A NOTE ON THE ASYMPTOTIC BEHAVIOR OF 2D GRAVITY WATER WAVES

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ABSTRACT. In this note we explain how to derive an asymptotic formula for solutions of the 2d gravity water waves system expressed in Eulerian coordinates, using some of the bounds obtained by the authors in [IPu13]. The main ingredients of the proof are: 1) the uniform estimates for the Fourier transform of the profile of solutions obtained in [IPu13], 2) a refined linear estimate for the propagator $\exp(it|\partial_x|^{1/2})$, and 3) an argument similar to the one used by Hayashi and Naumkin in [HN98] in the context of NLS type equations.

1. INTRODUCTION

Let Ω_t be the region with free boundary occupied by a perfect fluid with velocity $v(t,z)$, $z \in \Omega_t$. Assume that $\Omega_t \subset \mathbb{R}^2$ is the region the graph of a function $h : \mathbb{R}_t \times \mathbb{R}_x \to \mathbb{R}$, that is $\Omega_t = \{(x,y) \in \mathbb{R}^2 \,:\, y \leq h(t,x)\}$ and $S_t = \{(x,y) \,:\, y = h(t,x)\}$. Let us denote by Φ the velocity potential: $\nabla\Phi(t,x,y) = v(t,x,y)$, for $(x,y) \in \Omega_t$. If $\phi(t,x) := \Phi(t,x,h(x,t))$ is the restriction of Φ to the boundary S_t, the equations of motion reduce to the following system for the unknowns[1] $h, \phi : \mathbb{R}_t \times \mathbb{R}_x \to \mathbb{R}$:

$$\begin{cases} \partial_t h = G(h)\phi \\ \\ \partial_t \phi = -h - \frac{1}{2}|\phi_x|^2 + \frac{1}{2(1+|h_x|^2)}(G(h)\phi + h_x\phi_x)^2 \end{cases} \tag{1.1}$$

with

$$G(h) := \sqrt{1+|h_x|^2}\,\mathcal{N}(h) \tag{1.2}$$

where $\mathcal{N}(h)$ is the Dirichlet-Neumann operator associated to the domain Ω_t.

2. GLOBAL EXISTENCE AND ASYMPTOTIC BEHAVIOR

Let $N_0 = 10^4$ and $N_1 = \frac{N_0}{2} + 4$. Let $S = \frac{1}{2}t\partial_t + \alpha\partial_\alpha$ be the scaling vector field. Given a time interval I and a function $f : I \times \mathbb{R} \to \mathbb{C}$ we define the norms

$$\|f(t)\|_{X_k} \stackrel{def}{=} \|f(t)\|_{H^k} + \|Sf(t)\|_{H^{k/2}}\,. \tag{2.1}$$

$$\|f(t)\|_Z \stackrel{def}{=} \sup_{\xi\in\mathbb{R}} \left| \left(|\xi|^\beta + |\xi|^{N_1+15}\right) \widehat{f}(\xi,t) \right|\,, \tag{2.2}$$

where $\beta = 10^{-2}$.

The following global existence result has been obtained in [IPu13, Theorem 1.1 (i)]:

Theorem 2.1. *Let $h_0(x) = h(0,x)$ be the initial height of the surface S_0, and let $\phi_0(x) = \phi(0,x)$ be the restriction to S_0 of the initial velocity potential. Assume that at the initial time one has*

$$\|(h_0, \Lambda\phi_0)\|_{H^{N_0+2}} + \|x\partial_x(h_0,\Lambda\phi_0)\|_{H^{N_0/2+1}} + \|h_0 + i\Lambda\phi_0\|_Z \leq \varepsilon_0\,, \tag{2.3a}$$

where Z is defined by (2.2) and $\Lambda \stackrel{def}{=} |\partial_x|^{1/2}$. Moreover, for $x \in \Omega_0$ let $v_0(x) = v(0,x)$, where v is the irrotational and divergence free velocity field of the fluid, and assume that

$$\||x|\nabla v_0\|_{H^{N_0/2}(\Omega_0)} \leq \varepsilon_0\,. \tag{2.3b}$$

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[1] We refer to [SS99, chap. 11] or [CS93] for the derivation of the water wave equations (1.1) from the free boundary Euler's equations.

