aligned} &(T_{\sqrt{1+\alpha}}\partial_y - iT_a + T_b)(T_{\sqrt{1+\alpha}}\partial_y - iT_a - T_b)\omega \\ &= \mathcal{Q} + \mathcal{C} + \mathcal{E}\omega - T_{(a^{(0)})^2 + (b^{(0)})^2}\omega - T_{\{\sqrt{1+\alpha}, b^{(1)}\}\varphi_{\leq -1}(\zeta)}i\partial_y\omega. \end{aligned}$$

The terms \mathcal{C} , $T_{(a^{(0)})^2 + (b^{(0)})^2}\omega$ and $T_{\{\sqrt{1+\alpha}, b^{(1)}\}\varphi_{\leq -1}(\zeta)}i\partial_y\omega$ are in $\varepsilon_1^3(1+t)^{-11\delta^2} [L_y^\infty \mathcal{O}_{3,1/2} \cap L_y^2 \mathcal{O}_{3,1}]$. Moreover, using Lemma 9.4, Lemma 8.8, and (9.35)–(9.38), we can verify that

$$\mathcal{E}\omega - [T_{2|\zeta|b_1^{(0)}} - T_{|\zeta|}T_{b_1^{(0)}} - T_{b_1^{(0)}}T_{|\zeta|}]\omega - [i[T_{\zeta \cdot \nabla h}, T_{|\zeta|}] + T_{\{\zeta \cdot \nabla h, |\zeta|\}\varphi_{\geq 0}(\zeta)}]\omega$$

is an acceptable cubic error, where $b_1^{(0)} := -\varphi_{\geq 0}(\zeta) \frac{\zeta_j \zeta_k \partial_j \partial_k h}{2|\zeta|^2}$. Indeed, most of the terms in \mathcal{E} are already acceptable cubic errors; the last three terms become acceptable cubic errors after removing the quadratic components corresponding to the symbols $\zeta \cdot \nabla h$ in $a^{(1)}$, $|\zeta|$ in $b^{(1)}$, and $b_1^{(0)}$ in $b^{(0)}$. As a consequence, $\mathcal{E}\omega - Q'_0 \in \varepsilon_1^3 \langle t \rangle^{-11\delta^2} [L_y^\infty \mathcal{O}_{3,1/2} \cap L_y^2 \mathcal{O}_{3,1}]$, where

$$\begin{aligned} \widehat{Q}'_0(\xi, y) &:= \frac{1}{4\pi^2} \int_{\mathbb{R}^2} \chi\left(\frac{|\xi - \eta|}{|\xi + \eta|}\right) q'_0(\xi, \eta) \widehat{h}(\xi - \eta) \widehat{u}(\eta, y) d\eta, \\ q'_0(\xi, \eta) &:= \frac{(|\xi| - |\eta|)^2 (|\xi| + |\eta|) (\xi \cdot \eta - |\xi||\eta|)}{|\xi + \eta|^2} \varphi_{\geq 0}\left(\frac{\xi + \eta}{2}\right) + \frac{(|\xi| - |\eta|)^2 (|\xi| + |\eta|)}{2} \varphi_{\leq -1}\left(\frac{\xi + \eta}{2}\right). \end{aligned}$$

The desired conclusions follow, using also the formula $\mathcal{Q} = 2\mathcal{H}(\nabla h, \nabla \partial_y u) + \mathcal{H}(\Delta h, \partial_y u)$ in (9.33), and the approximations $\partial_y u \approx |\nabla|u$, $\omega \approx u$, up to suitable quadratic errors. \square

In order to continue we want to invert the first operator in (9.40) which is elliptic in the domain under consideration.

