Geometry of the Einstein Equations, with a View to Calabi-Yau Manifolds

Steven Rayan
Notes by Evan Sundbo

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Preface

These notes comprise the course MATH 498/872, given by Dr. Steven Rayan in the winter term of 2018 at the University of Saskatchewan. The course will serve as an introduction to "modern geometry" as well as a window into research-level mathematics, not only in its content, but *how* it is performed. We will see the interactions between mathematics and physics (particularly viewing physics as inspiration) and will endeavor to break down artificial boundaries between mathematical subdisciplines

1 A General Introduction: What is Geometry?

The field of geometry is in some sense difficult to define. We can look back at its history to gain persepective:

- The "First Revolution of Geometry" (approx. 300 B.C.) was Euclid's study of plane geometry under strict axioms. This could be called the *axiomitization* of geometry.
- The "Second Revolution of Geometry" (approx. 1650 A.D.) was the discovery of algebraic equations common smooth shapes, which was made possible by Descartes' devolopment of the Cartesian coordinate system. This could be called the *algebraification* of geometry, and it is still incomplete.
- The "Third Revolution of Geometry" (approx. 1850 A.D.) was the realization that geometry occurs when a topological space is equipped with a metric (and the geometry is dependent on the metric). This is due mainly to Riemann, Poincaré, and Klein, and could be called the *definition* of geometry.
- The "Fourth Revolution of Geometry" (mid 20th century onward) was the recogntion of the fact that most (and perhaps the only) interesting geometries arise as solutions to equations from physics, and that most of these are complex. This could be called the *complexification* of geometry, and it is the era in which we are living right now.

Now, as this course ostensibly deals with Einstein's Equations, we should introduce them. Einstein's Equations can be expressed as $R = \alpha g$, where α is a real number, g is a metric, and R is a curvature. Note that curvature depends on g, so perhaps we should write $R(g) = \alpha g$. Thus, we are solving for g. Such a solution is called an *Einstein metric*. There are, in general, a huge number of possible metrics we could choose for a space, so choosing Einstein metrics cuts down our choices. In this course, we deal exclusively with the case $\alpha = 0$, i.e. $R(g) = 0$. Amongst such solutions are some special solutions called *Calabi-Yau metrics* (Calabi-Yau metrics are solutions to $R(g) = 0$ that have the additional property of being Kähler). The existence of Calabi-Yau metrics was an open problem from the early 1950s to the mid 1970s, when existence was proven by Shing-Tung Yau, for which he won the Fields Medal. It was realized in 1985 that Calabi-Yau metrics were exactly the metrics that were being sought after in string theory.

2 Geometry from Decorated Vector Spaces

2.1 Multilinear Functions and Scalar Product Spaces

To begin, we consider vector spaces V over the real numbers. Recall that if V is n-dimensional, then $V \cong \mathbb{R}^n$. This isomorphism can be thought of as a change of basis (if $\{b_1, \ldots, b_n\}$ is a basis, then we send b_1 to the first canonical basis vector in $\mathbb{R}^n$, b_2 to the second, et cetera).

We can create geometry on V by equipping V with some additional structure, called a *decoration*, which in this case is a multilinear function mapping k copies of V into $\mathbb{R}$. Recall that a function $S : V \times \ldots \times V \to \mathbb{R}$ is *multilinear* if

$$S(\vec{x}_1, \ldots, a\vec{x}_i + b\vec{y}_i, \ldots, \vec{x}_k) = aS(\vec{x}_1, \ldots, \vec{x}_i, \ldots, \vec{x}_k) + bS(\vec{x}_1, \ldots, \vec{y}_i, \ldots, \vec{x}_k) \tag{1}$$

for all i.

Example. The basic notion of geometry is afforded by the Pythagorean theorem: in $\mathbb{R}^n$, the length of $\vec{x}$ is given by $\|\vec{x}\| = \sqrt{x_1^2 + \ldots + x_n^2}$. This is truly geometry since we are measuring length of $\vec{x}$. The space $\mathbb{R}^n$ equipped with $\| \bullet \|$ is called *Euclidean space*. Note that $\|\vec{x}\|$ is certainly not bilinear, but it arises from $\|\vec{x}\|^2 = \vec{x}^T\vec{x}$, which can be thought of as taking the bilinear function $S : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ given by $S(\vec{x}, \vec{y}) = \vec{x}^T\vec{y}$ and specializing to $\vec{x} = \vec{y}$. Hence, Euclidean space can be thought of as $(\mathbb{R}^n, S)$, where $S(\vec{x}, \vec{y}) = \vec{x}^T\vec{y}$ and we obtain length as $\|\vec{x}\| = \sqrt{S(\vec{x}, \vec{x})}$.

In view of this, it is worth pursuing multilinear functions more formally. We define the set of all linear functions from a given vector space V to $\mathbb{R}$:

$$V^* = \{S : V \to \mathbb{R} \mid S \text{ is linear}\} \tag{2}$$

V^* is known as the *dual vector space* of V, and indeed it is a vector space: for $S, T \in V^*$, $S + T$ is linear, and for $k \in \mathbb{R}$, kS is linear, and the zero vector is the zero function $S(\vec{x}) = 0$ for all $\vec{x} \in V$. So, from a vector space V, we have formed a new one! We should calculate it's dimension, as that is one of the only identifying characteristics that vector spaces have. Choose a basis and identify V with $\mathbb{R}^n$: now, V^* is the set of linear maps $\mathbb{R}^n$ to $\mathbb{R}^n$. We know that each such map can be represented by a $1 \times n$ matrix, and thus V^* is n-dimensional over $\mathbb{R}$:

$$S(\vec{x}) = \begin{pmatrix} s_1 & s_2 & \dots & s_n \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \tag{3}$$

This also tells us that V and V^* can be thought of as the spaces of column and row vectors resepctively.

Suppose we want to consider bilinear functions $V \times V \to \mathbb{R}$. Denote this set by

$$V^* \otimes V^* = \{S : V \times V \to \mathbb{R} \mid S \text{ is bilinear}\} \tag{4}$$

Read $V^* \otimes V^*$ as "the tensor product of V^* with itself". This is again a vector space. What is it's dimension? Once again choose a basis, then $S \in V^* \otimes V^*$ is a bilinear function $\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$. Such maps are represented by $1 \times n \times n$ tensors, which are simply $n \times n$ matrices:

$$S(\vec{x}, \vec{y}) = \vec{x}^T \begin{pmatrix} s_{11} & s_{12} & \dots & s_{1n} \\ s_{21} & \ddots & & \\ \vdots & & & \\ s_{n1} & & & s_{nn} \end{pmatrix} \vec{y} \tag{5}$$

And so we see that the dimension of $V^* \times V^*$ is n^2.

In general, the tensor product of V^* with itself k times, denoted $V^* \otimes \dots \otimes V^* = (V^*)^{\otimes k}$ is an n^k-dimensional vector space, whose elements are k-linear functions $V \times \dots \times V \to \mathbb{R}$. If we regard V as $\mathbb{R}^n$, then elements of $(V^*)^{\otimes k}$ are represented by $(k+1)$-tensors of size $1 \times n \times \dots \times n$ (which are effectively k-tensors of size n^k). Note that the vector spaces $(V^*)^{\otimes k}$ are associated canonically to V: that is, there is no special choices or extra data involved in constructing them.

Example. $V^* \otimes V^* \otimes V^*$ consists of tri-linear functions $S : V \times V \times V \to \mathbb{R}$, each of which can be represented by a 3-tensor A.

In this new language, Euclidean space consists of $V \cong \mathbb{R}^n$ and an element of $V^* \otimes V^*$, specifically the element $S(\vec{x}, \vec{y}) = \vec{x}^T \vec{y}$, which corresponds to the identity matrix $A = I_n$. This choice of matrix is in fact canonical: let P be a change of basis matrix for V (i.e. P is an $n \times n$ invertible matrix). Then P acts on A by conjugation to produce a representation of S in the new basis. In this case we have $P^{-1}AP = P^{-1}I_nP = I_n$, so our choice was unique. This is *not* true in general. Also, notice that in this special case

$$\begin{aligned} S(\vec{y}, \vec{x}) &= S(\vec{y}, \vec{x})^T \\ &= (\vec{y}^T \vec{x})^T \\ &= \vec{x}^T (\vec{y}^T)^T \\ &= \vec{x}^T \vec{y} \\ &= S(\vec{x}, \vec{y}) \end{aligned} \tag{6}$$

That is, S is a symmetric function.

The subspace of $V^* \otimes V^*$ consisting of symmetric bilinear functions $V \times V \to \mathbb{R}$ is denoted $\mathcal{S}^2(V^*)$. Elements S of $\mathcal{S}^2(V^*)$ are represented by symmetric matrices, as one might expect.

Example. For a 2-dimensional vector space V, elements of $\mathcal{S}^2(V^*)$ are represented by symmetric 2×2 matrices (these are matrices A which satisty $A^T = A$.). By writing

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{7}$$

we can see that $A \in \mathcal{S}^2(V^*)$ implies $b = c$, and thus that the dimension of $\mathcal{S}^2(V^*)$ of is 3.

We can produce other geometries on V by using symmetric matrices which are not the identity. This inspires the following definition:

Definition. The pair (V,S) is called a *scalar product space* if S is a nondegenerate element of $\mathcal{S}^2(V^*)$, where "nondegenerate" means that any matrix representing S is invertible. Note that if A represents S and A is invertible, then any other representation $P^{-1}AP$ is also invertible. The map S is known as the *scalar procuct*.

Now, we would like to say that the length of $\vec{x} \in V$ induced by S is $\|\vec{x}\|_S = \sqrt{S(\vec{x},\vec{x})}$, analagous to our definition for Euclidean space. However, $S(\vec{x},\vec{x})$ could be negative. Hence, we deal mostly with the length squared instead of length: $\|\vec{x}\|_S^2 = S(\vec{x},\vec{x})$.

Example. Consider $(\mathbb{R}^2,S)$ with S corresponding to $A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. We can see that $A^T = A$ and $\det(A) \neq 0$, so this is a true scalar product space. Evaluate

$$\begin{aligned} S(\vec{x},\vec{y}) &= \vec{x}^T \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \vec{y} \\ &= \begin{pmatrix} x_1 & x_2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \\ &= \begin{pmatrix} x_1 & x_2 \end{pmatrix} \begin{pmatrix} y_1 \\ -y_2 \end{pmatrix} \\ &= x_1y_1 - x_2y_2 \end{aligned} \tag{8}$$

Observe

$$\begin{aligned} \left\| \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\|_S^2 &= S\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right) = (1)(1) - (0)(0) = 1 \\ \left\| \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\|_S^2 &= S\left(\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right) = (0)(0) - (1)(1) = -1 \end{aligned} \tag{9}$$

Definition. When every nonzero vector in V has positive length squared, we say that S is *positive definite* and that (V,S) is a *positive definite inner product space* (or just *inner product space*).

When every nonzero vector in V has negative length squared, we say that S is *negative definite* and that (V,S) is a *negative definite inner product space*

If neither of these is true, then S is *indefinite*.

Note that we only use the term "inner product" when S is definite.

Recall the following: a symmetric matric A has only real eignevalues, and is always diagonalizable with mutually orthogonal eigenvectors (with respect to the Euclidean inner product). That is, if λ_i and λ_j are two eigenvalues of A, then they are both real, and if λ_i has eigenvector $\vec{v}_i$ and λ_j has $\vec{v}_j$, then $\vec{v}_i^T \vec{v}_j = 0$. Diagonalizability also implies that there exists a complete basis for V consisting of eigenvectors of the matrix.