Then there exists $\overline{\varepsilon}_0$ small enough, such that for any $\varepsilon_0 \leq \overline{\varepsilon}_0$, the initial value problem associated to (1.1) *admits a unique global solution with*

$$\sup_t \Big[(1+t)^{-p_0} \|(h(t), \phi_x(t))\|_{X_{N_0}} + \|h(t) + i\Lambda\phi(t)\|_{H^{N_1+10}} + \sqrt{1+t}\|(h(t), \Lambda\phi(t))\|_{W^{N_1+4,\infty}} \Big] \lesssim \varepsilon_0 \,,$$

where $p_0 = 10^{-4}$.

A similar global existence result has also been obtained by Alazard and Delort in [AD13]. The following statement about modified scattering is also contained in In [IPu13, Theorem 1.1 (ii)]:

Proposition 2.2. *With the above notations, let $u(t) \stackrel{def}{=} h(t) + i\Lambda\phi(t)$. Define*

$$G(\xi, t) := \frac{|\xi|^4}{\pi} \int_0^t |\widehat{u}(\xi, s)|^2 \frac{ds}{s+1} \,, \qquad t \in [0, T] \,.$$

Then there is $0 < p_1 < p_0$ such that

$$(1+t_1)^{p_1} \Big\| (1+|\xi|)^{N_1} \big[e^{iG(\xi,t_2)} e^{it_2\Lambda(\xi)} \widehat{u}(\xi, t_2) - e^{iG(\xi,t_1)} e^{it_1\Lambda(\xi)} \widehat{u}(\xi, t_1) \big] \Big\|_{L^2_\xi} \lesssim \varepsilon_0 \,, \tag{2.4}$$

for any $t_1 \leq t_2 \in [0, T]$. In particular, there is $w_\infty \in L^2$ with the property that

$$\sup_{t \in [0,\infty)} (1+t)^{p_1} \Big\| (1+|\xi|)^{N_1} e^{iG(\xi,t)} e^{it\Lambda(\xi)} \widehat{u}(\xi, t) - w_\infty(\xi) \Big\|_{L^2_\xi} \lesssim \varepsilon_0 \,. \tag{2.5}$$

The purpose of this note is to show how to derive an asymptotic formula for the solution $u(t, x)$ in physical coordinates from the bounds proven in [IPu13], in the course of the proofs of Theorem 2.1 and Proposition 2.2 above. More precisely, one can show

Proposition 2.3. *Let $u(t) := h(t) + i\Lambda\phi(t)$. Under the same assumption of Theorem 2.1 there exists a uniformly bounded function f_∞, such that*

$$\left| u(t,x) - \frac{e^{-it|t/4x|}}{\sqrt{1+|t|}} f_\infty\left(\frac{x}{t}\right) \exp\left(-\frac{i}{64} \frac{|f_\infty(x/t)|^2}{|x/t|^5} \log(1+|t|) \right) \right| \lesssim \varepsilon_0 (1+|t|)^{-1/2-p_1/2} \,. \tag{2.6}$$

The same asymptotic formula above has been derived in [AD13] under slightly different assumptions[2].

The proof of Proposition 2.3 is based on the following ingredients:

1. The bounds contained in [IPu13], which are stated in Propostion 4.1 below;
2. A refined linear estimate for the propagator $e^{it\Lambda}$, essentially contained in the proof of Lemma 2.3 in the author's work [IPu12];
3. A simple argument used by Hayashi and Naumkin [HN98] to deal with NLS type equations, as well as many other models, see for example [HN99a], [HN99b], and [HN99a][3].