Lemma 9.6. *Let $U := (T_{\sqrt{1+\alpha}}\partial_y - iT_a - T_b)\omega \in \varepsilon_1 [L_y^\infty \mathcal{O}_{1,-1/2} \cap L_y^2 \mathcal{O}_{1,0}]$, so*

$$(T_{\sqrt{1+\alpha}}\partial_y - iT_a + T_b)U = Q_0 + \tilde{\mathcal{C}}. \quad (9.44)$$

Define

$$\widehat{M_0[f, g]}(\xi) = \frac{1}{4\pi^2} \int_{\mathbb{R}^2} m_0(\xi, \eta) \widehat{f}(\xi - \eta) \widehat{g}(\eta) d\eta, \quad m_0(\xi, \eta) := \frac{q_0(\xi, \eta)}{|\xi| + |\eta|}. \quad (9.45)$$

Then, recalling the notation (8.7), and letting $U_0 := U|_{y=0}$, $u_0 := u|_{y=0} = \phi$, we have

$$P_{\geq -10}(U_0 - M_0[h, u_0]) \in \varepsilon_1^3 \langle t \rangle^{-\delta^2} \mathcal{O}_{3,3/2}. \quad (9.46)$$

Proof. Set

$$\tilde{U} := T_{(1+\alpha)^{1/4}} U \in \varepsilon_1 [L_y^\infty \mathcal{O}_{1,-1/2} \cap L_y^2 \mathcal{O}_{1,0}], \quad \sigma := \frac{b-ia}{\sqrt{1+\alpha}} = |\zeta| (1 + \varepsilon_1 \mathcal{M}_{N_3-1}^{0,1}). \quad (9.47)$$

Using (9.44) and Lemma 8.8, and letting $f := (1+\alpha)^{1/4} - 1 \in \varepsilon_1^2 \mathcal{O}_{2,0}$, we calculate

$$\begin{aligned} T_{(1+\alpha)^{1/4}} (\partial_y + T_\sigma) \tilde{U} &= Q_0 + \mathcal{C}_1, \\ \mathcal{C}_1 &:= \tilde{\mathcal{C}} + [T_f^2 - T_{f^2}] \partial_y U + [T_{f+1} T_\sigma T_{f+1} - T_{(f+1)^2 \sigma}] U \in \varepsilon_1^3 \langle t \rangle^{-11\delta^2} [L_y^\infty \mathcal{O}_{3,1/2} \cap L_y^2 \mathcal{O}_{3,1}]. \end{aligned}$$

Let $g = (1+f)^{-1} - 1 \in \varepsilon_1^2 \mathcal{O}_{2,0}$ and apply the operator T_{1+g} to the identity above. Using Lemma 8.8, it follows that

$$(\partial_y + T_\sigma) \tilde{U} = Q_0 + \mathcal{C}_2, \quad \mathcal{C}_2 \in \varepsilon_1^3 \langle t \rangle^{-11\delta^2} [L_y^\infty \mathcal{O}_{3,1/2} \cap L_y^2 \mathcal{O}_{3,1}]. \quad (9.48)$$

Notice that, using Lemma 9.4, (9.43), (9.45) and Lemma 8.2,

$$M_0[h, u] \in \varepsilon_1^2 [L_y^\infty \mathcal{O}_{2,5/2} \cap L_y^2 \mathcal{O}_{2,3}], \quad M_0[h, \partial_y u] \in \varepsilon_1^2 [L_y^\infty \mathcal{O}_{2,3/2} \cap L_y^2 \mathcal{O}_{2,2}]. \quad (9.49)$$

We define $V := \tilde{U} - M_0[h, u]$. Since

$$V = T_{(1+\alpha)^{1/4}} U - M_0[h, u] = T_{(1+\alpha)^{1/4}} (U - M_0[h, u]) + \mathcal{C}', \quad \mathcal{C}' \in \varepsilon_1^3 \langle t \rangle^{-11\delta^2} L_y^\infty \mathcal{O}_{3,3/2},$$

for (9.46) it suffices to prove that

$$P_{\geq -20} V(y) \in \varepsilon_1^3 \langle t \rangle^{-\delta^2} \mathcal{O}_{3,3/2} \quad \text{for any } y \in (-\infty, 0]. \quad (9.50)$$