Suppose that S is a scaler product on V and that A is a representative of S, and let $\vec{x}$ be any nonzero vector in V. If we let $\{\vec{v}_1, \dots, \vec{v}_n\}$ be a mutually orthogonal basis of eigenvectors from A, we can write $\vec{x} = a_1\vec{v}_1 + \ldots + a_n\vec{v}_n$. Then

$$\begin{aligned} \|\vec{x}\|_S^2 = S(\vec{x},\vec{x}) &= (a_1\vec{v}_1 + \ldots + a_n\vec{v}_n)^T A (a_1\vec{v}_1 + \ldots + a_n\vec{v}_n) \\ &= (a_1\vec{v}_1 + \ldots + a_n\vec{v}_n)^T (a_1\lambda_1\vec{v}_1 + \ldots + a_n\lambda_n\vec{v}_n) \\ &= \sum_{i,j=1}^{n} a_i a_j \lambda_j \vec{v}_i^T \vec{v}_j \\ &= \sum_{i=1}^{n} a_i^2 \lambda_i \vec{v}_i^T \vec{v}_i \end{aligned} \tag{10}$$

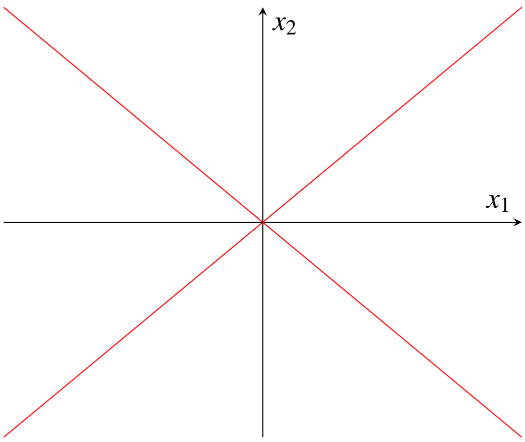
Since a_i^2 and $\vec{v}_i^T\vec{v}_i$ are both non-negative for all i (and cannot be zero for all i), we can say that if $\lambda_i > 0$ for all i, then $\|\vec{x}\|_S^2 > 0$. Additionally, if $\lambda_i < 0$ for all i, then $\|\vec{x}\|_S^2 < 0$. The converse of these statements is true as well: if S is positive definite, then $\lambda_i > 0$ for all i, and if S is negative definite, then $\lambda_i < 0$ for all i. It is a crucial point here that eigenvalues are basis independent.

To summarize:

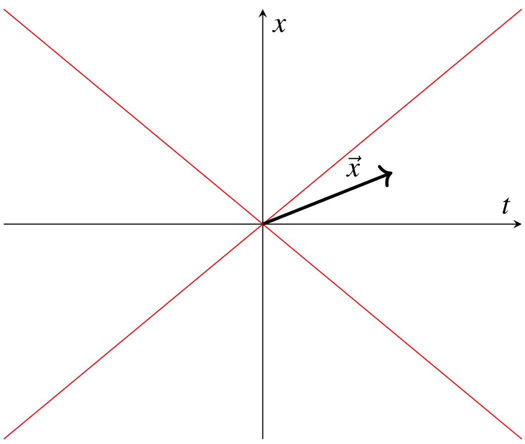
- S is positive definite if and only if A has only positive eigenvalues
- S is negative definite if and only if A has only negative eigenvalues
- S is indefinite if A has some positive and some negative eigenvalues (A will never have eigenavalues equal to 0 by invertibility).

Of particular physical intereset is the case when A has $n-1$ positive eigenvalues and 1 negative eigenvalue. In this case, (V,S) is said to be a *Lorentzian space*, and V is sometimes denoted $V^{n-1,1}$ or $\mathbb{R}^{n-1,1}$ to emphasize the Lorentz scalar product.

Example. Recall the space $V \cong \mathbb{R}^2$ with S given by $A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. This scalar product has $2-1=1$ positive eigenvalues and 1 negative eigenvalue, and hence we can look (V,S) as Lorentzian space $(\mathbb{R}^{1,1},S)$. Also recall that $S(\vec{x},\vec{y}) = x_1y_1 - x_2y_2$, $\left\| \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\|_S^2 = 1$, and $\left\| \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\|_S^2 = -1$. We can also calculate $\left\| \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\|_S^2 = (1)(1)-(1)(1) = 0$, which shows that we can have nonzero vectors with length squared equal to zero. This phenomenon occurs in indefinite scalar product spaces. In this case, any vector of the form $\vec{x} = \begin{pmatrix} a \\ a \end{pmatrix}$ or $\vec{x} = \begin{pmatrix} a \\ -a \end{pmatrix}$ will have length squared equal to zero. If we draw the following picture



then values along the red lines correspond to $\| \bullet \|_S^2 = 0$, values in the top and bottom quadrants correspond to $\| \bullet \|_S^2 < 0$, and values in the right and left quadrants correspond to $\| \bullet \|_S^2 > 0$. To more nicely appeal to physicists sensibilities, let $x = x_1$, $x_2 = t$, and rotate the picture by $\frac{\pi}{2}$:



where here the values along the red lines correspond to $\| \bullet \|_S^2 = 0$, values in the top and bottom quadrants correspond to $\| \bullet \|_S^2 > 0$, and values in the right and left quadrants correspond to $\| \bullet \|_S^2 < 0$. If we consider the vector $\vec{x} = (t_0, x_0)$ as the displacement of a particle from $x = 0$ to $x = x_0$ in the time t. The average speed of this particle is $\frac{x_0}{t_0} < 1$. If (t_0, x_0) were above (or below) both of the red lines, then the average speed would be $\frac{x_0}{t_0} > 1$. Indeed, the sign of $\| \bullet \|_S^2$ distinguishes when speed is above or below 1.

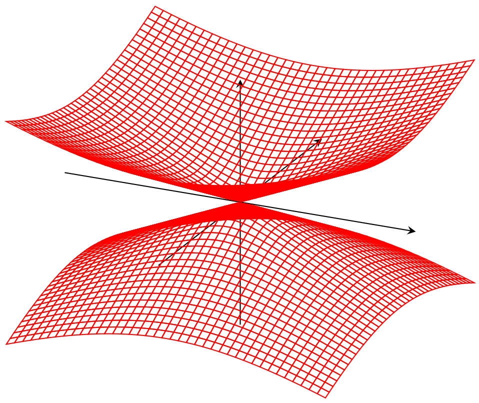
If we take 1 to be the absolute maximum speed possible (normalized), then having $||\vec{x}||_S^2 < 0$ corresponds to motion below this speed limit, and $||\vec{x}||_S^2 > 0$ corresponds to forbidden motion above this limit. This structure works well with special relativity, one of the axioms of which is constancy of speed of light.

Definition. If $||\vec{x}||_S^2 < 0$, then $\vec{x}$ is said to be *timelike*. If $||\vec{x}||_S^2 > 0$, then $\vec{x}$ is said to be *spacelike*. If $||\vec{x}||_S^2 = 0$, then $\vec{x}$ is said to be *lightlike* (or *null*).

Example. Consider the Lorentzian space $(\mathbb{R}^{2,1}, S)$ with S corresponding to

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \tag{11}$$

Here we draw the cone corresponding to $|| \bullet ||_S^2 = 0$



This cone is called the *light cone*.

A Lorentzian scalar product can be seen as creating time and space! Time is the direction of the eigenvector corresponding to the negative eigenvalue, and space is spanned by the eigenvectors corresponding to the positive eigenvalues.

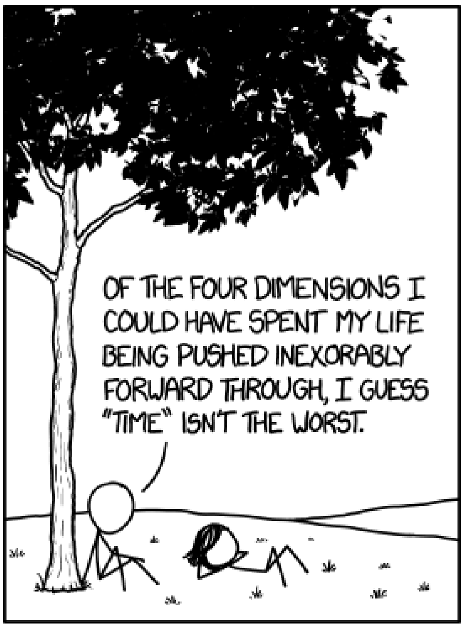


Figure 1: https://xkcd.com/1524/

In summary:

- If (V,S) is an n-dimensional scalar product space with n positive eigenvalues, then its all space, no time. That is, all vectors $\vec{x}$ are spacelike.
- If (V,S) is an n-dimensional scalar product space with $n-k$ positive eigenvalues and k negative eigenvalues, then we have $n-k$ space directions and k time directions.
- If (V,S) is an n-dimensional scalar product space with n negative eigenvalues, then its all time, no space. That is, all vectors $\vec{x}$ are timelike.

2.2 Symplectic Vector Spaces

Up to this point, we have been been considering symmetric bilinear functions. What if we instead use *anti-symmetric* (or *skew symmetric*) bilinear functions? We define the subspace $\wedge^2(V^*) \subset V^* \otimes V^*$ of antisymmetric binlinear functions. These are the functions represented by matrices B which satisfy $B^T = -B$. We read $\wedge^2(V^*)$ as "*wedge* V^*".

Let ω denote such a function. It is clear that $\omega(\vec{x},\vec{y}) = -\omega(\vec{y},\vec{x})$ for all $\vec{x},\vec{y} \in V$. Thus

$$\begin{aligned} \omega(\vec{x},\vec{x}) &= -\omega(\vec{x},\vec{x}) \\ 2\omega(\vec{x},\vec{x}) &= 0 \\ \omega(\vec{x},\vec{x}) &= 0 \end{aligned} \tag{12}$$

This tells us that ω can't give us a useful length. More generally, if $\vec{y} = k\vec{x}$ for some k, then $\omega(\vec{x},\vec{y}) = 0$:

$$\begin{aligned} \omega(\vec{x},\vec{y}) &= \omega(\vec{x},k\vec{x}) \\ &= -\omega(k\vec{x},\vec{x}) \\ &= -k\omega(\vec{x},\vec{x}) \\ &= -\omega(\vec{x},k\vec{x}) \\ &= -\omega(\vec{x},\vec{y}) \end{aligned} \tag{13}$$

which implies $\omega(\vec{x},\vec{y}) = 0$.

Let's take a closer look at the representatives B of ω:

- $n=1$: $B = (b)$, so $B^T = -B$ implies $b = 0$, so $\det B = 0$.
- $n=2$: $B = \begin{pmatrix} 0 & b \\ -b & 0 \end{pmatrix}$. Here $\det B = b^2$, unless $b = 0$.
- $n=3$: $B = \begin{pmatrix} 0 & b & c \\ -b & 0 & d \\ -c & -d & 0 \end{pmatrix}$. $\det B = 0$.
- $n=4$: $\det B$ is not necessarily 0.
- $n=5$: $\det B$ is always 0.

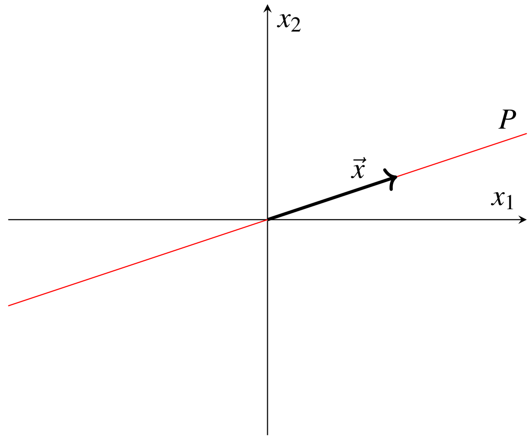
It is a fact that $\det B = 0$ whenever n is odd. So if we impose that $\omega(\vec{x},\vec{y}) = \vec{x}^T B\vec{y}$ is nondegenerate, then n must be even!

Definition. A *symplectic vector space* (V,ω) is an even-dimensional vector space $V \cong \mathbb{R}^{2n}$ equipped with a nondegenerate antisymmetric bilinear function $\omega \in \wedge^2(V^*)$. The function ω is refereed to as the *symplectic form*.

Example. Here we look at $(\mathbb{R}^2,\omega)$ with ω represented by $B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. This is clearly antisymmetric with determinant 1. We calculate

$$\begin{aligned}
\omega(\vec{x},\vec{y}) &= \vec{x}^T \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \vec{y} \\
&= \begin{pmatrix} x_1 & x_2 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \\
&= \begin{pmatrix} x_1 & x_2 \end{pmatrix} \begin{pmatrix} y_2 \\ -y_1 \end{pmatrix} \\
&= x_1 y_2 - x_2 y_1
\end{aligned} \tag{14}$$

For any vector $\vec{x} \in V$, we can define P to be the set of all vectors $\vec{y}$ such that $\vec{y} = k\vec{x}$ for some k.



So, if $\vec{y} \in P$, then $\omega(\vec{x},\vec{y}) = 0$. Conversely, if $\vec{y} \in \mathbb{R}^2$ and $\omega(\vec{x},\vec{y}) = 0$, then

$$\begin{aligned}
x_1 y_2 - x_2 y_1 &= 0 \\
x_1 y_2 &= x_2 y_1 \\
\frac{x_1}{x_2} &= \frac{y_1}{y_2}
\end{aligned} \tag{15}$$

Hence we can say $\omega(\vec{x},\vec{y}) = 0$ if and only if $\vec{y} \in P$. We will denote $\mathbb{R}^2 \setminus P$ by M. Clearly if $\vec{y} \in M$, then $\omega(\vec{x},\vec{y}) \neq 0$. This is telling us that ω is detecting which vectors are parallel to $\vec{x}$ and which are transverse to $\vec{x}$. That is, ω "detects" P and M by $\omega(\vec{x},\vec{z}) = 0$ if $\vec{z} \in P$ and $\omega(\vec{x},\vec{z}) \neq 0$ if $\vec{z} \in M$.