3. Linear Estimates

A important but simple ingredient in the proof of the Theorem 2.1, and in the result of [IPu12], was the following linear estimate, proved by the authors in Lemma 2.3 of [IPu12]:

Lemma 3.1. *For any $t \in \mathbb{R}$ we have*

$$\|e^{it\Lambda} f\|_{L^\infty} \lesssim (1+|t|)^{-1/2} \| \, |\xi|^{3/4} \widehat{f}(\xi)\|_{L^\infty_\xi} + (1+|t|)^{-5/8} \big[\|x\partial_x f\|_{L^2} + \|f\|_{H^2} \big] \,. \tag{3.1}$$

[2] In [AD13] the authors seem to assume stronger decay at spatial infinity than what is assumed in (2.3a). For example, a smooth intial data of the form $\varepsilon u_0 = \varepsilon(h_0 + i\Lambda\phi_0)$, with ε sufficiently small, and u_0 behaving at spatial infinity like $\sin(x)/x^2$, satisfies the hypotheses of Theorem 2.1. However, it does not comply with the hypotheses in [AD13], since $(x\partial_x)^2 u_0 \notin L^2$.

[3] This argument appears to be quite standard, and together with refined liner estimates allows one to derive an asymptotic formula for solutions in real space, from asymptotic statements about the fourier transform of their profiles. It seems to be applicable to every situtation where there is a logarithmic phase correction to scattering which only depends on the modulus of the transform of the solution.

A careful inspection of the proof of the above lemma shows that the following refinement holds:

Lemma 3.2. *For any $t, x \in \mathbb{R}$ we have*

$$\begin{aligned}\left| e^{-it\Lambda}f(x) - \frac{\sqrt{2}}{\sqrt{\pi i(1+|t|)}} e^{-i\frac{t^2}{4x}\operatorname{sign}\frac{x}{t}} \left(\frac{t^2}{4x^2}\right)^{3/4} \widehat{f}\left(\frac{t^2}{4x^2}\right) \right| \\ \lesssim (1+|t|)^{-11/20}\big[\||\xi|^{3/4}\widehat{f}\|_{L^\infty} + \|f\|_{H^3} + \|x\partial_x f\|_{L^2}\big] .\end{aligned} \tag{3.2}$$

For completeness we give below a detailed proof of the above statement.

Proof. We fix $\varphi : \mathbb{R} \to [0,1]$ an even smooth function supported in $[-8/5, 8/5]$ equal to 1 in $[-5/4, 5/4]$. Let

$$\varphi_k(x) := \varphi(x/2^k) - \varphi(x/2^{k-1}), \qquad k \in \mathbb{Z},\, x \in \mathbb{R}.$$

More generally, for any $m, k \in \mathbb{Z}$, $m \le k$, we define

$$\varphi_k^{(m)}(x) := \begin{cases} \varphi(x/2^k) - \varphi(x/2^{k-1}), & \text{if } k \ge m+1, \\ \varphi(x/2^k), & \text{if } k = m. \end{cases} \tag{3.3}$$

Let P_k, $k \in \mathbb{Z}$, denote the operator on $\mathbb{R}$ defined by the Fourier multiplier $\xi \to \varphi_k(\xi)$, and let $f_k := P_k f$. We then write

$$e^{-it\Lambda}f(x) = \sum_{k\in\mathbb{Z}} \frac{1}{2\pi}\int_{\mathbb{R}} e^{it\Psi(\xi;x,t)}\widehat{f_k}(\xi)\,d\xi \qquad \text{with} \qquad \Psi(\xi;x,t) \overset{def}{=} -\Lambda(\xi) + \frac{x}{t}\xi\,. \tag{3.4}$$

Sometimes we will just denote the phase by $\Psi(\xi)$. Let $\xi_0 \in \mathbb{R}$ denote the unique solution of the equation $\Psi'(\xi) = 0$, i.e.

$$\xi_0 \overset{def}{=} \operatorname{sign}(t/x)\frac{t^2}{4x^2}. \tag{3.5}$$

Set $C_0 \overset{def}{=} \sqrt{2}/\sqrt{\pi i}$. For (3.2) it suffices to prove that

$$\left| \sum_{k\in\mathbb{Z}} \frac{1}{2\pi}\int_{\mathbb{R}} e^{it\Psi(\xi)}\widehat{f_k}(\xi)\,d\xi - \frac{C_0}{\sqrt{t}} e^{i\frac{t^2}{4x}\operatorname{sign}\frac{x}{t}}|\xi_0|^{3/4}\widehat{f}(\xi_0)\right| \lesssim 1\,, \tag{3.6}$$

for any $t, x \in \mathbb{R}$, and any function f satisfying

$$(1+|t|)^{-11/20}\big[\||\xi|^{3/4}\widehat{f}\|_{L^\infty} + \|x\partial f\|_{L^2} + \|f\|_{H^3}\big] \le 1\,. \tag{3.7}$$