Using also (9.24) we verify that

$$\begin{aligned} (\partial_y + T_\sigma) V &= (\partial_y + T_\sigma) \tilde{U} - (\partial_y + |\nabla|) M_0[h, u] - T_{(\sigma-|\zeta|)} M_0[h, u] \\ &= \mathcal{C}_2 + M_0[h, |\nabla| u - \partial_y u] - T_{(\sigma-|\zeta|)} M_0[h, u] \\ &= \mathcal{C}_3 \in \varepsilon_1^3 \langle t \rangle^{-11\delta^2} [L_y^\infty \mathcal{O}_{3,1/2} \cap L_y^2 \mathcal{O}_{3,1}]. \end{aligned} \quad (9.51)$$

Letting $\sigma' := \sigma - |\zeta|$ and $V_k := P_k V$, $k \in \mathbb{Z}$, we calculate

$$(\partial_y + T_{|\zeta|}) V_k = P_k \mathcal{C}_3 - P_k T_{\sigma'} V.$$

We can rewrite this equation in integral form,

$$V_k(y) = \int_{-\infty}^y e^{(s-y)|\nabla|} [P_k \mathcal{C}_3(s) - P_k T_{\sigma'} V(s)] ds. \quad (9.52)$$

To prove the desired bound for the high Sobolev norm, let, for $k \in \mathbb{Z}$,

$$X_k := \sup_{y \leq 0} 2^{(N_0+3/2)k} \|V_k(y)\|_{L^2}.$$

Since $\sigma'/|\zeta| \in \varepsilon_1 \mathcal{M}_{N_3-1}^{0,1}$, it follows from Lemma 8.7 that, for any $y \leq 0$,

$$\begin{aligned} &2^{(N_0+3/2)k} \int_{-\infty}^y \|e^{(s-y)|\nabla|} P_k T_{\sigma'} V(s)\|_{L^2} ds \\ &\lesssim 2^{(N_0+3/2)k} \varepsilon_1 \sum_{|k'-k| \leq 4} \int_{-\infty}^y e^{(s-y)2^{k-4}} 2^k \|P_{k'} V(s)\|_{L^2} ds \lesssim \varepsilon_1 \sum_{|k'-k| \leq 4} X_{k'}. \end{aligned}$$

It follows from (9.52) that, for any $k \in \mathbb{Z}$

$$\begin{aligned} X_k &\lesssim \varepsilon_1 \sum_{|k'-k| \leq 4} X_{k'} + \sup_{y \leq 0} 2^{(N_0+3/2)k} \int_{-\infty}^y e^{(s-y)2^{k-4}} \|P_k \mathcal{C}_3(s)\|_{L^2} ds \\ &\lesssim \varepsilon_1 \sum_{|k'-k| \leq 4} X_{k'} + 2^{(N_0+1)k} \left[\int_{-\infty}^0 \|P_k \mathcal{C}_3(s)\|_{L^2}^2 ds \right]^{1/2}. \end{aligned}$$

We take l^2 summation in k , and absorb the first term in the right-hand side⁷ into the left-hand side, to conclude that

$$\left(\sum_{k \in \mathbb{Z}} X_k^2 \right)^{1/2} \lesssim \left[\sum_{k \in \mathbb{Z}} 2^{2(N_0+1)k} \int_{-\infty}^0 \|P_k \mathcal{C}_3(s)\|_{L^2}^2 ds \right]^{1/2} \lesssim \varepsilon_1^3 \langle t \rangle^{-11\delta^2} \langle t \rangle^{-2(5/6-20\delta^2)+\delta^2}, \quad (9.53)$$

where the last inequality in this estimate is a consequence of $\mathcal{C}_3 \in \varepsilon_1^3 \langle t \rangle^{-11\delta^2} L_y^2 \mathcal{O}_{3,1}$. The desired bound $\|P_{\geq -20} V(y)\|_{H^{N_0+3/2}} \lesssim \varepsilon_1^3 \langle t \rangle^{-11\delta^2} \langle t \rangle^{-2(5/6-20\delta^2)+\delta^2}$ in (9.50) follows.