Theorem 1. *(Darboux's Theorem at a point for symplectic vector spaces): If (V,ω) is a symplectic vector space of dimension $2n$, then there exists a basis of V in which ω is represented by*

$$\left(\begin{array}{c|c} 0_n & I_n \\ \hline -I_n & 0_n \end{array}\right)$$

This says that the first n basis vectors span a subspace P, the next n basis vectors span M, and the vectors are all mutually orthogonal with respect to the Euclidean inner product. In particular, it tells us that up to change of basis, there is only one symplectic vetor space of dimension $2n$.

This is interesting in physics as well: if we have n variables $x_1,\ldots,x_n$ (position coordinates) they generate n derivate operators $\frac{d}{dx_1},\ldots,\frac{d}{dx_n}$ (momenta). A symplectic form is an easy way of taking $2n$ coordinates and declaring which are positions and which are momenta.

2.3 Complex Vector Spaces

Complex vector spaces arise when we try to create $\mathbb{C}$ from $\mathbb{R}^2$ using the technique of multilinear of algebra. Recall $\mathbb{C} \cong \mathbb{R}^2$ as sets, but $\mathbb{C}$ has a special element i with the property $i^2 = -1$. We would like to turn i into a linear transformation of some kind, so we need to know exactly how i acts. The element i acts on $(x+iy) \in \mathbb{C}$ in the following way: $i(x+iy) = ix - y = -y + ix$. If we write these as vectors in $\mathbb{R}^2$, then $i\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -y \\ x \end{pmatrix}$. This can be expressed as a linear transformation $J:\mathbb{R}^2\to\mathbb{R}^2$ represented by a 2×2 matrix C that takes $\begin{pmatrix}x\\y\end{pmatrix}$ to $\begin{pmatrix}-y\\x\end{pmatrix}$. Hence $C=\begin{pmatrix}0&-1\\1&0\end{pmatrix}$. This is the matrix that "changes" $\mathbb{R}^2$ into $\mathbb{C}$. That is, $\mathbb{C}\cong(\mathbb{R}^2,J)$.

Note that $J\circ J$ is represented by $C^2=\begin{pmatrix}0&-1\\1&0\end{pmatrix}\begin{pmatrix}0&-1\\1&0\end{pmatrix}=\begin{pmatrix}-1&0\\0&-1\end{pmatrix}=-I_2$. So J is indeed the transformation analogue of i. The eigenvalues of J are solutions to $\det(C_\lambda I_2)$:

$$\begin{aligned}\det\begin{pmatrix}-\lambda&-1\\1&-\lambda\end{pmatrix}&=0\\-\lambda^2+1&=0\\\lambda^2&=-1\\\lambda&=\pm i\end{aligned}\tag{16}$$

One can recover i as an eigenvalue of J!

Definition. A pair (V,J) is a *complex vector space* if $V\cong\mathbb{R}^{2n}$ and J is a linear function $V\to V$ such that $J\circ J=J^2=-Id_V$. Here J is called the *complex structure*.

Note that this definition has no explicit nondegeneracy condition: $J^2=-Id_V$ is in fact a stronger condition. We write $V^*\otimes V=\{\text{linear functions } V\to V\}$, so $J\in V^*\otimes V$.

Theorem 2. *(Darboux's Theorem at a point for complex vector spaces): If (V,J) is a complex vector space of dimension $2n$, then there exists a basis of V in which J is represented by*

$$\left(\begin{array}{c|c}0_n&-I_n\\\hline I_n&0_n\end{array}\right)$$

This shows that on a vector space, there are only cosmetic differences between symplectic structures and complex structures. We will see that on manifolds they are truly different.

2.4 Kähler Vector Spaces

Now suppose that an even-dimensional vector space V possesses both a symplectic form ω and a complex structure J. We can think of this as a triple (V,ω,J). Note that if $\vec{x}\in V$, then $J(\vec{x})\in V$ by definition. So, we can compose ω and J in two different ways: $\omega(\vec{x},J(\vec{y}))$ or $\omega(J(\vec{x}),J(\vec{y}))$. We don't consider $\omega(J(\vec{x}),\vec{y})$ since it is the same, essentially, as $\omega(\vec{x},J(\vec{y}))$: $\omega(J(\vec{x}),\vec{y})=-\omega(\vec{x},J(\vec{y}))$ by skew symmetry. We refer to $\omega(\vec{x},J(\vec{y}))$ as $\omega\circ J$.

Definition. We say that J *preserves* ω (or that ω and J are *compatible*) if $\omega(J(\vec{x}),J(\vec{y}))=\omega(\vec{x},\vec{y})$.

If (V,J) is a complex vector space, denote by $\wedge^2_J(V^*)$ the skew symmetric linear functions which are compatible with J. We call $\wedge^2_J(V^*)$ the *Kähler cone* of J.

Given (V,J), take a nondegenerate $\omega\in\wedge^2_J(V^*)$. We can see that $\omega\circ J$ is a bilinear function:

$$\begin{aligned}\omega\circ J: V\times V&\to V\times V\ \to\mathbb{R}\\(\vec{x},\vec{y})&\mapsto(\vec{x},J(\vec{y}))\mapsto\omega(\vec{x},J(\vec{y}))\end{aligned}\tag{17}$$

Hence $\omega\circ J\in V^*\otimes V^*$. Consider how this acts on $(\vec{y},\vec{x})$:

$$\begin{aligned}(\omega\circ J)(\vec{y},\vec{x})&=\omega(\vec{y},J(\vec{x}))\\&=\omega(J(\vec{y}),J^2(\vec{x}))\\&=\omega(J(\vec{y}),-\vec{x})\\&=-\omega(J(\vec{y}),\vec{x})\\&=\omega(\vec{x},J(\vec{y}))\\&=(\omega\circ J)(\vec{x},\vec{y})\end{aligned}\tag{18}$$

Thus, $\omega \circ J$ is symmetric ! That is, (V, ω, J) being compatible implies $\omega \circ J \in \mathcal{S}(V^*)$. We say that J *symmetrizes* its compatible ω's.

What matrix represents $\omega \circ J$? If B represents ω and C represents J, we can compute

$$\begin{aligned}(\omega \circ J)(\vec{x}, \vec{y}) &= \omega(\vec{x}, J(\vec{y})) \\ &= \vec{x}^T B(J(\vec{y}) \\ &= \vec{x}^T B(C\vec{y}) \\ &= \vec{x}^T (BC)\vec{y})\end{aligned} \tag{19}$$

to see that $\omega \circ J$ is represented by BC, and since B and C are both invertible, BC is invertible and hence $\omega \circ J$ is nondegenerate. That is, $\omega \circ J$ is a scalar product on V! This leads to the following definition:

Definition. A *Kähler vector space* (V, ω, J) is an even-dimensional vector space V equipped with a complex structure $J \in V^* \otimes V$, $J^2 = -Id_V$, and a compatible symplectic form $\omega \in \wedge^2_J(V^*)$ called a *Kähler form*. Together, ω and J induce a scalar product on V by $S = \omega \circ J$, called the *Kähler product*.

Example. Consider $(\mathbb{R}^2, \omega, J)$, where ω and J are the symplectic and complex structures on $\mathbb{R}^2$ represented in the standard way: $B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $C = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. To show that these give $\mathbb{R}^2$ the structure of a Kähler vector space, it is enough to show that ω and J are compatible. Recall that $\omega(\vec{x}, \vec{y}) = x_1y_2 - x_2y_1$, $J(\vec{x}) = \begin{pmatrix} -x_2 \\ x_1 \end{pmatrix}$ and $J(\vec{y}) = \begin{pmatrix} -y_2 \\ y_1 \end{pmatrix}$. Thus,

$$\begin{aligned}\omega(J(\vec{x}), J(\vec{y})) &= \begin{pmatrix} -x_2 & x_1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} -y_2 \\ y_1 \end{pmatrix} \\ &= \begin{pmatrix} -x_2 & x_1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \\ &= x_1y_2 - x_2y_1 \\ &= \omega(\vec{x}, \vec{y})\end{aligned} \tag{20}$$

So ω and J are compatible, and thus ω is a Kähler form for J and $S = \omega \circ J$ is a Kähler product. What is this Kähler product exactly? It is the scalar product whose matrix is $BC = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Hence, the Kähler structure on $\mathbb{R}^2$ gives it the Euclidean inner product.

Note that eigenvalues of BC are not related to those of B and C in a simple way, and thus $\omega \circ J$ is in no way guarenteed to be an inner product.

Definition. If $\omega \circ J$ is positive definite, then ω is a *positive Kähler form* for J, and we write $\omega \in \wedge^2_{J^+}(V^*)$. We call $\wedge^2_{J^+}(V^*)$ the *positive cone* of J.

In particular, we have the following inclusions:

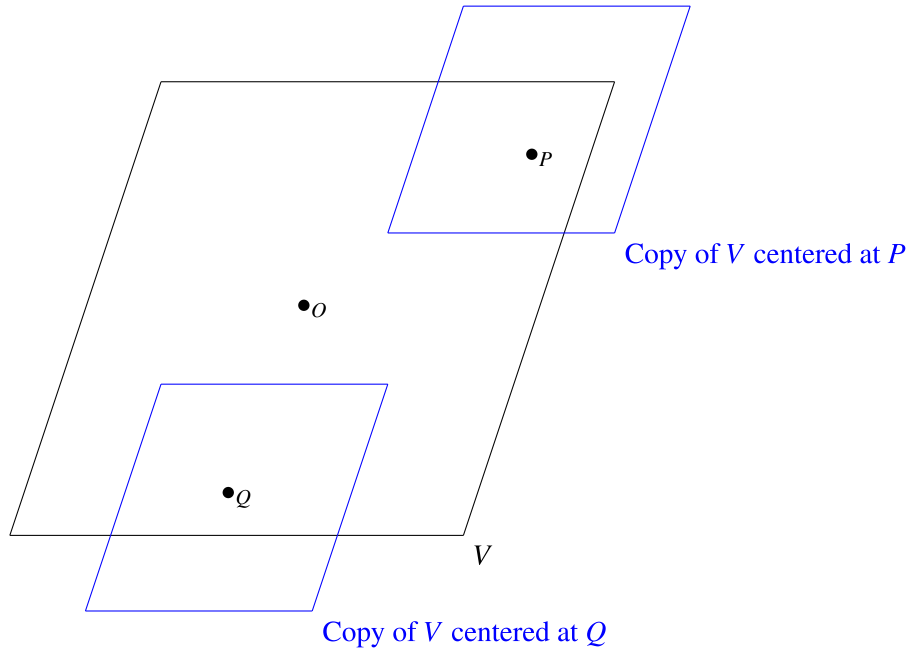
$$\wedge^2_{J^+}(V^*) \subset \wedge^2_J(V^*) \subset \wedge^2(V^*) \subset V^* \otimes V^* \tag{21}$$

2.5 Vector Bundles Defined by Vector Spaces

Now let us revisit length: suppose e stand at the origin of a vector space V. To measure the length squared $\|\vec{x}\|^2_S$ of a vector $\vec{x}$, we simply apply the scalar product S. If we walk to $P \in V$ and find a new vector $\vec{y}$ to measure, how do we calculate $\|\vec{y}\|^2_S$? We have to put $\vec{y}$ into "standard position" by translating it: we apply S to the vector

$$\begin{pmatrix} y_1 - p_1 \\ \vdots \\ y_n - p_n \end{pmatrix} \tag{22}$$

Another way of thinking about this is to translate the whole vector space: keep the vector $\vec{y}$ fixed but translate V so that it is centered at P. We don't want to completely forget about our position, though; that is, we want to remember P. We want to keep both the old and new vector spaces! This is in some sense more natural that translating vectors around. We can think of each point in V as being equipped with its own copy of V, centered at that point.



(23)

Now, what exactly have we created? As a set, (23) is $V \times V$, the set of pairs $(\vec{x}, \vec{y})$. For each $P \in V$, there is a copy of V.

Example. $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$ as a set.

As a vector space, $V \times V$ has dimension $2n$. We denote this vector space $V \oplus V$ since, strictly speaking, $V \times V$ does not have a vector space structure, it is simply a set. Read $V \oplus V$ as "the direct sum of V with itself".

Example. $\mathbb{R}^2 = \mathbb{R} \oplus \mathbb{R}$ as a vector space.

So, (23) is $V \oplus V$ as a vector space, but it has even more structure than that. Note that the example $\mathbb{R}^2 = \mathbb{R} \oplus \mathbb{R}$ has an internal symmetry: neither $\vec{x}$ nor $\vec{y}$ is more important. However, in (23), we *do* care about the original vector space, as it is the space in which we are moving around in, while the other V's are just for measuring vectors. This asymmetry is an extra structure on top of $V \oplus V$.