Using only the bound $\|f\|_{H^2} \lesssim (1+|t|)^{11/20}$, we estimate first the contribution of small frequencies,

$$\sum_{2^k \le 2^{10}(1+|t|)^{-11/10}} \Big| \int_{\mathbb{R}} e^{i\Psi(\xi;x,t)}\widehat{f_k}(\xi)\,d\xi\Big| \lesssim \sum_{2^k \le 2^{10}(1+|t|)^{-11/10}} 2^{k/2}\|\widehat{f_k}\|_{L^2} \lesssim 1\,,$$

and the contribution of large frequencies,

$$\sum_{2^k \ge 2^{-10}(1+|t|)^{11/50}} \Big| \int_{\mathbb{R}} e^{i\Psi(\xi;x,t)}\widehat{f_k}(\xi)\,d\xi\Big| \lesssim \sum_{2^k \ge 2^{-10}(1+|t|)^{11/50}} 2^{k/2}\|\widehat{f_k}\|_{L^2} \lesssim 1\,.$$

Therefore, for (3.6) it suffices to prove that

$$\Big| \sum_{\substack{2^k \le 2^{-10}(1+|t|)^{11/50} \\ 2^k \ge 2^{10}(1+|t|)^{-11/10}}} \frac{1}{2\pi}\int_{\mathbb{R}} e^{it\Psi(\xi)}\widehat{f_k}(\xi)\,d\xi - \frac{C_0}{\sqrt{t}} e^{i\frac{t^2}{4x}\operatorname{sign}\frac{x}{t}}|\xi_0|^{3/4}\widehat{f}(\xi_0)\Big| \lesssim 1\,. \tag{3.8}$$

We estimate first the nonstationary contributions. Using (3.7) we see that $\|\widehat{P_k f}\|_{L^2} + 2^k\|\partial(\widehat{P_k f})\|_{L^2} \lesssim |t|^{5/8}$. Note that if $2^{-k/2+4} \leq |x/t|$ or $|x/t| \leq 2^{-k/2-4}$ one has $|\Psi'(\xi)| \gtrsim 2^{-k/2}$. Then we integrate by parts to estimate

$$\Big|\int_{\mathbb{R}} e^{it\Psi(\xi)}\widehat{f}_k(\xi)\,d\xi\Big| \lesssim |t|^{-1}2^{k/2}\cdot\|\partial(\widehat{f}_k)\|_{L^1} + |t|^{-1}2^{-k/2}\cdot\|\widehat{f}_k\|_{L^1} \lesssim 1.$$

Therefore, for (3.8) it suffices to prove

$$\Big|\sum_k \frac{1}{2\pi}\int_{\mathbb{R}} e^{it\Psi(\xi)}\widehat{f}_k(\xi)\,d\xi - \frac{C_0}{\sqrt{t}}e^{i\frac{t^2}{4x}\operatorname{sign}\frac{x}{t}}|\xi_0|^{3/4}\widehat{f}\,(\xi_0)\Big| \lesssim 1, \tag{3.9}$$

where the above sum is only over those indeces k such that

$$2^k \in [2^{10}(1+|t|)^{-11/10}, 2^{-10}(1+|t|)^{11/50}] \cap [2^{-8}t^2/x^2, 2^8t^2/x^2]\,.$$