The proof of the bound for the weighted norms is similar. For $k \in \mathbb{Z}$ let

$$Y_k := \sup_{y \leq 0} 2^{(N_3+3/2)k} \sum_{j \leq N_1} \|\Omega^j V_k(y)\|_{L^2}.$$

As before, we have the bounds,

$$2^{(N_3+3/2)k} \int_{-\infty}^y \|e^{(s-y)|\nabla|} \Omega^j P_k T_{\sigma'} V(s)\|_{L^2} ds \lesssim \varepsilon_1 \sum_{|k'-k| \leq 4} [Y_{k'} + \langle t \rangle^{6\delta^2} X_{k'}],$$

for any $y \in (-\infty, 0]$ and $j \leq N_1$, and therefore, using (9.52),

$$Y_k \lesssim \varepsilon_1 \sum_{|k'-k| \leq 4} Y_{k'} + \varepsilon_1 \langle t \rangle^{6\delta^2} \sum_{|k'-k| \leq 4} X_{k'} + \sum_{j \leq N_1} 2^{(N_3+1)k} \left[\int_{-\infty}^0 \|\Omega^j P_k \mathcal{C}_3(s)\|_{L^2}^2 ds \right]^{1/2}.$$

As before, we take the l^2 sum in k and use (9.53) and the hypothesis $\mathcal{C}_3 \in \varepsilon_1^3 \langle t \rangle^{-11\delta^2} L_y^2 \mathcal{O}_{3,1}$. The desired bound $\|P_{\geq -20} V(y)\|_{H_{\Omega}^{N_1, N_3+3/2}} \lesssim \varepsilon_1^3 \langle t \rangle^{-4\delta^2} \langle t \rangle^{-2(5/6-20\delta^2)+\delta^2}$ in (9.50) follows.

Finally, for the L^∞ bound, we let, for $k \in \mathbb{Z}$,

$$Z_k := \sup_{y \leq 0} 2^{(N_2+3/2)k} \sum_{j \leq N_1/2} \|\Omega^j V_k(y)\|_{L^\infty}.$$

As before, using (9.52) it follows that

$$Z_k \lesssim \varepsilon_1 \sum_{|k'-k| \leq 4} Z_{k'} + \sum_{j \leq N_1/2} 2^{(N_2+1)k} \left[\int_{-\infty}^0 \|\Omega^j P_k \mathcal{C}_3(s)\|_{L^\infty}^2 ds \right]^{1/2}.$$

After taking l^2 summation in k it follows that

$$\left(\sum_{k \in \mathbb{Z}} Z_k^2 \right)^{1/2} \lesssim \sum_{j \leq N_1/2} \left[\sum_{k \in \mathbb{Z}} 2^{2(N_2+1)k} \int_{-\infty}^0 \|\Omega^j P_k \mathcal{C}_3(s)\|_{L^\infty}^2 ds \right]^{1/2} \lesssim \varepsilon_1^3 \langle t \rangle^{-11\delta^2} \langle t \rangle^{-5/2+45\delta^2},$$

⁷To make this step rigorous, one can modify the definition of X_k to $X'_k := \sup_{y \leq 0} 2^{(N_0+3/2)\min(k, K)} \|V_k(y)\|_{L^2}$, in order to make sure that $\sum_k (X'_k)^2 < \infty$, and then prove uniform estimates in K and finally let $K \rightarrow \infty$.

where the last inequality is a consequence of $\mathcal{C}_3 \in \varepsilon_1^3 \langle t \rangle^{-11\delta^2} L_y^2 \mathcal{O}_{3,1}$. The desired bound on $\|P_{\geq -20} V(y)\|_{\widetilde{W}_\Omega^{N_1/2, N_2+3/2}}$ in (9.50) follows, once we recall that only the sum over $2^{|k|} \leq \langle t \rangle^8$ is relevant when estimating the $\widetilde{W}_\Omega^{N_1/2, N_2+3/2}$ norm; the remaining frequencies are already accounted for by the stronger Sobolev norms. \square

We are now ready to obtain the parilinearization of the Dirichlet-Neumann operator.