Definition. Consider $V \oplus V$ for some vector space V. If we declare one of the summands V to be the *base* V_B and the other to be the *fibre* V_F, then $V_B \oplus V_F$ is called the *vector bundle defined by* V. The base is the vector space of positions and the fibre is the vector space for measuring vectors.

A few remarks:

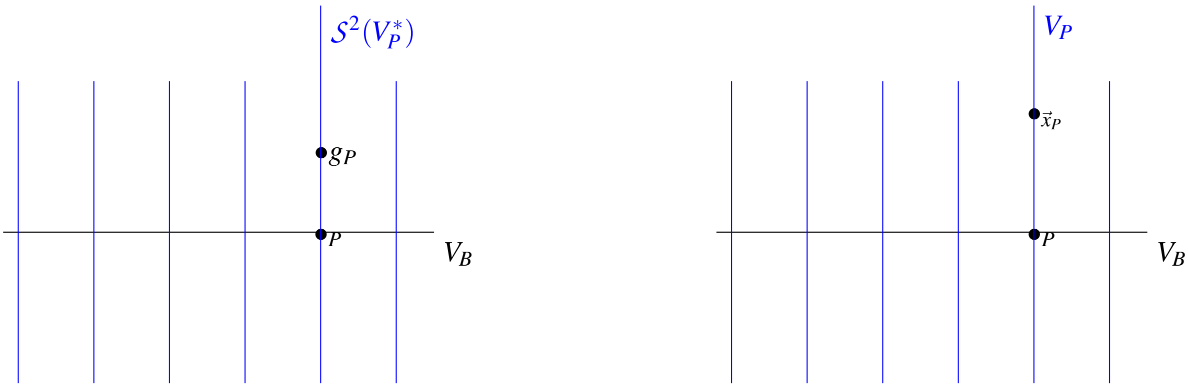
- We often denote V_F over $P \in V_B$ by V_P to emphasize where this fibre lives.
- We also sometimes denote $V_B \oplus V_F$ by U.
- V_B doesn't need a scalar product, since measurements occur in V_F. In fact, V_B need not even be a vector space, only a set!
- A symplectic form on an even-dimensional vector space V induces a vector bundle structure in V with the position subspace as the base and the momentum subspace as the fibre.

- We could be more general and turn $W \oplus V$, where W and V are vector spaces of different dimensions, into a vector bundle with base V and fibre W. This is the *vector bundle of W over V*. Here, however, there is no natural way of identifying vectors in V with ones in the fibre.

Now we will consider the scalar product S on V_P. Is there a reason that we would have to use the same S at each P? From a mathematical point of view, there is no good reason to insist that S be constant with respect to P. Appropriately, we allow S to depend on $P \in V_B$, that is, $S_P \in \mathcal{S}^2(V_P^*)$.

Definition. As P varies, S_P creates a family of scalar products, one for each fibre V_P. This family is known as a *metric* on V, and is usually denoted by g. On a single fibre V_P, g is simply S_P. but we often write g_P. If g is positive definite for each $P \in V_B$, then g is a *Riemannian metric*. If g_P is Lorentzian for every $P \in V_B$, then g is a *Lorentzian metric*.

We can think of each $P \in V_B$ as having a vector space $\mathcal{S}^2(V_P^*)$ attached to it, in which g_P lives. This gives rise to another bundle $\mathcal{S}^2(V_P^*) \oplus V_B$, which we denote by $\mathcal{S}^2(U^*)$. This is a bundle of spaces of symmetric bilinear functions, one such space for each $P \in V_B$.



(24)

So g_p measures $\vec{x}_P$, et cetera.

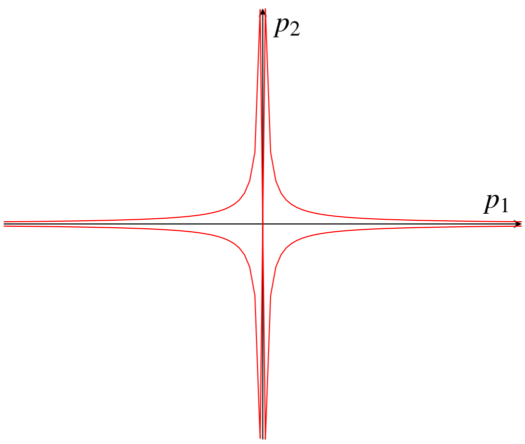
Definition. The choice of a vector $\vec{x}_P$ in each V_P is a *section* of a bundle U. Hence, the metric is a section of $\mathcal{S}^2(U^*)$.

Normally, we ask that the metric g be smooth (meaning smooth as a function of P). That is, the map from V_B to $\mathcal{S}^2(U^*)$ given by $P \mapsto A_P$ where A_P is the matrix which represents g_P, is smooth.

Example. Consider the bundle $U = \mathbb{R}^2_B \oplus \mathbb{R}^2_F$. A metric g on $\mathbb{R}^2$ is a section of $\mathcal{S}^2(U^*) = \mathcal{S}^2((\mathbb{R}^2)^*) \oplus \mathbb{R}^2_B$ which is smooth and nondegenerate at each point in the base.

From each $\mathcal{S}^2((\mathbb{R}^2)^*)$, let us take $g_P = g_{(p_1,p_2)}$ given by $A_{(p_1,p_2)} = \begin{pmatrix} 1 & p_1p_2 \\ p_1p_2 & 1 \end{pmatrix}$. What kind of metric is g? Eigenvalues of $A_{(p_1,p_2)}$ are given by $(1-\lambda)^2 - p_1^2p_2^2 = 0$ which reduces to $\lambda = 1 \pm p_1p_2$. Thus,

- If $0 \leq |p_1p_2| < 1$ then $A_{(p_1,p_2)}$ has two positive eigenvalues and hence g is Riemannian.
- If $|p_1p_2| > 1$ then $A_{(p_1,p_2)}$ has one positive and one negative eigenvalue and hence g is Lorentzian.
- If $|p_1p_2| = 1$, then $A_{(p_1,p_2)}$ has at least one zero eigenvalue and is degenerate.



Here, the red curve is the degeneracy locus. Points P on the "interior" have fibres $\mathbb{R}^2_P$ equipped with g_P Riemannian. Points P on the "exterior" have fibres $\mathbb{R}^2_P$ equipped with g_P Lorentzian. Note that the Euclidean metric occurs along the axes.

At the point $(2,1) \in \mathbb{R}^2_B$, we have $g_{(2,1)}(\vec{x},\vec{y}) = \begin{pmatrix} x_1 & x_2 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$, so we can calculate, for example,

$$\begin{aligned} \left\| \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\|^2_{g_{(2,1)}} &= g_{(2,1)} \left(\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right) \\ &= \begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ &= 6 \end{aligned} \tag{25}$$

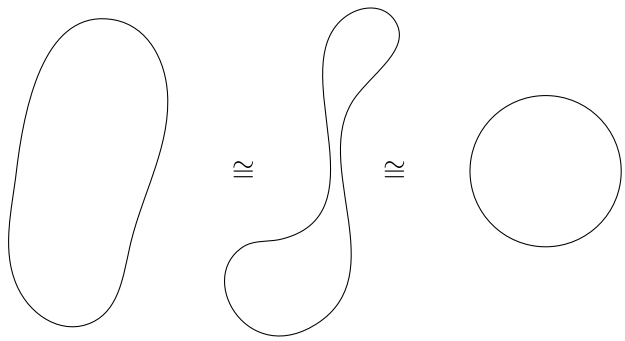
Finally, we should note that we could also equip V with a varying J_P and ω_P, and hence have a varying Kähler structure. Doing so requires further conditions on J_P and ω_P, called *integrability conditions*, which we will see later.

3 Geometry on Smooth Manifolds

3.1 Smooth Manifolds

Now we would like to replace the base V_B with something more general. Which features of V_B do we wish to keep? Chiefly, the base needs to have *coordinates* $p = (p_1, \ldots, p_n)$ that keep track of which which fibre we are working in. We also want to keep being "tricked" into thinking that our world (the base) is flat. Of course, it is not, but what we see immediately around us seems to be.

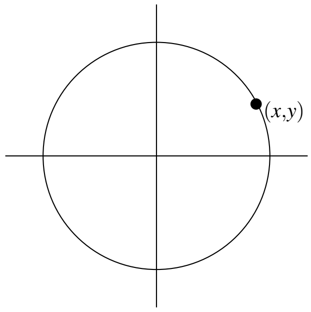
Suppose that we have the circle as our base. This is usually denoted S^1, the emph1-sphere. Note that S^1 is not necessarily the "round" or unit circle, the set points all equidistant from a given point. These are simply presentations of the 1-sphere: to a mathematician, S^1 is used to denote the closed loop.



(26)

That is, we permit deformations of our picture which do not "tear" it. (Here, "≅" is a homeomorphism). All the above pictures belong to the same homeomorphism class. For example, the usual round circle is a representative of the class S^1. How can we tell the difference between a closed and an open loop? If we embed them in the plane without self-intersections, then the closed loop has an interior and exterior, while the open loop does not (the closed loop divides the plane while the open loop does not). A curve is homeomorphic to S^1 if and only if a non-self-crossing embedding of that curve into the plan has an interior and exterior.

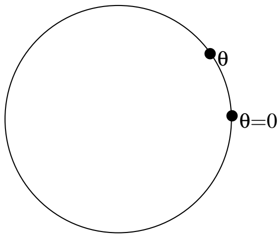
So, we are starting with curves and we want to coordinatize them. For example, to put coordinates on S^1, we could embed it in $\mathbb{R}^2$ as the round circle and use the induced (or *extrinsic*) coordinates from $\mathbb{R}^2$.



(27)

But these coordinates depend on each other as $x^2+y^2=1$. So these coordinates "remember" the embedding. Also, we should only need one coordinate to tell us where we are on the circle. Even further, our S^1 may not live in $\mathbb{R}^2$, but somewhere more exotic.

How could we put a single coordinate on S^1 without immersing it in some ambient space? Let's try this: First, homeomorph S^1 so that it is round. Then put an angle (or polar coordinate) θ on S^1. Now we are using θ to assign each point on S^1 to number in the interval $[0,2\pi)$. This is equivalent to taking the interval $[0,2\pi)$ and gluing the ends to make a coordinatized closed loop.



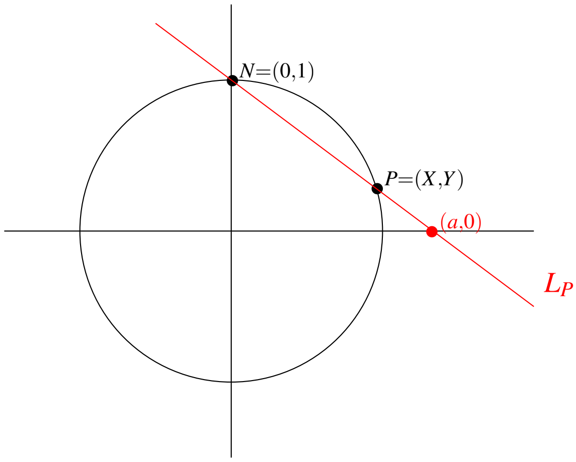
(28)

Now we could certainly put a vector bundle structure on S^1 and measure a vector $\vec{x}\in V_\theta$ using g_θ. Suppose we had a family of vectors $\vec{x}_\theta \in V_\theta$ varying with theta. Suppose we want to konw how fast $\vec{x}_\theta$ is changing with respect to θ at some θ_0. Roughly, this is some kind of derivative $\frac{d\vec{x}}{d\theta}\big|_{\theta=\theta_0}$, which is some limit $\lim_{\theta\to\theta_0}\frac{\vec{x}_\theta-\vec{x}_{\theta_1}}{\theta-\theta_0}$. Note that if $\theta_0=0$, then this is $\lim_{\theta\to 0}\frac{\vec{x}_\theta-\vec{x}_0}{\theta}$, which only makes sense as a one-sided limit! In general, $\varepsilon-\delta$ limits are only well-defined if there is a ball (or, in this case, an interval) around the value which we are approaching.

The problem here is that $[0,2\pi)$ is half-closed as an interval, which is an obstruction to analysis. Coordinates must come from an open set on $\mathbb{R}^n$, where n is as small as possible. Let us try extending the interval by a small amount, and use $(-\varepsilon,2\pi)$. Here, the problem is that if we assign more than one θ to a single point, g_θ can have two different values depending on θ, and how can we know which one to use?

The solution is the following: don't try to cover all of S^1 with an open set from $\mathbb{R}^1$, but cover it with patches that overlap each other but not themselves, and have a way of translating between them.