In these cases clearly $|\xi_0| \approx 2^k$. Let l_0 denote the largest integer with the property that

$$2^{l_0} \leq 2^{3k/4}|t|^{-9/20}$$

and estimate the left-hand side of (3.9) by

$$\Big|\sum_k \frac{1}{2\pi}\int_{\mathbb{R}} e^{it\Psi(\xi)}\widehat{f}_k(\xi)\,d\xi - \frac{C_0}{\sqrt{t}}e^{-i\frac{t^2}{4x}\operatorname{sign}\frac{x}{t}}|\xi_0|^{3/4}\widehat{f}\,(\xi_0)\Big| \leq |J_{l_0}| + \sum_{l>l_0}^{k+100}|J_l|\,, \tag{3.10}$$

where, with the notation in (3.3), we have defined

$$\begin{aligned} J_l(t,x) &\overset{def}{=} \int_{\mathbb{R}} e^{it\Psi(\xi)}\widehat{f}_k(\xi)\varphi_l^{(l_0)}(\xi-\xi_0)\,d\xi\,, \quad \text{for} \quad l > l_0\,,\\ J_{l_0}(t,x) &\overset{def}{=} \int_{\mathbb{R}} e^{it\Psi(\xi+\xi_0)}\widehat{f}_k(\xi+\xi_0)\varphi_{l_0}^{(l_0)}(\xi) - \frac{C_0}{\sqrt{t}}e^{-i\frac{t^2}{4x}\operatorname{sign}\frac{x}{t}}|\xi_0|^{3/4}\widehat{f}\,(\xi_0)\;d\xi\,. \end{aligned} \tag{3.11}$$

To estimate J_l for $l > l_0$, notice that $|\Psi'(\xi)| \gtrsim 2^{-3k/2}2^l$ whenever $|\xi| \approx 2^k$ and $|\xi-\xi_0| \approx 2^l$. We can then integrate by parts to estimate

$$\begin{aligned} |J_l| &\lesssim \frac{1}{|t|2^{-3k/2}2^l}\big[2^{-l}\|\widehat{f}_k(\xi)\cdot\mathbf{1}_{[0,2^{l+4}]}(|\xi-\xi_0|)\|_{L^1_\xi} + \|\partial(\widehat{f}_k)(\xi)\cdot\mathbf{1}_{[0,2^{l+4}]}(|\xi-\xi_0|)\|_{L^1_\xi}\big]\\ &\lesssim |t|^{-1}2^{3k/2}2^{-l}\big[\|\widehat{f}_k\|_{L^\infty_\xi} + 2^{l/2}\|\partial(\widehat{f}_k)\|_{L^2}\big]\\ &\lesssim |t|^{-9/20}2^{3k/4}2^{-l} + |t|^{-9/20}2^{k/2}2^{-l/2} \lesssim 1\,. \end{aligned} \tag{3.12}$$

To estimate J_{l_0} we first notice that since $\Psi'(\xi_0) = 0$ then

$$\big|\Psi(\xi+\xi_0) - \Psi(\xi_0) - \xi^2/(8\xi_0^{3/2})\big| \lesssim 2^{-5k/2}|\xi|^3\,.$$

Therefore, if we define

$$J_1 \overset{def}{=} e^{it\Psi(\xi_0)}\frac{1}{2\pi}\int_{\mathbb{R}} e^{it\xi^2/(8\xi_0^{3/2})}\widehat{f}_k(\xi+\xi_0)\varphi(\xi/2^{l_0})\,d\xi$$

we see that

$$\big|J_{l_0} - J_1\big| \lesssim |t|2^{-5k/2}2^{7l_0/2}\|\widehat{f}_k\|_{L^2} \lesssim 1\,, \tag{3.13}$$

having used the L^2 bound in (3.7).

We then define

$$J_2 \overset{def}{=} e^{it\Psi(\xi_0)}\frac{1}{2\pi}\int_{\mathbb{R}} e^{it\xi^2/(8\xi_0^{3/2})}\widehat{f}(\xi_0)\varphi(\xi/2^{l_0})\,d\xi$$

and estimate

$$\big|J_1 - J_2\big| \lesssim 2^{3l_0/2}\|\partial\widehat{f}_k\|_{L^2} \lesssim 1\,, \tag{3.14}$$

having used the weighted L^2 bound in (3.7).