Proof of Proposition 9.1. Recall that $G(h)\phi = (1 + |\nabla h|^2)\partial_y u|_{y=0} - \nabla h \cdot \nabla u|_{y=0}$, see (9.12), and $B = \partial_y u|_{y=0}$. All the calculations below are done on the interface, at $y = 0$. We observe that, using Corollary 9.7,

$$\begin{aligned} P_{\leq 6}((1 + |\nabla h|^2)\partial_y u - \nabla h \cdot \nabla u) &= P_{\leq 6}(\partial_y u - \nabla h \cdot \nabla u) + \varepsilon_1^3 \mathcal{O}_{3,3/2} \\ &= P_{\leq 6}(|\nabla|\omega - \operatorname{div}(T_V h)) + P_{\leq 6}(\operatorname{div}(T_V h) + |\nabla|T_{|\nabla|\omega} h + Q[h, \omega]) + \varepsilon_1^3 \mathcal{O}_{3,3/2}. \end{aligned} \quad (9.54)$$

Thus low frequencies give acceptable contributions. To estimate high frequencies we compute

$$\begin{aligned} &(1 + |\nabla h|^2)\partial_y u - \nabla h \cdot \nabla u \\ &= T_{1+\alpha}\partial_y u - T_{\nabla h}\nabla u - T_{\nabla u}\nabla h + T_{\partial_y u}\alpha + \mathcal{H}(\alpha, \partial_y u) - \mathcal{H}(\nabla h, \nabla u) \\ &= T_{1+\alpha}\partial_y \omega - T_{\nabla h}\nabla \omega - T_{\nabla u}\nabla h + T_{\nabla h}T_{\partial_y u}\nabla h \\ &+ (T_{\partial_y u}\alpha - 2T_{\nabla h}T_{\partial_y u}\nabla h) + T_{1+\alpha}T_{\partial_y^2 u}h - T_{\nabla h}T_{\nabla \partial_y u}h + \mathcal{H}(\alpha, \partial_y u) - \mathcal{H}(\nabla h, \nabla u). \end{aligned}$$

Using Lemma 9.6 with $U = (T_{\sqrt{1+\alpha}}\partial_y - iT_a - T_b)\omega$ and (9.49), Lemma 8.7, and Lemma 8.8, we find that

$$\begin{aligned} T_{1+\alpha}\partial_y \omega &= T_{\sqrt{1+\alpha}}(iT_a\omega + T_b\omega + M_0[h, u] + \mathcal{C}') + (T_{1+\alpha} - T_{\sqrt{1+\alpha}}^2)\partial_y \omega \\ &= T_{\sqrt{1+\alpha}}(T_b + iT_a)\omega + M_0[h, u] + \mathcal{C}'', \end{aligned}$$

where \mathcal{C}'' satisfies $P_{\geq -6}\mathcal{C}'' \in \varepsilon_1^3 \mathcal{O}_{3,3/2}$. Therefore, with $V = \nabla u - \partial_y u \nabla h$,

$$\begin{aligned} (1 + |\nabla h|^2)\partial_y u - \nabla h \cdot \nabla u &= T_{\sqrt{1+\alpha}}(T_b + iT_a)\omega + M_0[h, u] + \mathcal{C}'' \\ &- T_{\nabla h}\nabla \omega - \operatorname{div}(T_V h) + \mathcal{C}_1 + \mathcal{C}_2 - \mathcal{H}(\nabla h, \nabla u), \end{aligned} \quad (9.55)$$

with cubic terms $\mathcal{C}_1, \mathcal{C}_2$ given explicitly by

$$\begin{aligned} \mathcal{C}_1 &= (T_{\partial_y u}\alpha - 2T_{\nabla h}T_{\partial_y u}\nabla h) + \mathcal{H}(\alpha, \partial_y u), \\ \mathcal{C}_2 &= (T_{\operatorname{div} V} + T_{1+\alpha}T_{\partial_y^2 u} - T_{\nabla h}T_{\nabla \partial_y u})h + (T_{\nabla h}T_{\partial_y u} - T_{\partial_y u}\nabla h)\nabla h. \end{aligned}$$

Notice that $\operatorname{div} V + (1 + \alpha)\partial_y^2 u - \nabla h \nabla \partial_y u = 0$, as a consequence of (9.13). Using also Lemma 8.8 it follows that $\mathcal{C}_1, \mathcal{C}_2 \in \varepsilon_1^3 \mathcal{O}_{3,3/2}$.

