Topology of the Moduli Space of Twisted Higgs Bundles on $\mathbb{P}^1$ via Quiver Representations

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Geometry and Physics of Gauge Theories at Infinity
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Let X be a Riemann surface with genus g.

Definition

An **L-twisted Higgs bundle** on X is a pair (E, Φ), where E is a holomorphic vector bundle on X and Φ is an L-valued endomorphism of E, $\Phi : E \to E \otimes L$, where L is a holomorphic line bundle on X of degree t.

Higgs Bundles and Stability

We say that two Higgs bundles (E, Φ) and (E', Φ') are equivalent if E and E' are isomorphic as vector bundles and $\Phi = \Psi\Phi'\Psi^{-1}$ for some $\Psi \in H^0(X, \mathrm{Aut}(E))$.

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$$\begin{array}{ccc} E & \xrightarrow{\quad\Psi\quad} & E' \\ \Phi \big\downarrow & & \big\downarrow \Phi' \\ E \otimes L & \xrightarrow{\Psi \otimes 1_L} & E' \otimes L \end{array}$$

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$$\begin{CD} E @>{\Psi}>> E' \\ @V{\Phi}VV @VV{\Phi'}V \\ E \otimes L @>{\Psi \otimes 1_L}>> E' \otimes L \end{CD}$$

We let $\mathcal{M}_{X,L}(r, d)$ be the moduli space of L-twisted Higgs bundles on X of rank r and degree d.

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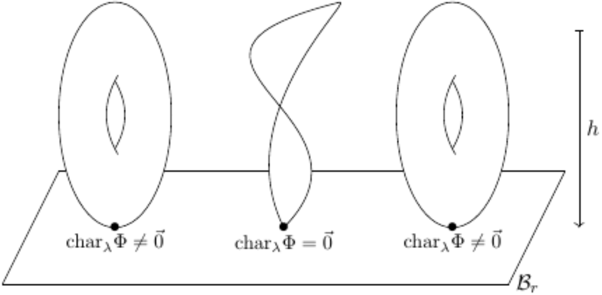
A Higgs bundle (E, Φ) is called **stable** if $\mu(U) < \mu(E)$ for all Φ-invariant proper subbundles U of E. Otherwise it is **unstable**. Note that U is Φ-invariant if $\Phi(U) \subseteq U \otimes L$.

The Hitchin Fibration

There is a well-known and very useful characterization of $\mathcal{M}_{X,L}(r,d)$ known as the Hitchin fibration:



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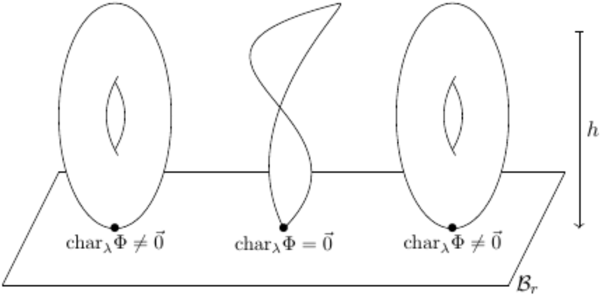

The base of this fibration is $\mathcal{B}_r = \bigoplus_{i=1}^{r} H^0(X, L^{\otimes i})$, effectively the space of possible characteristic polynomials of Φ. $\mathcal{B}_r$ is known as the Hitchin base. The map h that sends (E, Φ) to $\mathrm{char}_\lambda \Phi$ is known as the Hitchin map.

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Morse Theory

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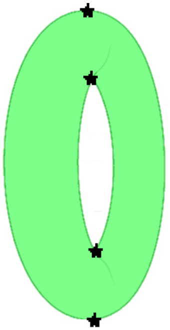

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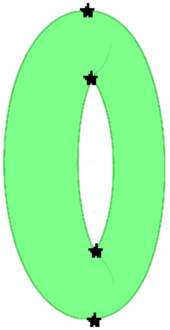

This can tell us that the Poincaré polynomial of the torus is $y^2 + 2y + 1$.

Morse Theory

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When the critical “points” are submanifolds instead of points, the theory is called Morse-Bott Theory. In this case, we will also need to know the Poincaré polynomials of these submanifolds.

