Hypertoric Hitchin Systems

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Some Definitions

We will be working over $\mathbb{C}$ to not complicate things.

Definition

An *toric variety* is an irreducible variety X such that

(i) $(\mathbb{C}^*)^n$ is an open dense subset of X

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Toric varieties have a close connection to combinatorics (fans, polytopes, etc.).

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Let's ask for a bit more structure:

Definition

An *symplectic toric variety* is a triple (X, ω, H) where X is a compact toric variety, ω is a symplectic form on X and H is a corresponding moment map $H : X \to \mathbb{R}^n$.

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$\mathbb{P}^n$ is symplectic toric.

The symplectic toric varieties are in one-to-one correspondence with a special class of polytopes.

Delzant polytopes

Definition

A *Delzant polytope* Δ in $\mathbb{R}^n$ is a convex polytope which is

(i) simple, meaning there are n edges meeting at each vertex

(ii) rational, meaning that the edges meeting at the vertex p are of the form $p + tu_i$, $t \geq 0$, $u_i \in \mathbb{Z}^n$

(iii) smooth, meaning that for each vertex, the u_i can be chosen to be a $\mathbb{Z}$-basis for $\mathbb{Z}^n$.

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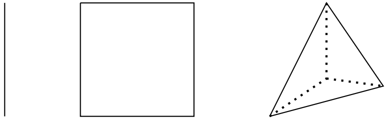
Examples:



Delzant polytopes

Delzant's Theorem

Symplectic toric manifolds are classified by Delzant polytopes:

$$\{\textit{symplectic toric manifolds}\} \longleftrightarrow \{\textit{Delzant polytopes}\}$$

$$(M, \omega, H) \mapsto H(M)$$

Delzant polytopes

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 $\longleftrightarrow$ $\mathbb{P}^1$

Delzant polytopes

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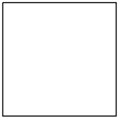

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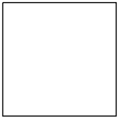

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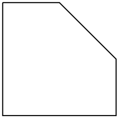

$$\longleftrightarrow \qquad \mathrm{Bl}_x(\mathbb{P}^1 \times \mathbb{P}^1)$$

Hyperkähler structure

Definition

A $4n$-dimensional manifold is *hyperkähler* is it has a Riemannian metric g which is Kähler with respect to three complex structures I, J, and K which satisfy quaternionic relations

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Examples: the Hilbert scheme of k points, the moduli space of Higgs bundles (Hitchin systems), Nakajima quiver varieties.

Hypertoric varieties

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Examples: If X is symplectic toric, then T^*X can be given hypertoric structure.

- Toric varieties can be written as Kähler quotients by vector spaces
- Hypertoric varieties can be written as hyperkähler quotients by tori.

Hypertoric big picture

{Hypertoric varieties}

{"additive" hypertoric varieties}
$(T^*\mathbb{C}^n /\!\!/\!\!/ T_\mathbb{R})$

{"multiplicative" hypertoric varieties}
$((T^*\mathbb{C}^n)^\circ /\!\!/\!\!/ T_\mathbb{R})$

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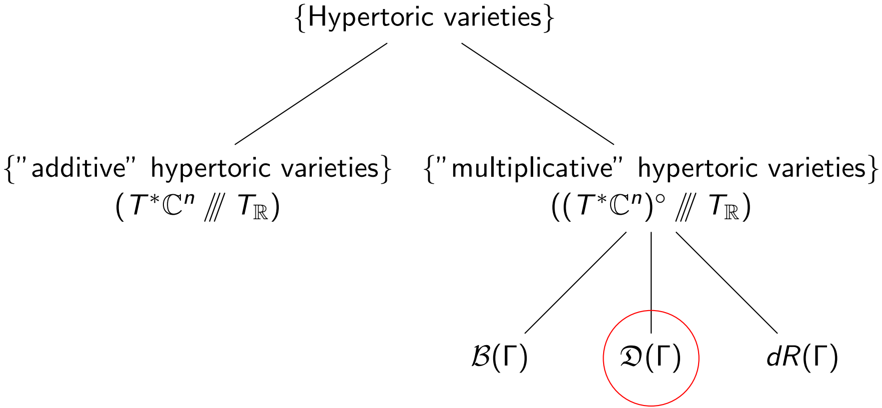
$\mathcal{B}(\Gamma)$ $\mathfrak{D}(\Gamma)$ $dR(\Gamma)$

Hypertoric big picture



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The Tate Curve

The building block of all hypertoric Hitchin systems is $\mathfrak{D}(\circlearrowright)$, the Tate curve.