Moreover, using the formulas (9.36), (9.38), Lemma (8.5), and Lemma 8.8, we see that

$$\begin{aligned} T_{\sqrt{1+\alpha}}T_b\omega &= T_{b\sqrt{1+\alpha}}\omega + \frac{i}{2}T_{\{\sqrt{1+\alpha}, b\}}\omega + E(\sqrt{1+\alpha} - 1, b)\omega \\ &= T_{\lambda^{(1)}}\omega + T_{b^{(0)}\sqrt{1+\alpha}}\omega + \frac{i}{2}T_{\{\sqrt{1+\alpha}, b^{(1)}\}}\omega + \varepsilon_1^3 \mathcal{O}_{3,3/2} \end{aligned}$$

where $\lambda^{(1)}$ is the principal symbol in (9.5). Similarly, using (9.35), (9.37),

$$\begin{aligned} iT_{\sqrt{1+\alpha}}T_a\omega - T_{\nabla h}\nabla \omega &= T_{i\zeta \cdot \nabla h}\omega - T_{\nabla h}\nabla \omega + iT_{a^{(0)}\sqrt{1+\alpha}}\omega - \frac{1}{2}T_{\{\sqrt{1+\alpha}, a\}}\omega + iE(\sqrt{1+\alpha} - 1, a)\omega \\ &= \frac{1}{2}T_{\Delta h}\omega + iT_{a^{(0)}\sqrt{1+\alpha}}\omega - \frac{1}{2}T_{\{\sqrt{1+\alpha}, a^{(1)}\}}\omega + \varepsilon_1^3 \mathcal{O}_{3,3/2}. \end{aligned}$$

Summing these last two identities and using (9.35)-(9.38) we see that

$$T_{\sqrt{1+\alpha}}T_b\omega + iT_{\sqrt{1+\alpha}}T_a\omega - T_{\nabla h}\nabla\omega = T_{\lambda^{(1)}}\omega + T_m\omega + \varepsilon_1^3\mathcal{O}_{3,3/2} \quad (9.56)$$

where

$$\begin{aligned} m &:= b^{(0)}\sqrt{1+\alpha} - \frac{1}{2}\{\sqrt{1+\alpha}, a^{(1)}\} + \frac{1}{2}\Delta h \\ &= \frac{(1+\alpha)^{3/2}}{2\lambda^{(1)}}\left\{\frac{\lambda^{(1)}}{\sqrt{1+\alpha}}, \frac{\zeta \cdot \nabla h}{1+\alpha}\right\}\varphi_{\geq 0}(\zeta) - \frac{1}{2}\left\{\sqrt{1+\alpha}, \frac{\zeta \cdot \nabla h}{\sqrt{1+\alpha}}\right\} + \frac{1}{2}\Delta h \\ &= \lambda^{(0)} - \frac{1}{2}\left\{\sqrt{1+\alpha}, \frac{\zeta \cdot \nabla h}{\sqrt{1+\alpha}}\right\}\varphi_{\leq -1}(\zeta) + \frac{\Delta h}{2}\varphi_{\leq -1}(\zeta). \end{aligned} \quad (9.57)$$

We conclude from (9.55) and (9.56) that

$$P_{\geq 7}((1 + |\nabla h|^2)\partial_y u - \nabla h \nabla u) = P_{\geq 7}(T_{\lambda_{DN}}\omega - \operatorname{div}(T_V h) + M_0[h, u] - \mathcal{H}(\nabla h, \nabla u) + \varepsilon_1^3\tilde{\mathcal{O}}_{3,3/2}).$$