The S^1 Action

There is a natural action of S^1 on $\mathcal{M}_{X,L}(r,d)$ given by

$$\theta \cdot (E, \Phi) = (E, e^{i\theta}\Phi),$$

whose "height function" is

$$f((E, \Phi)) = \|\Phi\|^2.$$

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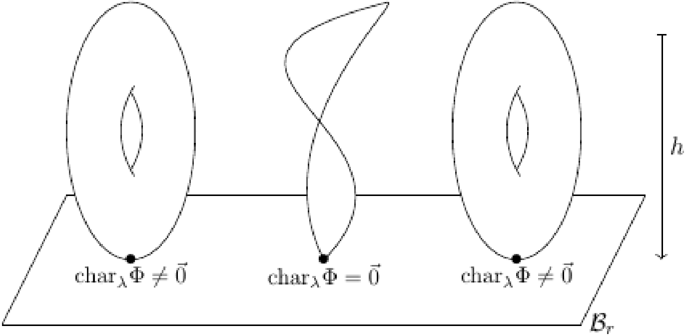
All the S^1 fixed points lie in the "nilpotent cone"

$$h^{-1}(0) := \{(E,\Phi) : \mathrm{char}_\lambda \Phi = \lambda^r\}.$$

That is, all the topological information of $\mathcal{M}_{X,L}(r,d)$ is in this particular fibre:



The S^1 Action in the Nilpotent Cone

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$$E = U_1 \oplus \cdots \oplus U_n \quad \text{and} \quad \Phi = \begin{pmatrix} 0 & 0 & \dots & \dots & 0 \\ \phi_1 & 0 & & & \vdots \\ 0 & \phi_2 & \ddots & & \\ \vdots & & \ddots & & \vdots \\ 0 & \dots & \dots & \phi_{n-1} & 0 \end{pmatrix}$$

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That is, (E, Φ) looks like

$$U_1 \xrightarrow{\phi_1} U_2 \xrightarrow{\phi_2} \dots \xrightarrow{\phi_{n-1}} U_n.$$

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These objects are separated into distinct submanifolds of the nilpotent cone decided by the rank and degree of the subbundles U_i.

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Quivers and Representations

Definition

A **labelled A-type quiver** is a directed, labelled graph of the form

$$\bullet_{r_1,d_1} \longrightarrow \bullet_{r_2,d_2} \longrightarrow \cdots \longrightarrow \bullet_{r_n,d_n}$$

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Definition

A **representation** of a quiver Q in the category $\mathcal{C}$ is the assignment of an object of $\mathcal{C}$ to each of the vertices of Q (possibly subject to labels), and a morphism of $\mathcal{C}$ to each of the arrows.

Quivers and Representations

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$$U_1 \xrightarrow{\phi_1} U_2 \xrightarrow{\phi_2} \dots \xrightarrow{\phi_{n-1}} U_n.$$

An Example

Example: Let $X = \mathbb{P}^1$, $L = \mathcal{O}(4)$, and $Q = \bullet_{1,2} \longrightarrow \bullet_{2,-1} \longrightarrow \bullet_{1,-2}$.

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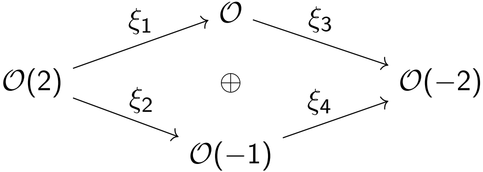

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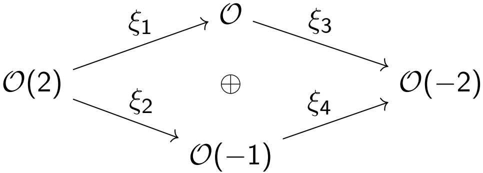

The bundle we assign to $\bullet_{2,-1}$ splits by the Birkhoff-Grothendieck Theorem. There are other ways to split a rank 2, degree 1 bundle, but in this case none of those correspond to stable representations.

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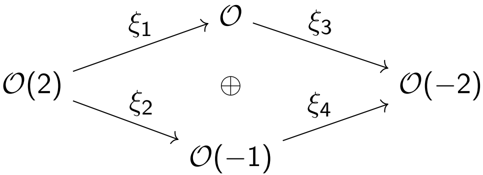

Example continued: We have

$$\xi_1 \in H^0(\mathbb{P}^1, \mathcal{O}(2)^* \otimes \mathcal{O} \otimes \mathcal{O}(4)) \cong \mathbb{C}^3$$
$$\xi_2 \in H^0(\mathbb{P}^1, \mathcal{O}(2)^* \otimes \mathcal{O}(-1) \otimes \mathcal{O}(4)) \cong \mathbb{C}^2$$
$$\xi_3 \in H^0(\mathbb{P}^1, \mathcal{O}^* \otimes \mathcal{O}(-2) \otimes \mathcal{O}(4)) \cong \mathbb{C}^3$$
$$\xi_4 \in H^0(\mathbb{P}^1, \mathcal{O}(-1)^* \otimes \mathcal{O}(-2) \otimes \mathcal{O}(4)) \cong \mathbb{C}^4$$

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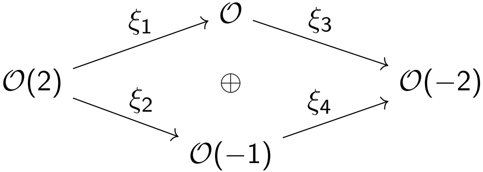

Example continued: Let's consider stability. The total slope is $\mu(\mathcal{O}(2) \oplus \mathcal{O} \oplus \mathcal{O}(-1) \oplus \mathcal{O}(-2)) = \frac{-1}{4}$.