Eventually, for (3.9) it suffices to show

$$\big|J_2 - \frac{C_0}{\sqrt{t}} e^{-i\frac{t^2}{4x}\mathrm{sign}\frac{x}{t}} |\xi_0|^{3/4} \widehat{f}\,(\xi_0)\ d\xi\big| \lesssim 1\,. \tag{3.15}$$

Using the general formula

$$\int_{\mathbb{R}} e^{-ax^2}\,dx = \sqrt{\pi}/\sqrt{a}\,, \qquad a \in \mathbb{C}\,,\ \mathrm{Re}\,a > 0\,,$$

we see that

$$\int_{\mathbb{R}} e^{it\xi^2/(8\xi_0^{3/2})} e^{-\xi^2/2^{l_0}}\,d\xi = 2|\xi_0|^{3/4}\sqrt{2\pi}/\sqrt{it} + O\left(2^{3k/2} t^{-9/10}\right)\,,$$

so that

$$\begin{aligned} J_2 &= e^{it\Psi(\xi_0)} \frac{1}{2\pi} \int_{\mathbb{R}} e^{it\xi^2/(8\xi_0^{3/2})} \widehat{f}(\xi_0)\varphi(\xi/2^{l_0})\,d\xi \\ &= \frac{e^{-i\frac{t^2}{4x}\mathrm{sign}\frac{t}{x}}}{\sqrt{i2\pi t}} 2|\xi_0|^{3/4}\widehat{f}(\xi_0) + O\left(\||\xi|^{3/4}\widehat{f}\|_{L^\infty} 2^{3k/4} t^{-9/10}\right) \end{aligned} \tag{3.16}$$

The desired bound (3.9) follows from (3.10)-(3.11), the estimates (3.12), (3.13), (3.14), and the identity (3.16) which gives (3.15). This completes the proof of the lemma. □

4. Asymptotic Behavior in Physical Space

4.1. **Notations and bounds from** [IPu13]**.** Here we gather some of the bounds proved by the authors in [IPu13], which hold for the global solutions of the water wave system obtained in Theorem 2.1. We refer to section 4 of [IPu13] for the precise statement containing these estimates. We will then use these bounds in the final section to prove Proposition 2.3.

Proposition 4.1. *Let $h = h(t)$ and $\phi = \phi(t)$ be the global solutions given by Theorem 2.1. The following hold:*

1) (Proposition 4.1 of [IPu13]*). There exist bilinear operators A and B such that if*

$$H \stackrel{def}{=} h + A(h,h) \quad , \quad \Psi \stackrel{def}{=} \phi + B(h,\phi)\,, \tag{4.1}$$

then the function V defined by

$$V \stackrel{def}{=} H + i\Lambda\Psi \tag{4.2}$$

satisfies

$$\partial_t V + i\Lambda V = C\left(h, |\partial_x|\phi\right) \tag{4.3}$$

where C is a nonlinearity consisting of cubic and higher order terms.

2) (Estimate (5.27) in the proof of Proposition 4.2 of [IPu13]*). The bilinear operators A and B above satisfy the following bounds:*

$$\|A(h,h)\|_{W^{N_1+4,\infty}} + \|\Lambda B(h,\phi)\|_{W^{N_1+4,\infty}} \lesssim \varepsilon_0 (1+t)^{-4/5}\,. \tag{4.4}$$

3) (Proposition 4.3 of [IPu13]*). Let V be the function defined by* (4.2) *and satisfying* (4.3)*, and define*

$$f(t,x) \stackrel{def}{=} \left(e^{it\Lambda} V(t)\right)(x)\,. \tag{4.5}$$

Then, there exists $p_0 \leq 10^{-3}$ such that

$$\sup_{t\in[0,T]} (1+t)^{-5p_0} \left[\|x\partial_x f(t)\|_{H^{N_0/2-20}} + \|f(t)\|_{H^{N_0/2-20}}\right] \lesssim \varepsilon_0\,. \tag{4.6}$$