Moreover, the symbol of the bilinear operator $M_0[h, u] - \mathcal{H}(\nabla h, \nabla u)$ is

$$\frac{q_0(\xi, \eta)}{|\xi| + |\eta|} + \left[1 - \chi\left(\frac{|\xi - \eta|}{|\xi + \eta|}\right) - \chi\left(\frac{|\eta|}{|2\xi - \eta|}\right)\right](\xi - \eta) \cdot \eta,$$

where q_0 is defined in (9.42). The symbol bounds (9.7) follow. Combining this with (9.54), we finish the proof. \square

9.3. Taylor expansion. We conclude this section with a simple expansion of the Dirichlet-Neumann operator that identifies the linear and the quadratic terms.

Corollary 9.7. (i) Assume that $\|\langle \nabla \rangle h\|_{\mathcal{O}_{1,0}} + \|\|\nabla\|^{1/2}\psi\|_{\mathcal{O}_{1,0}} \lesssim \varepsilon_1$ and $\mathbf{e}_a = 0$, $\mathbf{e}_b = 0$, and define u as in Lemma 9.4. Then we have an expansion

$$\partial_y u = |\nabla|u + \nabla h \cdot \nabla u + N_2[h, u] + \mathcal{E}^{(3)}, \quad \|\mathcal{E}^{(3)}\|_{L_y^2\mathcal{O}_{3,0} \cap L_y^\infty\mathcal{O}_{3,-1/2}} \lesssim \varepsilon_1^3 \langle t \rangle^{-11\delta^2}, \quad (9.58)$$

where

$$\mathcal{F}\{N_2[h, \phi]\}(\xi) = \frac{1}{4\pi^2} \int_{\mathbb{R}^2} n_2(\xi, \eta) \widehat{h}(\xi - \eta) \widehat{\phi}(\eta) d\eta, \quad n_2(\xi, \eta) := \xi \cdot \eta - |\xi||\eta|, \quad (9.59)$$

In particular,

$$\|G(h)\psi - |\nabla|\psi - N_2[h, \psi]\|_{\mathcal{O}_{3,-1/2}} \lesssim \varepsilon_1^3 \langle t \rangle^{-11\delta^2}. \quad (9.60)$$

Moreover

$$\|n_2^{k,k_1,k_2}\|_{S_\Omega^\infty} \lesssim 2^{\min\{k,k_1\}} 2^{k_2}, \quad (\Omega_\xi + \Omega_\eta)n_2 \equiv 0. \quad (9.61)$$

(ii) As in Proposition 2.2, assume that $(h, \phi) \in C([0, T] : H^{N_0+1} \times \dot{H}^{N_0+1/2, 1/2})$ is a solution of the system (2.1) with $g = 1$ and $\sigma = 1$, $t \in [0, T]$ is fixed, and (9.2) holds. Then

$$\|\partial_t(G(h)\phi) - |\nabla|\partial_t\phi\|_{\mathcal{O}_{2,-2}} \lesssim \varepsilon_1^2. \quad (9.62)$$

Proof. (i) Let $u^{(1)} := e^{y|\nabla|}\psi$ and $Q_a^{(1)} := \nabla u^{(1)} \cdot \nabla h$, $Q_b^{(1)} := \mathcal{R}(\partial_y u^{(1)} \nabla h)$. It follows from (9.18)–(9.19) and Lemma 9.4 (more precisely, from (9.22), (9.24), and (9.27)) that

$$\begin{aligned} &\|\|\nabla\|^{1/2}(u - u^{(1)})\|_{L_y^\infty\mathcal{O}_{2,0}} + \|\|\nabla\|(u - u^{(1)})\|_{L_y^2\mathcal{O}_{2,0}} \\ &\quad + \|\partial_y(u - u^{(1)})\|_{L_y^\infty\mathcal{O}_{2,-1/2}} + \|\partial_y(u - u^{(1)})\|_{L_y^2\mathcal{O}_{2,0}} \lesssim \varepsilon_1^2. \end{aligned} \quad (9.63)$$