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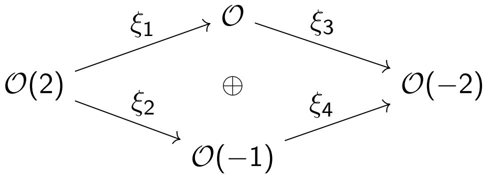

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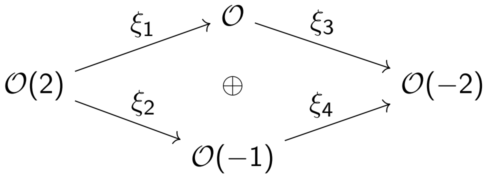

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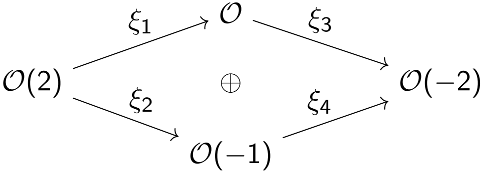

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$$\mathbb{C}^2 \setminus \{0\} \times \mathbb{C}^3 \setminus \{0\} \times (\mathbb{C}^3 \times \mathbb{C}^4) \setminus \{(0,0)\}$$

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To obtain the moduli space, we need to "mod out" by the equivalance relation mentioned earlier. This manifests as the automorphism groups acting at each node of the quiver:

$$\begin{array}{ccccc} \mathrm{Aut}(\mathcal{O}(2)) & & \mathrm{Aut}(\mathcal{O} \oplus \mathcal{O}(-1)) & & \mathrm{Aut}(\mathcal{O}(-2)) \\ \circlearrowleft & & \circlearrowleft & & \circlearrowleft \\ \mathcal{O}(2) & \xrightarrow{\xi_1 \oplus \xi_2} & \mathcal{O} \oplus \mathcal{O}(-1) & \xrightarrow{\xi_3 \oplus \xi_4} & \mathcal{O}(-2) \end{array}$$

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From this we can calculate

$$\mathcal{M}_{\mathbb{P}^1,\mathcal{O}(4)}(Q) = \mathbb{P}^1 \times \mathbb{P}^2 \times \mathbb{P}^5$$

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$(1, k, 1)$ Quivers

Theorem (Rayan, S.)

Let Q be a quiver of type $(1, k, 1)$ and let $\mathbf{a}$ be the splitting type of U_2. The projective closure of $\mathcal{M}^{\Delta}_{\mathbb{P}^1, \mathcal{O}(t)}(Q, \mathbf{a})$ is

$$\overline{\mathcal{M}^{\Delta}_{\mathbb{P}^1, \mathcal{O}(t)}}(Q, \mathbf{a}) \cong \mathbb{P}^q \times \prod_{j=1}^{i'} \mathrm{Gr}\Big(s_j, d_3 - a_j + t + 1 - \sum_{k=1}^{j-1} s_k(a_k - a_j + 1)\Big) \times \prod_{j=i'+1}^{m} \mathrm{Gr}\Big(s_j, a_j - d_1 + t + 1 - \sum_{k=j}^{m-1} s_k(a_k - a_j + 1)\Big)$$

where

$$q = \sum_{j=1}^{i'} s_j(d_3 - a_j + t + 1) + \sum_{j=i'+1}^{m} s_j(a_j - d_1 + t + 1) - 1 - \sum_{j=1}^{i'} \sum_{k=i'+1}^{m} s_j s_k(a_j - a_k + 1).$$

Argyle Quivers

Definition

An **argyle quiver** is an A-type quiver labelled as

$$\bullet_{1,d_1} \longrightarrow \bullet_{r_2,d_2} \longrightarrow \bullet_{1,d_3} \longrightarrow \cdots \longrightarrow \bullet_{r_{n-1},d_{n-1}} \longrightarrow \bullet_{1,d_n}$$

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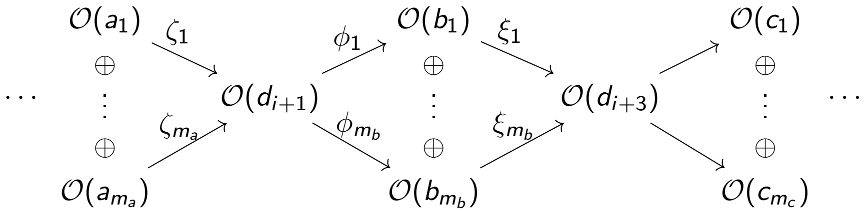
One can imagine that a representation of an argyle quiver would look something like