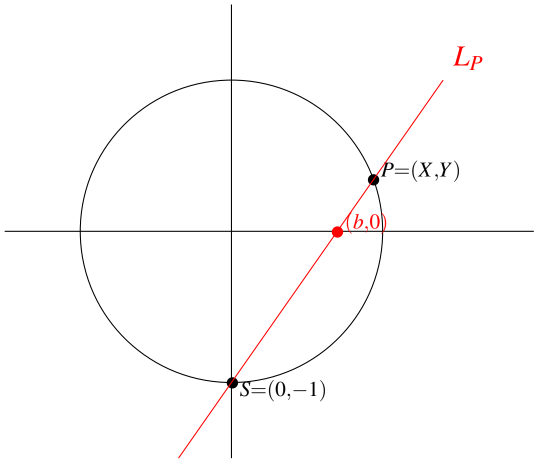
Example. We will cover the circle with two *patches* (open sets from $\mathbb{R}^1$) that each miss exactly one point. Let us embed S^1 into $\mathbb{R}^2$ again, just temporarily.



(29)

In this picture, a is the coordinate assigned to the point P. The line L_p has equation $\frac{-(1-Y)}{X}x+1=y$, and $(a,0)$ is on L_P, thus $\frac{Y-1}{X}a-1=0$, and so $a(X,Y)=\frac{X}{1-Y}$. Hence, if $P\in S^1$ has coordinates X,Y coming from an embedding in $\mathbb{R}^2$, then $a(P)=\frac{X}{1-Y}$, a single number associated to P. We can obtain all values of $a\in\mathbb{R}$ from this map, and the map is one-to-one. The only issue (which is by construction) is that the point $P=N$ has no value. That is, $a: S^1\setminus\{N\}\hookrightarrow\!\!\!\!\rightarrow\mathbb{R}$.

Next, we do the same sort of projection operation, but from the south pole:



(30)

to obtain $b(P)=\frac{X}{1+Y}$, and $b: S^1\setminus\{S\}\hookrightarrow\!\!\!\!\rightarrow\mathbb{R}$. Now, the final hurdle is to understand how we can translate from a to b. First projecting from the north pole, we have $\frac{-1}{a}X+1=Y$. In addition, we know $X^2+Y^2=1$, and thus

$$X^2+\left(\frac{-1}{a}X+1\right)^2=1 \implies \left(1+\frac{1}{a^2}\right)X^2-\frac{2X}{a}=0 \tag{31}$$

and then either $X=0$ or $X=\frac{2a}{a^2+1}$. The restriction $a\neq 0$ makes $X=0$ inadmissable, so we must have $X=\frac{2a}{a^2+1}$. The map we are constructing is from $\mathbb{R}\setminus\{0\}$ to $S^1\setminus\{N,S\}$. Now, $\frac{-1}{a}X+1=Y$ becomes $\frac{-1}{a}\left(\frac{2a}{a^2+1}\right)+1=Y$, which implies $Y=\frac{a^2-1}{a^2+1}$.

That is, given $a\in\mathbb{R}\setminus\{0\}$, the corresponding point $P\in S^1$ has $\mathbb{R}^2$-induced coordinates $\left(\frac{2a}{a^2+1},\frac{a^2-1}{a^2+1}\right)$. Now, by applying the b coordinate map, we obtain

$$b=\frac{\frac{2a}{a^2+1}}{1+\frac{a^2-1}{a^2+1}}=\frac{2a}{(a^2+1)\left(1+\frac{a^2-1}{a^2+1}\right)}=\frac{2a}{a^2+1+a^2-1}=\frac{1}{a} \tag{32}$$

So, what we have is a map

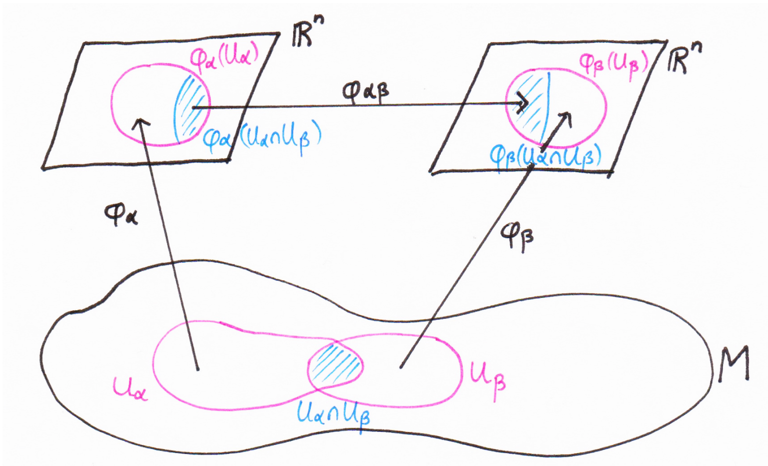
$$\begin{aligned} \mathbb{R}\setminus\{0\}\to S^1\setminus\{N,S\}\ &\to\mathbb{R}\setminus\{0\} \\ a\longmapsto\left(\frac{2a}{a^2+1},\frac{a^2-1}{a^2+1}\right)&\mapsto\frac{1}{a} \end{aligned} \tag{33}$$

The north pole coordinate is $\varphi_N(P)=\frac{X}{1-Y}=a(X,Y)$, and the south pole coordinate is $\varphi_S(P)=\frac{X}{1+Y}=b(X,Y)$. Moreover, $(\varphi_S\circ\varphi_N^{-1})(a)=\frac{1}{a}$.

This example inspires the definition of a manifold:

Definition. A *smooth manifold* is a set M for which there exists a collection of subsets $U_\alpha\subset M$ such that

- $\bigcup_\alpha U_\alpha=M$. That is, $\{U_\alpha\}$ covers M.
- there is a map $\varphi_\alpha: U_\alpha\hookrightarrow\mathbb{R}^n$ such that $\mathrm{Im}(\varphi_\alpha)$ is an open ball in $\mathbb{R}^n$ for each α. The maps φ_α are called *charts* and the set of all φ_α is called the *atlas*. The number n is the *dimension* of M.
- for each α,β, the map $\varphi_{\alpha\beta}:=\phi_\beta\circ\varphi_\alpha^{-1}:\varphi_\alpha(U_\alpha\cap U_\beta)\to\varphi_\beta(U_\alpha\cap U_\beta)$ exists and is smooth as a map between subsets of $\mathbb{R}^n$. The maps $\varphi_{\alpha\beta}$ are called *transition functions*.



Example. S^1 is a 1-dimensional manifold with atlas consisting of two charts φ_N and φ_S, with transition function $\varphi_{NS}(a) = \frac{1}{a}$. Moreover, $\frac{d^k(\varphi_{NS})}{d^k a}$ exists for all $a \in \mathbb{R} \setminus \{0\}$, so the manifold is smooth.

Note that $\varphi_{\alpha\beta}$ existing and being smooth for all α, β implies $\varphi_{\beta\alpha}$ exists and is smooth. That is, $\varphi_{\alpha\beta}$ has to have a smooth inverse. Thus, the transition functions satisfy the strong inverse function theorem:

Theorem 3. *(The Strong Inverse Function Theorem)*

If U and V are open subsets of $\mathbb{R}^n$ and $F : U \to V$ is a smooth, onto function, then F has a smooth inverse $F^{-1} : V \to U$ if and only if the derivative of F is invertible at every point in U.

If this is satisfied, then F^{-1} has the same property, i.e. that the derivative of F^{-1} is invertible at every point in V.

This works with the definition of manifold in the following way: We ask that, for each α, β, the function $\varphi_{\alpha\beta} = \varphi_\alpha^{-1} \circ \varphi_\beta$ from $\varphi_\alpha(U_\alpha \cap U_\beta)$ to $\varphi_\beta(U_\alpha \cap U_\beta)$ is smooth. (Implicitly, this asks that φ_α^{-1} exists. Also, when $U_\alpha \cap U_\beta$ is empty, the function $\varphi_{\alpha\beta}$ is the empty function, which is trivially smooth.)

By definition, $\varphi_{\alpha\beta}$ is surjective onto $\varphi_\beta(U_\alpha \cap U_\beta)$. Also, the symmetry of the definition means that we are also asking that $\varphi_{\beta\alpha}$, which by definition is the inverse of $\varphi_{\alpha\beta}$, exists and is smooth. Hence, the definition of manifold is satisfied when the smooth, onto function $\varphi_{\alpha\beta} : \varphi_\alpha(U_\alpha \cap U_\beta) \to \varphi_\beta(U_\alpha \cap U_\beta)$. is invertible with smooth inverse.

This is equivalent to the strong inverse function theorem holding for $\varphi_{\alpha\beta}$ and $\varphi_{\beta\alpha}$, and so if we have a manifold structure, then the derivatives of $\varphi_{\alpha\beta}$ and $\varphi_{\beta\alpha}$ are everywhere invertible on $\varphi_\alpha(U_\alpha \cap U_\beta)$ and $\varphi_\beta(U_\alpha \cap U_\beta)$, respectively.

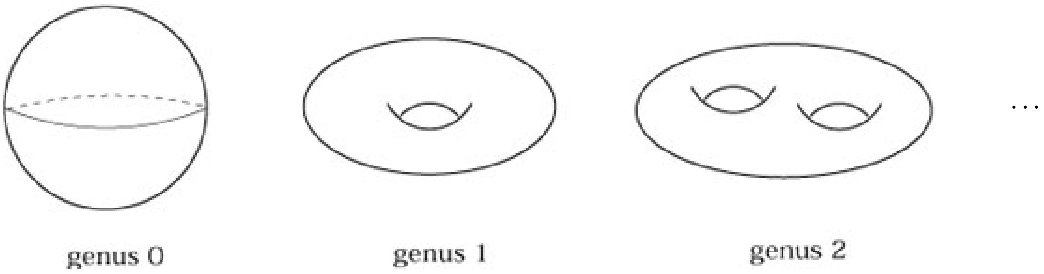
The set M that we started with in the definition of manifold is a "pre-topological space". A topology is a collection of subsets of M which we declare to be the open sets of M, and which satisfy a few other properties[1]. The definition of manifold should technically include that the U_α form a sufficiently "nice" (second countable Hausdorff) topology on M. For our purposes, we will say that the subsets U_α in our manifold structure become the open sets of M. This is known as the *induced topology*. Note that M is open in M, since $M = \bigcup_\alpha U_\alpha$. So we should denote a manifold by (M, U_α) to emphasize the structure we have placed on it: indeed, manifold structure is not unique in general. But of course we often just write M.

Example. All vector spaces V are manifolds: cover V by $U = V \cong \mathbb{R}^n$. This gives coordinates and no need for transition functions.

Example. Every 1-dimensional connected pre-topological space that admits a smooth manifold structure is homeomorphic either to the open loop or S^1 (this includes knots).

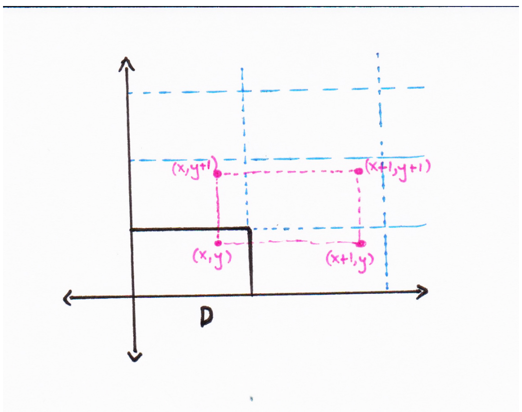
[1] A *topology* $\mathcal{T}$ for a set X is a nonempty collection of subsets of X called the *open sets* such that $\varnothing, X \in \mathcal{T}$, the union of any family of open sets is open, and the intersection of a finite number of open sets is open.

Example. Every 2-dimensional compact connected oriented pre-topological space admitting a smooth manifold structure is one of the following:



The *genus* g is a topological invariant that classifies such surfaces. All the surfaces of genus $g \geq 0$ can be made into manifolds.

Example. The easiest way to see the manifold structure on the torus T^2 (the genus 1 surface) is as the quotient $\mathbb{R}^2/\mathbb{Z}^2$.



Any point in $\mathbb{R}^2$ can be obtained from a point in the fundamental domain D by the action $+(k, j)$, $k, j \in \mathbb{Z}$. So if we look at points in $\mathbb{R}^2$ up to equivalence by this action, what is left over is the square with identified sides, which we know is isomorphic to the torus. Thus, the torus receives a manifold structure from $\mathbb{R}^2$. This begs an interesting question: is the torus actually "curved"? We know that other surfaces receive a manifold structure by gluing: genus 2 curves come from the octagon, etc. However, there is no octagonal lattice on $\mathbb{R}^2$ and thus the genus 2 surface *cannot* be obtained as $\mathbb{R}^2$ modulo a lattice. So the torus appears to be "flat" while the genus 2 surface is not.