Therefore, using Lemma 8.2, for $d \in \{a, b\}$,

$$\|Q_d - Q_d^{(1)}\|_{L_y^\infty\mathcal{O}_{3,-1/2}} + \|Q_d - Q_d^{(1)}\|_{L_y^2\mathcal{O}_{3,0}} \lesssim \varepsilon_1^3 \langle t \rangle^{-12\delta^2}. \quad (9.64)$$

Therefore, using (9.18)–(9.19) and (9.26),

$$\left\| \partial_y u - |\nabla|u - \nabla h \cdot \nabla u - \int_{-\infty}^y |\nabla|e^{-|s-y||\nabla|}(Q_b^{(1)}(s) - Q_a^{(1)}(s))ds \right\|_{L_y^2 \mathcal{O}_{3,0} \cap L_y^\infty \mathcal{O}_{3,-1/2}} \lesssim \varepsilon_1^3 \langle t \rangle^{-11\delta^2}.$$

Since

$$\mathcal{F}\{Q_b^{(1)}(s) - Q_a^{(1)}(s)\}(\xi) = \frac{1}{4\pi^2} \int_{\mathbb{R}^2} \left[\eta \cdot (\xi - \eta) - \frac{\xi \cdot (\xi - \eta)}{|\xi|} |\eta| \right] \widehat{h}(\xi - \eta) e^{s|\eta|} \widehat{\psi}(\eta) d\eta,$$

we have

$$\begin{aligned} & \mathcal{F}\left\{ \int_{-\infty}^y |\nabla|e^{-|s-y||\nabla|}(Q_b^{(1)}(s) - Q_a^{(1)}(s))ds \right\}(\xi) \\ &= \frac{1}{4\pi^2} \int_{\mathbb{R}^2} \left[\eta \cdot (\xi - \eta) - \frac{\xi \cdot (\xi - \eta)}{|\xi|} |\eta| \right] \frac{|\xi|}{|\xi| + |\eta|} \widehat{h}(\xi - \eta) e^{y|\eta|} \widehat{\psi}(\eta) d\eta \\ &= \mathcal{F}\{N_2[h, u^{(1)}]\}(\xi). \end{aligned}$$

Moreover, using the assumption $\|\langle \nabla \rangle h\|_{\mathcal{O}_{1,0}} \lesssim \varepsilon_1$ and the bounds (9.63), we have

$$\|N_2[h, u - u^{(1)}]\|_{L_y^2 \mathcal{O}_{3,0} \cap L_y^\infty \mathcal{O}_{3,-1/2}} \lesssim \varepsilon_1^3 \langle t \rangle^{-11\delta^2},$$

as a consequence of Lemma 8.2. The desired identity (9.58) follows. The bound (9.60) follows using also the identity (9.12).

(ii) We define $u = u(x, t, y)$ as in (9.11), let $v = \partial_t u$, differentiate (9.13) with respect to t , and find that v satisfies (9.14) with

$$\mathbf{e}_a = \nabla_x u \cdot \nabla_x \partial_t h - 2\partial_y u \nabla_x h \cdot \nabla_x \partial_t h, \quad \mathbf{e}_b = \mathcal{R}(\partial_y u \nabla_x \partial_t h).$$

In view of (9.60),

$$\|\partial_t h\|_{\mathcal{O}_{1,-1/2}} + \|\partial_t \phi\|_{\mathcal{O}_{1,-1}} \lesssim \varepsilon_1.$$

Therefore the triplet $(\partial_t \phi, \mathbf{e}_a, \mathbf{e}_b)$ satisfies (9.23) with $p = -3/2$. Therefore, using (9.24),

$$\|\partial_y v - |\nabla|v\|_{L_y^\infty \mathcal{O}_{2,-2}} \lesssim \varepsilon_1^2,$$

and the desired bound (9.62) follows using also (9.12). \square

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