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Let $\mathbb{C}^2_n$ be a copy of $\mathbb{C}^2$ with coordinates x_n, y_n. Then let

$$\tilde{\mathfrak{D}}(\overset{\bullet}{\bigcirc}) = \bigsqcup_{n \in \mathbb{Z}} \mathbb{C}^2_n / \sim,$$

with relations given by $x_n y_n = x_{n+1} y_{n+1}$ and $x_n = y_{n+1}^{-1}$.

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We have a map $\tilde{q} : \tilde{\mathfrak{D}}(\bigcirc) \to \mathbb{C}$ given by $\tilde{q}(x_n, y_n) = x_n y_n$.

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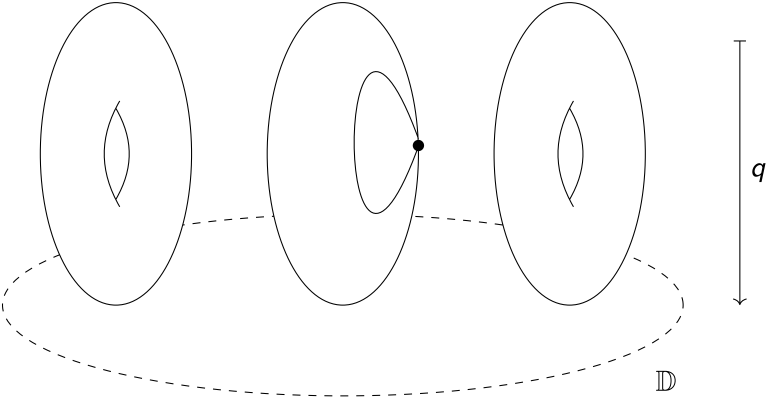
The generic fibre of $\tilde{q}$ is $\mathbb{C}^*$, and the fibre over zero is an infinite chain of copies of $\mathbb{P}^1$.

There is an action of $\mathbb{Z}$ which "shifts over by one" on $\tilde{q}^{-1}(\mathbb{D})$. We define

$$\mathfrak{D}(\bigcirc) = \tilde{q}^{-1}(\mathbb{D})/\mathbb{Z}.$$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We can also idenfity $\mathfrak{D}(\bigcirc)$ with $T^*\mathbb{C}^\circ = \{(z,w) : zw + 1 \neq 0\}$.

In general, start with an embedding of tori $(\mathbb{C}^*)^k = T \to D = (\mathbb{C}^*)^n$.

Construction of $\mathfrak{D}(\Gamma)$

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D acts on $\mathfrak{D}(\bigcirc)^n = (T^*\mathbb{C}^n)^\circ = \{(z_i, w_i) : z_i w_i + 1 \neq 0\}$, and we get a "moment map"

$$\begin{aligned} (\mu_\mathbb{C}, \mu_\mathbb{R}) : (T^*\mathbb{C}^n)^\circ &\longrightarrow D^\vee \times \mathfrak{d}^*_\mathbb{R} \\ (z_i, w_i)_{i=1}^n &\mapsto ((z_i w_i + 1), (|z_i|^2 + |w_i|^2))_{i=1}^n \end{aligned}$$

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\end{aligned}
$$

Compose the above maps with $D^\vee \to T^\vee$ to get a map μ_{HK} from $(T^*\mathbb{C}^n)^\circ$ to $T^\vee \times \mathfrak{t}_{\mathbb{R}}^* \cong T_{\mathbb{R}}^\vee \times \mathfrak{t}^*$.