Example. The spheres S^n all recieve manifold structures from stereographic projection.

Example. Cartesian products of manifolds are manifolds: if M, N are manifolds covered by U_α, W_α respectively, then $M \times N$ is covered by $U_\alpha \times W_\alpha$.

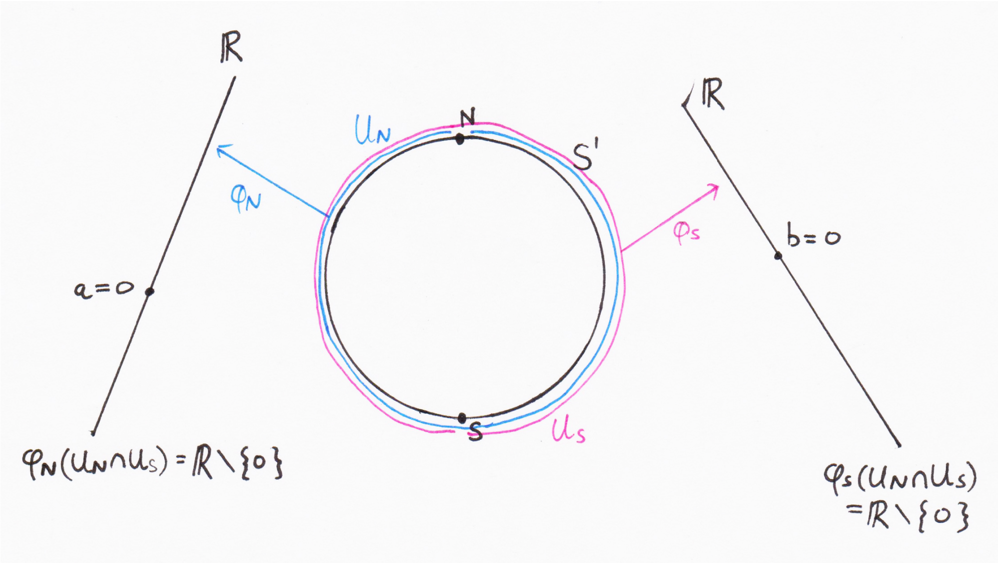
Example. In many cases, manifolds arise through restriction from a larger manifold: $N \hookrightarrow M$. Under good conditions, the manifold structure restricts. Such "good conditions" include when $M \cong \mathbb{R}^n$ and N is cut out of M by smooth equations: that is, $N = \{\vec{x} \in \mathbb{R}^n | F(\vec{x}) = \vec{y}\}$, where $F(\vec{x})$ is a smooth function.

Example. Using what we calculated earlier, we can draw out the manifold structure on S^1 very explicitly:

Thus, $\varphi_{NS} = \varphi_S \circ \varphi_N : \mathbb{R} \setminus \{0\} \to \mathbb{R} \setminus \{0\}$, $a \mapsto \frac{1}{a}$.

3.2 Vector Bundles on Manifolds

Now we want to be able to put a vector bundle struture on M. We could do this by putting a copy of a vector space at every point in φ_α. However, if a point p is in both U_α and U_β, then we are going to need some sort of transition funtion.



That is, locally, the bundle looks like $U_\alpha \times \mathbb{R}^r$ for each α. If $p \in U_\alpha \cap U_\beta$, then its associated fibre is not unique: we have both $V_{\varphi_\alpha(p)}$ and $V_{\varphi_\beta(p)}$. How do we convert from on to the other? The map we are looking for should be an invertible, linear transformation from $V_{\varphi_\alpha(p)} \cong \mathbb{R}^r$ to $V_{\varphi_\beta(p)} \cong \mathbb{R}^r$ (that is, an $r \times r$ matrix). For each $\vec{a} \in \varphi_\alpha(U_\alpha \cap U_\beta)$, we want an invertible $r \times r$ matrix that we will call $E_{\alpha\beta}(\vec{a})$.

$$\begin{tikzcd}
V_{\vec{a}} \arrow[rr, "E_{\alpha\beta}(\vec{a})"] \arrow[d] & & V_{\varphi_{\alpha\beta}(\vec{a})} \arrow[d] \\
\vec{a} \in \varphi_\alpha(U_\alpha \cap U_\beta) \arrow[rr, "\varphi_{\alpha\beta}"] & & \varphi_{\alpha\beta}(\vec{a}) \in \varphi_\beta(U_\alpha \cap U_\beta) \\
& U_\alpha \cap U_\beta \arrow[ul, "\varphi_\alpha"] \arrow[ur, "\varphi_\beta"'] &
\end{tikzcd} \tag{34}$$

Note that the data of a vector bundle here is really just the collection of transition functions $E_{\alpha\beta}$. That is, once we pick our $r \times r$ $E_{\alpha\beta}$, this tells us that the fibres are copies of $\mathbb{R}^r$. We can write this as

$$E_{\alpha\beta} : \varphi_\alpha(U_\alpha \cap U_\beta) \to \mathrm{GL}(r, \mathbb{R}) \tag{35}$$

where $\mathrm{GL}(r,\mathbb{R})$ is the *general linear group*: the group of invertible $r \times r$ matrices with real entries.

Here, we will take an aside to make an interesting observation: namely, that $\mathrm{GL}(r,\mathbb{R})$ is manifold itself, as well as a group. This can be seen by writing $\mathrm{GL}(r,\mathbb{R}) = \{\vec{x} \in \mathbb{R}^{r^2} | \det(\vec{x}) \neq 0\}$, since the determinant is a smooth function, and so $\mathrm{GL}(r,\mathbb{R})$ is cut smoothly out of $\mathbb{R}^{r^2}$, and so it is a submanifold. This is an example of a *Lie group*: a group which is also a manifold.

Example. S^1 is another example of a Lie group: to see this embed S^1 in $\mathbb{C}$ as $e^{i\theta}$ for $\{\theta \in \mathbb{R}\}$. Now if $g = e^{i\theta} \in S^1$ and $h = e^{i\psi} \in S^1$, then $gh = e^{i\theta}e^{i\psi} = e^{i(\theta+\psi)} \in S^1$. The identity of this group is $e^{i(0)}$, and the inverse of $e^{i\theta}$ is $e^{i(-\theta)}$. When regarded as a group, S^1 is usually denoted $U(1)$.

Recall that the transition functions for a vector bundle on a manifold are $E_{\alpha\beta} : \varphi_\alpha(U_\alpha \cap U_\beta) \to \mathrm{GL}(r,\mathbb{R})$. We can ask for this assignment to be smooth, which leads to the following definition:

Definition. Given a manifold M, a *smooth vector bundle* $V \to M$ of *rank* r is a collection of smooth functions $E_{\alpha\beta} : \varphi_\alpha(U_\alpha \cap U_\beta) \to \mathrm{GL}(r,\mathbb{R})$ for each α, β that index the cover of M.

A few remarks:

- If $r = 1$, the $V \to M$ is said to be a *line bundle* on M (all the fibres are copies of the real line).

- If $E_{\alpha\beta} = \mathrm{Id}_{\mathbb{R}}$ for all α, β and all $\vec{a} \in \varphi_\alpha(U_\alpha \cap U_\beta)$ (or if M has only a single chart) then the bundle is the *trivial bundle* of rank r over M. This is often denoted $M \times \mathbb{R}^r$.

Every bundle has a trivial bundle of every rank $r \geq 1$. Are there any other "natural" bundles on a manifold M? Recall that we already have natural invertible matrices associate to every point $\vec{a} \in \varphi_\alpha(U_\alpha \cap U_\beta)$: For each $\vec{a} \in \varphi_\alpha(U_\alpha \cap U_\beta)$, the derivative $D_{\vec{a}}\varphi_{\alpha\beta}$ is an invertible linear transformation, and so its matrix (the Jacobian) is invertible. Recall the Jacobian $J_{\vec{a}}\varphi_{\alpha\beta}$:

$$J_{\vec{a}}\varphi_{\alpha\beta} = \begin{pmatrix} \frac{\partial}{\partial a_1}(\varphi_{\alpha\beta})_1 & \cdots & \frac{\partial}{\partial a_1}(\varphi_{\alpha\beta})_1 \\ \vdots & \ddots & \vdots \\ \frac{\partial}{\partial a_1}(\varphi_{\alpha\beta})_n & \cdots & \frac{\partial}{\partial a_n}(\varphi_{\alpha\beta})_n \end{pmatrix} \tag{36}$$

where $(\varphi_{\alpha\beta})_i$ is the i-th entry of $\varphi_{\alpha\beta}$. This gives us a map $\vec{a} \mapsto J_{\vec{a}}\varphi_{\alpha\beta}$. We can use this as $E_{\alpha\beta}$! The bundle defined by this choice of $E_{\alpha\beta}$ is called the *tangent bundle* of M, and is denoted $TM \to M$. The tangent bundle has rank equal to the dimension of M.

TM plays a special role: it "linearizes" M. The fibres of TM, denoted $T_{\vec{a}}M \cong \mathbb{R}^n$ are called *tangent spaces* to M. Elements $\vec{x} \in T_{\vec{a}}M$ are called *tangent vectors* to M.

Example. $M = V \cong \mathbb{R}^n$, a vector space. As a manifold it has no transition functions, so every bundle is trivial. Thus, $TM = TV$ is trivial of rank n. That is, $TM \cong V \times \mathbb{R}^n \cong V \oplus V \cong V_F \oplus V_B$.

Example. Consider $M = S^1$ with φ_N and φ_S as charts. We have a single transition function $\varphi_{NS}(a) = \frac{1}{a}$ and the dimension of M is 1. Thus the Jacobian is 1×1: $J_a(\varphi_{NS}) = \left(\frac{d}{da}\varphi_{NS}\right) = \left(\frac{d}{da}\frac{1}{a}\right) = \left(\frac{-1}{a^2}\right)$. Thus, $TS^1 \to S^1$ is the line bundle defined by $a \mapsto \frac{-1}{a^2}$ for all $a \in \varphi_N(U_N \cap U_S) = \mathbb{R} \setminus \{0\}$.

Say we have a vector $x = 2$ in T_3M, which is the north-pole tangent space associated to $p \in S^1$. The corresponding south-pole tangent space is $T_{\frac{1}{3}}M$. To get the corresponding vector in this space, we must apply $E_{NS}(3) = \left(\frac{-1}{3^2}\right) = \left(\frac{-1}{9}\right)$. Thus, the corresponding vector in $T_{\frac{1}{3}}M$ is $(E_{NS}(3))(2) = \frac{-2}{9}$.

This construction actually captures the original idea that we were seeking: TM is the collection of "imaginary" vector spaces that we believe to be around us, and the balls $\varphi_\alpha(U_\alpha)$ are the bits of $\mathbb{R}^n$ in our immediate vicinity.

Now consider: what is really happening to $T_{\vec{a}}M$ when it is mapped to $T_{\vec{b}}M$ (where $\vec{b} = \varphi_{\alpha\beta}(\vec{a})$)? The map $\varphi_{\alpha\beta}$ "writes" $\vec{b}$ coordinates in terms of the $\vec{a}$ coordinates. We can think of the $\vec{b}$ entries as functions of the $\vec{a}$-entries: $b_i(a_j)$. The entries of the Jacobian $J_{\vec{a}}\varphi_{\alpha\beta}$ are $\frac{db_i}{da_j}$ when $\varphi_{\alpha\beta}$ are viewed this way. The natural action of $J_{\vec{a}}\varphi_{\alpha\beta}$ is to change $\frac{d}{db_i}$ to $\frac{db_i}{da_j}\frac{d}{db_i} = \frac{d}{da_j}$.

So from this point of view, $T_{\vec{a}}M$ consists of vectors made up of operators $\frac{d}{da_i}$. More formally, $T_{\vec{a}}M = \mathrm{span}\left\{\frac{d}{da_1}\big|_{\vec{a}}, \ldots, \frac{d}{da_n}\big|_{\vec{a}}\right\}$.

Example. For $M = S^1$, we have $T_{\vec{a}}S^1 = \mathrm{span}\left\{\frac{d}{da}\big|_{a=\varphi_N(p)}\right\} = \left\{k\frac{d}{da}\big|_{a=\varphi_N(p)} \middle| k \in \mathbb{R}\right\}$. One can think of $\frac{d}{da}\big|_a$ as the generator of the tangent space. The vector $\vec{x} = 2$ in T_3S^1 could then be thought of as $2\frac{d}{da}\big|_{a=3}$, and its counterpart in $T_{\frac{1}{3}}S^1$ would be $\frac{-2}{9}\frac{d}{da}\big|_{b=\frac{1}{3}}$.