4) (Lemma 6.1 of [IPu13]*). Let*

$$\begin{aligned} L(\xi,t) &\overset{def}{=} \frac{|\xi|^4}{\pi}\int_0^t \big|\widehat{f}(\xi,s)\big|^2 \frac{ds}{s+1}, \\ g(\xi,t) &\overset{def}{=} e^{iL(\xi,t)}\widehat{f}(\xi,t). \end{aligned} \tag{4.7}$$

Then there exists $p_1>0$ such that, for any $m\in\{1,2,\dots\}$ and any $t_1\le t_2\in[2^m-2,2^{m+1}]$,

$$\Big\|(|\xi|^\beta+|\xi|^{N_1+15})\,(g(\xi,t_2)-g(\xi,t_1))\Big\|_{L^\infty_\xi} \lesssim \varepsilon_0 2^{-p_1 m}\,. \tag{4.8}$$

4.2. **Proof of Proposition 2.3.** The proof of Proposition 2.3 relies on the bounds listed in the previous section, and on an argument similar to the one used by Hayashi and Naumkin to treat nonlinear Schrödinger equations, see [HN98, pp. 381-383], and several other models, see for example [HN99a], [HN99b], and [HN99a].

Step 1: Consequences of (4.8). First notice that from the definition of g in (4.7), $|\widehat{f}(\xi,t)|=|g(\xi,t)|$. Therefore, (4.8) implies that for any $m\in\{1,2,\dots\}$ and any $t_1\le t_2\in[2^m-2,2^{m+1}]$,

$$\Big\|(|\xi|^\beta+|\xi|^{N_1+15})\,\Big(\big|\widehat{f}(\xi,t_2)\big|^2-\big|\widehat{f}(\xi,t_1)\big|^2\Big)\Big\|_{L^\infty_\xi} \lesssim \varepsilon_0 2^{-p_1 m}\,. \tag{4.9}$$

From the estimate (4.8) we see that g as defined in (4.7), is a Cauchy sequence in time, with values in L^∞_ξ. We can then define its limit

$$g_\infty(\xi) \overset{def}{=} \lim_{t\to\infty} g(\xi,t)\,.$$

This satisfies the property

$$\Big\|(|\xi|^\beta+|\xi|^{N_1+15})\big(g(\xi,t)-g_\infty(\xi)\big)\Big\|_{L^\infty_\xi} \lesssim \varepsilon_0(1+t)^{-p_1} \tag{4.10}$$

and

$$\Big\|(|\xi|^\beta+|\xi|^{N_1+15})\big(\big|\widehat{f}(\xi,t)\big|^2-|g_\infty(\xi)|^2\big)\Big\|_{L^\infty_\xi} \lesssim \varepsilon_0(1+t)^{-p_1}\,. \tag{4.11}$$

Step 2: Convergence of the phase $L(t)$. For notational convenience we define the space

$$\mathcal{L}^k \overset{def}{=} L^\infty\big((1+|\xi|^k)d\xi\big)\,. \tag{4.12}$$

Set

$$\begin{aligned} A(\xi,t) &\overset{def}{=} \frac{|\xi|^4}{\pi}\int_0^t \Big(|\widehat{f}(\xi,s)|^2-|\widehat{f}(\xi,t)|^2\Big)\frac{ds}{s+1} \\ &= L(\xi,t)-\frac{|\xi|^4}{\pi}|\widehat{f}(\xi,t)|^2\log(t+1)\,. \end{aligned} \tag{4.13}$$

From the definition of A we have

$$\begin{aligned} &A(\xi,t_2)-A(\xi,t_1) \\ &= \frac{|\xi|^4}{\pi}\int_{t_1}^{t_2}\Big(|\widehat{f}(\xi,s)|^2-|\widehat{f}(\xi,t_2)|^2\Big)\frac{ds}{s+1} - \frac{|\xi|^4}{\pi}\log(1+t_1)\Big[|\widehat{f}(\xi,t_1)|^2-|\widehat{f}(\xi,t_2)|^2\Big] \end{aligned}$$

so that using (4.11) above we get

$$\begin{aligned} &\|A(\xi,t_2)-A(\xi,t_1)\|_{\mathcal{L}^{N_1+1}} \\ &\lesssim \int_{t_1}^{t_2}\varepsilon_0(1+s)^{-p_1}\frac{ds}{s+1} + \varepsilon_0\log(1+t_1)(1+t_1)^{-p_1} \lesssim \varepsilon_0(1+t_1)^{-p_1/2}\,. \end{aligned}$$