Construction of $\mathfrak{D}(\Gamma)$

Definition

For $\mu_{HK} : \mathfrak{D}(\overset{\bullet}{\bigcirc})^n \to T_{\mathbb{R}}^{\vee} \times \mathfrak{t}^*$, the *hypertoric Hitchin system* or *Dolbeault hypertoric variety* associated to Γ is

$$\mathfrak{D}(\Gamma) = \mathfrak{D}(\overset{\bullet}{\bigcirc})^n /\!\!/\!\!/_{(\beta,0)} T_{\mathbb{R}} = \mu_{HK}^{-1}(\beta, 0)/T_{\mathbb{R}},$$

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The data of the embedding $T \to D$ and generic $\beta \in T_{\mathbb{R}}^{\vee}$ give a periodic hyperplane arrangement, which in turn we associate with a graph Γ.

Alternatively:

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$$\begin{array}{ccc} \bigsqcup \mathfrak{X}_i & \longrightarrow & q^{-1}(0) \\ \downarrow & & \downarrow \\ \bigsqcup H(\mathfrak{X}_i) & \longrightarrow & \left(\bigsqcup H(\mathfrak{X}_i)\right)/\sim \end{array}$$

Structure of $\mathfrak{D}(\Gamma)$

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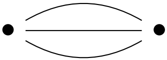
The generic fibres are complex Lagrangians with the structure of abelian varieties.

Example

Let

$\Gamma =$ 

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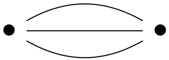
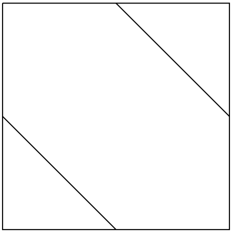
$\Gamma =$ 

Then the periodic hyperplane arrangement associated to Γ is



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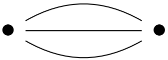
$\Gamma =$ 

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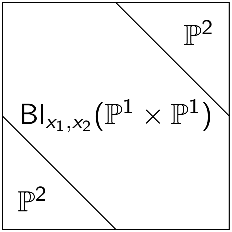

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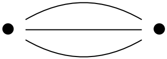
$\Gamma =$ 

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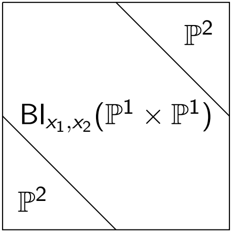

Thus the irreducible components $\mathfrak{X}_i$ of $q^{-1}(0)$ in $\mathfrak{D}(\Gamma)$ are $\mathbb{P}^2$, $\mathbb{P}^2$, and $\mathrm{Bl}_{x_1,x_2}(\mathbb{P}^1 \times \mathbb{P}^1)$.

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The Moduli Space of Higgs Bundles

Definition

A *Higgs bundle* on a Riemann surface Σ of genus at least 2 is a pair (E, Φ) where E is a holomorphic vector bundle on X and Φ is a map $E \to E \otimes K_\Sigma$.

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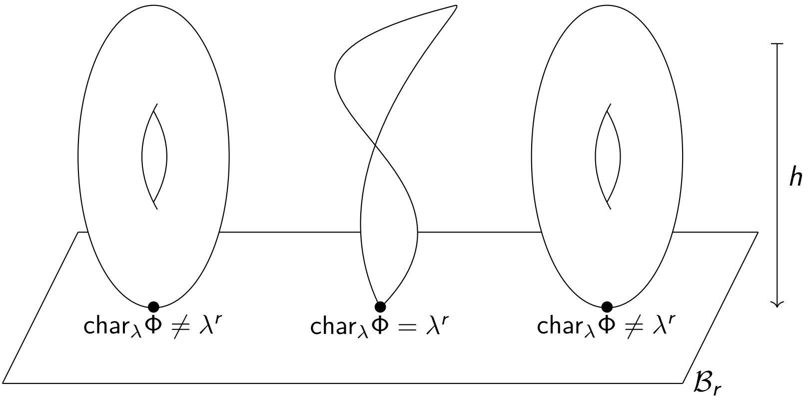
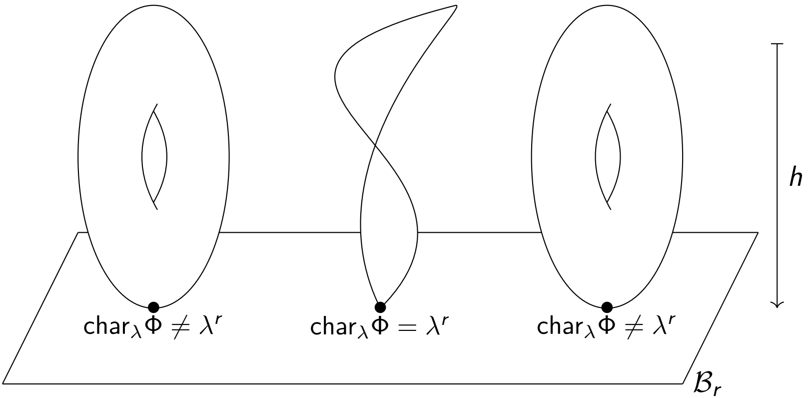
The moduli space of stable Higgs bundles on Σ of rank r and degree d is denoted $\mathcal{M}_\Sigma(r, d)$.