We could use these to differentiate functions on the circle (that is, $f : S^1 \to \mathbb{R}$). For example, let $f(a) = a^2$ on $S^1 \setminus \{N, S\}$. The vector $\vec{x}$ acts on this function by

$$\frac{d}{da}(a^2)\big|_{a=3} = 2(2a)\big|_{a=3} = 12 \tag{37}$$

On the other chart, we have

$$\frac{-2}{9}\frac{d}{db}(a^2)\big|_{b=\frac{1}{3}} = \frac{-2}{9}\frac{d}{db}\left(\frac{1}{b^2}\right)\big|_{b=\frac{1}{3}} = \frac{-2}{9}\left(2\frac{-1}{b^3}\right)\big|_{b=\frac{1}{3}} = \frac{4}{9}(27) = 12 \tag{38}$$

This illustrates that we can indeed to analysis on M, and the answers we get should not depend on coordinates.

The metric (the family of scalar products) will live on TM: to each tangent space $T_{\vec{a}}M$, we assign a scalar product $g_{\vec{a}} : T_{\vec{a}}M \times T_{\vec{a}}M \to \mathbb{R}$. We will certainly want the $g_{\vec{a}}$ to be compatible across transitions: $g_{\vec{a}}(\vec{x}_{\vec{a}}, \vec{y}_{\vec{a}}) = g_{\vec{b}}(\vec{x}_{\vec{b}}, \vec{y}_{\vec{b}})$, where $p \in U_\alpha \cap U_\beta$, $\vec{a} \in \varphi_\alpha(p)$, $\vec{b} = \varphi_{\alpha\beta}(\vec{a})$, $\vec{x}_{\vec{a}}, \vec{y}_{\vec{a}} \in T_{\vec{a}}M$, and $\vec{x}_{\vec{b}}, \vec{y}_{\vec{b}} \in T_{\vec{b}}M$. We can write $\vec{x}_{\vec{b}} = (J_{\vec{a}}\varphi_{\alpha\beta})\vec{x}_{\vec{a}}$, so $\vec{x}_{\vec{b}}$ is the vector corresponding to $\vec{x}_{\vec{a}}$ in $T_{\varphi_{\alpha\beta}(\vec{a})}M = T_{\vec{b}}M$. Likewise for $\vec{y}_{\vec{a}}$ and $\vec{y}_{\vec{b}}$. So, $g_{\vec{a}}(\vec{x}_{\vec{a}}, \vec{y}_{\vec{a}}) = g_{\vec{b}}(\vec{x}_{\vec{b}}, \vec{y}_{\vec{b}})$ says that

$$\begin{aligned}
\vec{x}_{\vec{a}}^T A_{\vec{a}} \vec{y}_{\vec{a}} &= \vec{x}_{\vec{b}}^T A_{\vec{b}} \vec{y}_{\vec{b}} \\
&= \left(J_{\vec{a}}\varphi_{\alpha\beta}\vec{x}_{\vec{a}}\right)^T A_{\vec{b}} \left(J_{\vec{a}}\varphi_{\alpha\beta}\vec{y}_{\vec{a}}\right) \\
&= \vec{x}_{\vec{a}}^T (J_{\vec{a}}\varphi_{\alpha\beta})^T A_{\vec{b}} (J_{\vec{a}}\varphi_{\alpha\beta})\vec{y}_{\vec{a}}
\end{aligned} \tag{39}$$

So we can see how the matrix of the scalar product must transform: $A_{\vec{a}} = (J_{\vec{a}}\varphi_{\alpha\beta})^T A_{\vec{b}} (J_{\vec{a}}\varphi_{\alpha\beta})$. This rule tells us how the metric transforms from coordinate patch to coordinate patch.

- Note that we will sometimes use a superscript α to remind ourselves that $\vec{a} \in \varphi_\alpha(U_\alpha)$, i.e. that $\vec{a}$ is an α-coordinate.

Example. Let us put a metric on $M = S^1$ and see this in practice. Take g_a^N with representative matrix $A_a^N = (1)$, constant for all $a \in \varphi_N(U_N)$. Now if $b = \varphi_{NS}(a) = \frac{1}{a}$, the A_b^S must satisfy $A_a^N = (J_a\varphi_{NS})^T A_b^S J_a \varphi_{NS}$. That is, $(1) = \left(\frac{-1}{a^2}\right)^T A_b^S \left(\frac{-1}{a^2}\right)$, and so we must have $A_b^S = \left(a^4\right) = \left(\frac{1}{b^4}\right)$. So we could describe g completely by $g = (g_a^N, g_b^S)$ where $g_a^N = (1)$ and $g_b^S = \left(\frac{1}{b^4}\right)$.

Now consider, at a point $a = 2$, the vector $x_2^N = 4$. The length of x_2^N is given by

$$\begin{aligned}
\|x_2^N\|_g &= \sqrt{g_2^N(x_2^N, x_2^N)} \\
&= \sqrt{{x_2^N}^T A_2^N x_2^N} \\
&= \sqrt{(4)(1)(4)} \\
&= 4
\end{aligned} \tag{40}$$

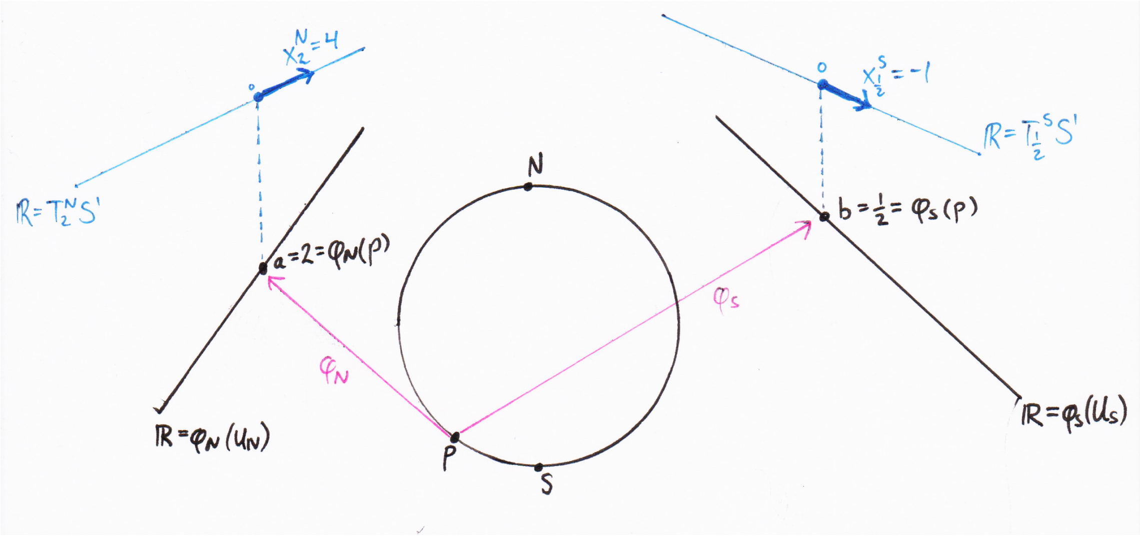
We can also measure it in the south-pole coordinates, but first we must transform it according to our rules:

$$\begin{aligned}
x_{\frac{1}{2}}^S &= E_{NS}(2)x_2^N \\
&= (J_2\varphi_{NS})x_2^N \\
&= \left(\frac{-1}{a^2}\right)\Bigg|_{a=2}(4) \\
&= -1
\end{aligned} \tag{41}$$

And then we can calculate the length of $x_{\frac{1}{2}}^S = -1$ to be

$$\begin{aligned}
\|x_{\frac{1}{2}}^S\|_g &= \sqrt{g_{\frac{1}{2}}^S(x_{\frac{1}{2}}^S, x_{\frac{1}{2}}^S)} \\
&= \sqrt{{x_{\frac{1}{2}}^S}^T A_{\frac{1}{2}}^S x_{\frac{1}{2}}^S} \\
&= \sqrt{(-1)\left(\frac{1}{(\frac{1}{2})^4}\right)(-1)} \\
&= \sqrt{16} = 4
\end{aligned} \tag{42}$$

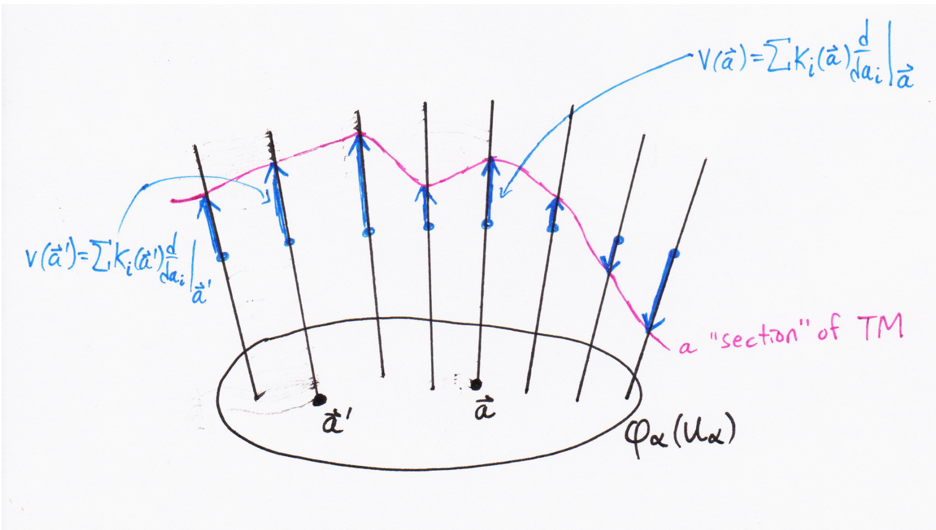
Note that on S, the metric is non-Euclidean but gives lengths that agree with those from N.



3.3 Metrics, Sections, Vector Fields, and One-forms

Since $g_{\vec{a}}^{\alpha} : T_{\vec{a}}^{\alpha}M \times T_{\vec{a}}^{\alpha}M \to \mathbb{R}$ is bilinear, symmetric, and nondegenerate, it belongs to $\mathcal{S}^2\left((T_{\vec{a}}^{\alpha}M)^*\right) \subset (T_{\vec{a}}^{\alpha}M)^* \otimes (T_{\vec{a}}^{\alpha}M)^*$. Inspired by this, we can create more bundles: in particular, we define a bundle $T^*M \to M$ whose fibres are the dual vector spaces $(T_{\vec{a}}^{\alpha}M)^* = \{\text{linear maps } T_{\vec{a}}^{\alpha}M \to \mathbb{R}\}$ with transition functions $E_{\alpha\beta}(\vec{a}) = \left(J_{\vec{a}}\varphi_{\alpha\beta}\right)^{-1}$. This bundle is called the *cotangent bundle of M*.

Recall that $T_{\vec{a}}^{\alpha}M = \operatorname{span}\left\{\frac{d}{da_1}\big|_{\vec{a}}, \dots, \frac{d}{da_n}\big|_{\vec{a}}\right\}$. that is, we are thinking of tangent vectors as being made up of these derivative operators. If we don't specify $\vec{a}$, we get a whole family of operators $\frac{d}{da_1}, \dots, \frac{d}{da_n}$ that do not belong to any particular tangent space until we specify $\vec{a}$. We can take a linear combination $v = k_1\frac{d}{da_1} + \dots + k_n\frac{d}{da_n}$, where the k_i are functions $k_i : \varphi_\alpha(U_\alpha) \to \mathbb{R}$. Hence, if we evaluate v at $\vec{a}$, we get the tangent vector $v(\vec{a}) = k_1(\vec{a})\frac{d}{da_1}\big|_{\vec{a}} + \dots + k_n(\vec{a})\frac{d}{da_n}\big|_{\vec{a}} \in T_{\vec{a}}^{\alpha}$.



Sections of $TM \to M$ are called *vector fields*. In a similar way, we can define operators that generate sections of $T^*M \to M$: namely, $da_1, \dots, da_n$. These are linear maps defined as follows, at each $\vec{a}$:

$$\left(da_i\big|_{\vec{a}}\right)\left(\frac{d}{da_j}\bigg|_{\vec{a}}\right) = \delta_{ij} = \begin{cases} 1, \text{if } i = j \\ 0, \text{if } i \neq j \end{cases} \tag{43}$$

This is consistent with $da_i\big|_{\vec{a}} \in \{\text{linear maps } T_{\vec{a}}^{\alpha}M \to \mathbb{R}\}$. A linear combination $\theta = \sum_{i=1}^{n} k_i da_i$ (where $k_i : \varphi_\alpha(U_\alpha) \to \mathbb{R}$) is a section of $T^*M \to \mathbb{R}$, and is called a *one-form*. Vector fields and one-forms are dual to each other in the sense that one forms take in vector fields and "spit out" numbers.