Therefore, $A(\xi,t)$ is a Cauchy sequence in time, and we can define its limit

$$A_\infty(\xi) \stackrel{def}{=} \lim_{t\to\infty} A(\xi,t)\,,$$

satisfying the property

$$\|A(\xi,t) - A_\infty(\xi)\|_{\mathcal{L}^{N_1+1}} \lesssim \varepsilon_0(1+t)^{-p_1/2}\,.$$

From this, the definition of A in (4.13), and (4.11), it follows that

$$\left\| L(\xi,t) - A_\infty(\xi) - \frac{|\xi|^4}{\pi}|g_\infty(\xi)|^2\log(t+1)\right\|_{\mathcal{L}^{N_1+1}} \lesssim \varepsilon_0(1+t)^{-p_1/2}\,. \tag{4.14}$$

Step 3: Convergence of $\widehat{f}(t)$ and definition of $\alpha(t)$. From the definition of g and L in (4.7), and the bounds (4.10) and (4.14) it follows that

$$\left\| |\xi|^{3/4}\widehat{f}(\xi,t) - |\xi|^{3/4}g_\infty(\xi)\exp\left(-iA_\infty(\xi) - i\frac{|\xi|^4}{\pi}|g_\infty(\xi)|^2\log(t+1)\right)\right\|_{\mathcal{L}^{N_1}} \lesssim \varepsilon_0(1+t)^{-p_1/2}\,. \tag{4.15}$$

Setting

$$\alpha_\infty(\xi) \stackrel{def}{=} |\xi|^{3/4}g_\infty(\xi)\exp\left(-iA_\infty(\xi)\right) \tag{4.16}$$

we have

$$\left\| |\xi|^{3/4}\widehat{f}(\xi,t) - \alpha_\infty(\xi)\exp\left(-i\frac{|\xi|^{5/2}}{\pi}|\alpha_\infty(\xi)|^2\log(t+1)\right)\right\|_{\mathcal{L}^{N_1}} \lesssim \varepsilon_0(1+t)^{-p_1/2}\,. \tag{4.17}$$

Step 4: Asymptotics for $V(t)$. Let V be the function defined by (4.2), and related to f via (4.5). Combining the linear dispersive estimate (3.2) with (4.17) above, and the bounds (4.6) on the Sobolev and weighted Sobolev norms of f, one see that

$$\begin{aligned} \left| V(t,x) - \frac{\sqrt{2}}{\sqrt{\pi i(1+t)}}e^{it|t/x|}\alpha_\infty\left(t^2/4x^2\right)\exp\left(-i\frac{|t/2x|^5}{\pi}\left|\alpha_\infty\left(t^2/4x^2\right)\right|^2\log(t+1)\right)\right| & \\ \lesssim \varepsilon_0(1+t)^{-1/2-p_1/2}\,. & \end{aligned} \tag{4.18}$$

Step 5: Asymptotics for $u = h + i\Lambda\phi$. Eventually, one can easily obtain an asymptotic estimate like (4.18) for the actual solution $u(t,x)$. Indeed, using the relations (4.1) and (4.2), together with the L^∞ bounds given by (4.4), one sees that

$$\|u(t) - V(t)\|_{W^{N_1,\infty}} \lesssim \|A(h,h) + i\Lambda B(h,\phi)\|_{W^{N_1,\infty}} \lesssim \varepsilon_0(1+t)^{-4/5}\,.$$

The proof of (2.3) follows by the triangular inequality, and appropriately defining

$$f_\infty(\xi) \stackrel{def}{=} \frac{\sqrt{2}}{\sqrt{\pi i}}\,\alpha_\infty\left(1/(4\xi^2)\right)$$

The proof of Proposition 2.3 is complete. □

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