The Hitchin Fibration





The space $\mathcal{B}_r = \bigoplus_{i=1}^r H^0(\Sigma, K_\Sigma^{\otimes i})$ paramatrizes possible characteristic polynomials of Φ. Think of $b \in \mathcal{B}_r$ as a map $b : \mathrm{Tot}(K_\Sigma) \to \mathrm{Tot}(K_\Sigma)^{\otimes r}$.

Spectral Curves

Definition

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That is, $\tilde{\Sigma}_a = \left\{ z \in \mathrm{Tot}(K_\Sigma) : b(z) = (\mathrm{char}_\lambda \Phi)(z) = 0 \right\}$.

The BNR Correspondence

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For $\tilde{\Sigma}_b$ smooth, $\text{Jac}(\tilde{\Sigma}_b) \cong h^{-1}(b)$.

For $\tilde{\Sigma}_b$ with rational components and nodal singularities, $\overline{\text{Jac}}(\tilde{\Sigma}_b) \cong h^{-1}(b)$.

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If we have a curve $\tilde{\Sigma}_b$ with rational components and nodal singularities, we can consider its dual graph Γ (a vertex for each component, an edge for each node).

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The compactified Jacobian of a curve C with rational components and nodal singularities can be obtained by gluing the toric varieties associated to the periodic hyperplane arrangement arising from the dual graph of C in the prescribed way.

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$$C \rightsquigarrow \Gamma \rightsquigarrow \text{per. hpa} \rightsquigarrow \text{sympl toric vars glued together} = \overline{\text{Jac}}(C)$$

Upshot

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The hypertoric Hitchin system $\mathfrak{D}(\Gamma)$ is a model for a small neighbourhood of a fibre of the classical Hitchin system that has spectral curve with dual graph Γ.

They are not expected to be isomorphic as complex manifolds, but they at least have the same cohomology.

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Idea

The approach here is to utilize the toric structure of $q^{-1}(0)$ and employ the machinery of derived categories to calculate the cohomology of $\mathfrak{D}(\Gamma)$.

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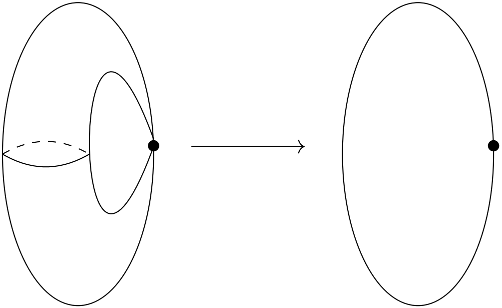
This is a work in progress!

Cohomology of $\mathfrak{D}(\,\circ\,)$

Let f be the map from the nodal elliptic curve X to its associated periodic hyperplane arrangement Y.



Cohomology of $\mathfrak{D}$()

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It is not too difficult to show that $R^0 f_* \mathbb{Q} = \mathbb{Q}_{S^1}$ and $R^1 f_* \mathbb{Q} = j_! \mathbb{Q}$.

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$$H^0(X, \mathbb{Q}) = H^0(S^1, R^0 f_*\mathbb{Q}) = \mathbb{Q}$$

$$H^1(X, \mathbb{Q}) = H^1(S^1, R^0 f_*\mathbb{Q}) \oplus H^0(S^1, R^1 f_*\mathbb{Q}) = \mathbb{Q}$$

$$H^2(X, \mathbb{Q}) = H^1(S^1, R^1 f_*\mathbb{Q}) = \mathbb{Q}$$

Does this work in general? We should check $\mathfrak{D}($  $)$.

The End

Thank you!