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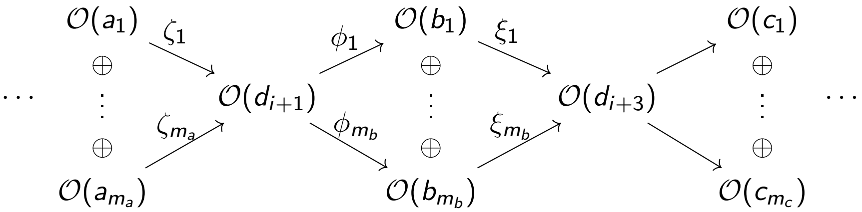
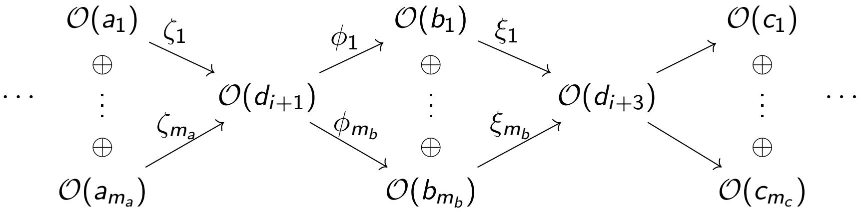


It is a bunch of $(1, k, 1)$ quiver representations stuck together!

Argyle?


$\cdots$
$\mathcal{O}(a_1)$
ζ_1
$\oplus$
$\vdots$
$\oplus$
ζ_{m_a}
$\mathcal{O}(a_{m_a})$
$\mathcal{O}(d_{i+1})$
ϕ_1
ϕ_{m_b}
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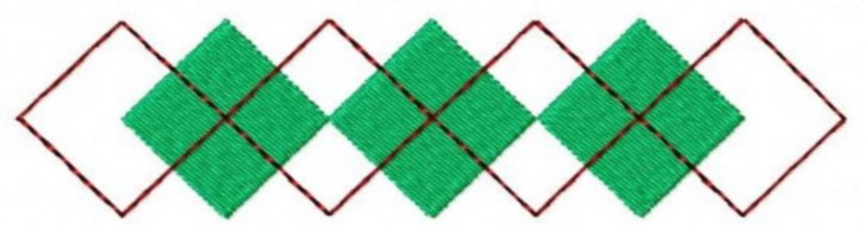
$\mathcal{O}(c_{m_c})$
$\cdots$



Argyle Quivers

Theorem (Rayan, S.)

Given a general argyle quiver Q with $\mathbf{a}_i$ the splitting type of U_i, the projective closure of the regular part of the moduli space of representations of Q in the category of $\mathcal{O}(t)$-twisted holomorphic vector bundles over $\mathbb{P}^1$ is

$$\begin{aligned}&\overline{\mathcal{M}^{\Delta}_{\mathbb{P}^1,\mathcal{O}(t)}}(Q,\mathbf{a}_2,\mathbf{a}_4,\dots,\mathbf{a}_{n-1})\\&\quad=\overline{\mathcal{M}'^{\Delta}_{\mathbb{P}^1,\mathcal{O}(t)}}(\bullet_{1,d_1}\longrightarrow\bullet_{r_2,d_2}\longrightarrow\bullet_{1,d_3},\mathbf{a}_2)\times\dots\\&\qquad\dots\times\overline{\mathcal{M}'^{\Delta}_{\mathbb{P}^1,\mathcal{O}(t)}}(\bullet_{1,d_{n-2}}\longrightarrow\bullet_{r_{n-1},d_{n-1}}\longrightarrow\bullet_{1,d_n},\mathbf{a}_{\mathbf{n-1}})\end{aligned}$$

where

$$\overline{\mathcal{M}'^{\Delta}_{\mathbb{P}^1,\mathcal{O}(t)}}(\bullet_{1,d_i}\longrightarrow\bullet_{r_{i+1},d_{i+1}}\longrightarrow\bullet_{1,d_{i+2}},\mathbf{a}_{i+1})$$

is the projective closure of the moduli space of the quiver

$$\bullet_{1,d_i}\longrightarrow\bullet_{r_{i+1},d_{i+1}}\longrightarrow\bullet_{1,d_{i+2}}$$

with splitting type of U_{i+1} given by $\mathbf{a}_{i+1}$, with stability condition induced by Q.

The End

Questions?