Going further, one can define a bundle $\mathcal{S}^2(T^*M)$ whose fibre at $\vec{a} \in \varphi_\alpha(U_\alpha)$ is $\mathcal{S}^2\left((T^\alpha_{\vec{a}}M)^*\right)$. In other words, a metric g is a section of $\mathcal{S}^2(T^*M)$. From a global point of view, g takes in two vector fields v, w and at each point $p \in M$ returns the number $g_p(v_p, w_p)$ (we say p here instead of $\vec{a}$ since g is coordinate-invariant).

To summarize,

Bundle	Section
TM	vector fields $v = \sum_{i=1}^n k_i \frac{d}{da_i}$
T^*M	one-forms $\theta = \sum_{i=1}^n k_i da_i$
$\mathcal{S}^2(T^*M)$	metrics $g = \sum_{i,j=1}^n g_{ij} da_i \otimes da_j$

The notation $g = \sum_{i,j=1}^n g_{ij} da_i \otimes da_j$ comes about because the metric is a section of $\mathcal{S}^2(T^*M) \subset T^*M \otimes T^*M$, and $T^*M \otimes T^*M$ has a basis given by $da_i \otimes da_j$. So g can be written as a linear combination of these $g = \sum_{i,j=1}^n g_{ij} da_i \otimes da_j$, where $g_{ij} : \varphi_\alpha(U_\alpha) \to \mathbb{R}$ and $g_{ij} = g_{ji}$. This way of writing g is called a *line element*, from which we can reconstruct the associated matrix representation.

3.4 The Problem of Mercury and the Einstein Equations

In 1859 it was realized that the observed orbit of the planet Mercury did not match with that predicted by Newtonian mechanics. At first it was hypothesized that another planet (preemptively named Vulcan) was perturbing this orbit. When astronomers could not find Vulcan or resolve this disparity by any other means, a revolutionary idea was needed.

Einstein resolved this with the following idea: suppose that the universe is a manifold M and a particle moves from p to q in M, and traces out a curve γ. The curve γ could be parametrized by "time" t (normalized): that is, $\gamma : [0,1] \to M$, $\gamma(0) = p$, $\gamma(1) = q$. At each point along γ, the particle has a velocity denoted $\dot{\gamma}$. So, $\gamma(t)$ is a point in M and $\dot{\gamma}(t)$ is a point in $T^\alpha_{\varphi_\alpha(\gamma(t))}M$. That is, $\dot{\gamma}$ is a vector field on M. Now if M is equipped with a metric g, we can define the *(arc)length* of $\gamma(t)$ by $L_g(\gamma) = \int_0^1 \sqrt{g_{\gamma(t)}(\dot{\gamma}(t), \dot{\gamma}(t))}dt$.

Consider the following: If one drops an object near the earth, under gravity it will take the shortest path to the ground. This is an example of the *principle of least action*: If an event is to be selected by nature from a set of possible events E, and the set has a function $H : E \to \mathbb{R}$ that is bounded below, then in the absence of other constraints, the event that occurs is the one that minimizes H. This is not rigourous in the least, and is not theorem: it is a *principle* that is taken as a *postulate* or *ansatz*. Of all the paths the ball could take, it takes the one that minimizes the Euclidean length $L_{\text{Euclid}}(\gamma)$. Einstein postulated that particles on a manifold would act the same way, and move along a length-minimizing path. Now the question is, what metric g are we using? That is, how are we defining length? The Euclidean metric agreed with observations for all the planets aside from Mercury. So there were two possibilities: either the priciple does not hold, or the Euclidean metric is the incorrect one to use to measure length in our universe.

A paper in 1913 by Eintein and Grossmann proposed the following correspondance between ideas in geometry and physics:

Physics	Geometry
spacetime	manifold
gravitational field	metric g
path of motion	distance-minimizing curve (w.r.t g)

This could be thought of as general relativity in a nutshell. The next hurdle is to find out what the correct metric g should be. There are several ways to go about finding a condition on g that would make it "physical". Firstly, note that that the metric seems to be locally given by

$$A = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & & \\ \vdots & & 1 & \\ 0 & & & -1 \end{pmatrix} \tag{44}$$

This is the Lorentzian equivalent of the Euclidean metric, called the *Minkowski metric*

- **Idea 1** : **Poisson′s Equation**. The gravitational field should be in an equilibrium state. That is, the final state of an evolutionary (time-dependent) process that shaped it. The study of partial differential equations tells us that such a process is described by Poisson's equation: $\Delta g_{ij} = f_{ij}$, where $\Delta = \frac{d^2}{dx_1^2} + \ldots + \frac{d^2}{dx_n^2}$ and f_{ij} are fixed functions $\varphi_\alpha(U_\alpha) \to \mathbb{R}$. So what should the functions f_{ij} be? They could be determined by initial or boundary conditions, and another possiblity is f_{ij}, leading to $\Delta g_{ij} = 0$. This is known as *Laplace's Equation*. Supporting this idea is the fact that Newtonian gravity is also related to Poisson's equation': $\Delta\phi = 4\pi G\rho$, where ϕ is the Newtonian potential, G is the gravitational constant, and ρ is density of a gravitational source. When we take a point mass ($\rho = 0$), we get a solution $\phi(r) = \frac{-Gm}{r}$.

- **Idea 2** : **Critical points**. At an ordinary point in space p, it is difficult to detect "first-order" deformations from the Minkowski metric. Another way of saying this is that $D_p g_{ij} = 0$. That is, we may often find ourselves at a critical point of g. If we take the Taylor expansion of the components of g at p, we see

$$g_{ij} = \delta_{ij} + \frac{1}{2} D_p^2 g_{ij} + \ldots \tag{45}$$

where

$$\delta_{ij} = \begin{cases} 1, \text{if } i = j \neq n \\ 0, \text{if } i \neq j \\ -1, \text{if } i = j = n \end{cases} \tag{46}$$

What this is saying is that sufficiently close to p, $g_{ij} \approx \delta_{ij} + \frac{1}{2} D_p^2 g_{ij}$, or $g_{ij} \propto k D_p^2 g_{ij}$. Here we see a proprtionality between two symmetric bilinear transformations. Perhaps

$$g \propto \sum_{ij} D_p^2 g_{ij} \tag{47}$$

- **Idea 3** : **Curvature control**. Recall that the topology of a manifold does not imply anything (for now) about curvature. Curvature is a feature of the metric, and in fact there are natural tensors associated to g that measure curvature. One such object is the *Riemann tensor* $Rie(g)$, the existence of which is a consequence of the *Fundamental Thereom of Semi-Riemannian Geometry*, which comes from the existence of the *Levi-Civita Connection*, which is a derivative of vector fields induced by g. We will not define $Rie(g)$ formally, but we will list a few facts about it:

  - $Rie(g)$ is a section of $(T^*M)^{\otimes 4}$. That is, it is a map that takes in 4 vector fields and gives a value in $\mathbb{R}$.
  - If $Rie(g) = 0$, we say that (M, g) is flat.
  - If $Rie(g) \neq 0$, we say that (M, g) is not flat.
  - $Tr(Rie(g))$ is a section of $(T^*M)^{\otimes 2}$. That is, $Tr(Rie(g))$ is a bilinear function.
  - By the "Bianchi Identity", $Tr(Rie(g))$is symmetric, and thus $Tr(Rie(g)) \in \mathcal{S}^2(T^*M)$.

Further, we define the *Ricci tensor* $R(g)$ to be $Tr(Rie(g))$. If $R(g) = 0$, (M, g) is said to be *Ricci-flat*. This is a weaker condition that flat: $R(g) = 0$ or $R(g) \propto g$ allows us to control the curvature without making g "uninteresting".

So, what is $R(g)$? There does exist a formula which does not depend on the type of coordinates: we can represent $R(g)$ by an $n \times n$ matrix of entries R_{ij}, which are given by

$$R_{ij} = \sum_{k=1}^{n} \frac{d}{da_k} \Gamma^k_{ji} - \frac{d}{da_j} \Gamma^k_{ki} + \sum_{l=1}^{n} \sum_{k=1}^{n} \Gamma^k_{kl} \Gamma^l_{ji} - \Gamma^k_{jl} \Gamma^l_{ki} \tag{48}$$

These Γ^k_{ij} are functions called *Christoffel symbols* (colloquially, the Christ-awful equations) defined by

$$\Gamma^k_{ij} = \sum_{m=1}^{n} g^{km} \left(\frac{dg_{mi}}{da_j} + \frac{dg_{mj}}{da_i} - \frac{dg_{ij}}{da_m} \right) \tag{49}$$

where g^{km} is the (k, m)-th entry of the inverse matrix of the metric.

- **Idea 4** : **Minimizing the Einstein – Hilbert Action**. Both Hilbert and Einstein believed that g should be "action-minimizing" (that is, that g should minimize the "Einstein-Hilbert Action"). This idea leads to the equation
$$R(g) = \alpha g \tag{50}$$
for some $\alpha \in \mathbb{R}$. This is known as Einstein's Equation (or *Equations*, if thought of in terms of the entries). Note that if $n = 4$ (spacetime with 3 space dimensions and 1 time) then g and $R(g)$ are given by 4×4 matrices of functions of coordinates. By symmetry, there are $\binom{4+2-1}{2} = 10$ independent terms. This is why Einstein's Equations are often given as a set of 10 individual equations that must be solved. If g solves $R(g) = \alpha g$, then (M, g) is called an *Einstein Manifold*. Einstein's Equation is sometimes expressed as
$$R(g) - \frac{1}{2}s(g)g + \Lambda g = \frac{8\pi G}{c^4}T \tag{51}$$
where $s(g)$ is the scalar curvature, Λ is the cosmological constant (which was introduced to get solutions which did not expand and contract over time), G is the gravitational constant, c is the speed of light, and T is a symmetric bilinear tensor caled the stress-energy tensor. In mathematics we set T to 0, and $R(g) = \alpha g$ is called the *vacuum Einstein Equation(s)*.

With all these mind, the question is: can we find solutions to $R(g) = \alpha g$? The answer is yes!

Example. Consider $M = \mathbb{R}^n$ with g given by
$$A = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & \ddots & & \\ \vdots & & 1 & \\ 0 & & & -1 \end{pmatrix} \tag{52}$$
Recall that we can refer to this as $\mathbb{R}^{n-1,1}$. This is a solution (with $\alpha = 0$) since all the entries are constant.

Interestingly, Einstein conjectured no other exact solution to his equations could be found. In 1916, Karl Schwarzschild constructed the following:

Example. Let $M = \mathbb{R} \times \mathbb{R}_{>0} \times S^2$. This is a manifold: $\mathbb{R}$ is covered by itself, $U = \mathbb{R}$ with coordinate t, $\mathbb{R}_{>0}$ is covered by itself, $V = \mathbb{R}_{>0}$ with coordinate r, and S^2 is covered by U_N, U_S, S^2 coordinatized by θ and φ (the latitude and longitude). Note that this is not the "usual" structure on S^2. Here t is time, and (r, θ, φ) are spherical coordinates. That is, at a fixed time $t = t_0$ and fixed radius $r = r_0$, we are on a fixed sphere of radius r_0, with latitude given by $\theta \in [\frac{-\pi}{2}, \frac{\pi}{2}]$ and longitude given by $\varphi \in [0, 2\pi]$. For $M > 0$ constant, g is given by
$$A_{(t,r,\theta,\varphi)} = \begin{pmatrix} \frac{2M}{r} - 1 & 0 & 0 & 0 \\ 0 & \left(1 - \frac{2M}{r}\right)^{-1} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2 \sin^2\theta \end{pmatrix} \tag{53}$$
The constant M has an interpretation as the mass of the object generating the gravity, and the spacetime is centered at this mass. We will need to know the eigenvalues of this matrix: since it is diagonal, the eigenvalues are exactly the entries. Hence, it is Lorentzian. We would like to know, in particular, when $g_{tt} < 0$: this is when $r > 2M$.

- $r = 2M$ is a critical value for g. This is called the *Schwarzschild radius*.
- When $r > 2M$, the negative eigenvalue is in the time direction.
- When $r < 2M$, the negative eigenvalue is in the r direction, and so r is the time direction? This is called *causality change*.

The metric "looks" degenerate or nonexistent at $\theta = 0$ or $r = 2M$, but this is only a defect of the coordinates we chose (recall latitude and longitude do not come from a natural manifold structure on S^2). Note that g models a universe near an object of mass M that is gravitating. We take it to a spherical shell of any radius greater than $2M$. Can it be demonstrated that this a solution to $R(g) = \alpha g$ for some α? Yes, we used a computer (MAPLE) to compute $R(g)$ and found that $R(g) = 0$. Thus, (M,g) is Ricci-flat (and is consequently an Einstein manifold for $\alpha = 0$). This g solves the problem of Mercury when we take $2M = 2.95 \times 10^3$m (twice the mass of sun) and $r = 6.96 \times 10^8$m (the radius of the sun). Specifically, there is a distance-minimizing curve around the "star" that agrees remarkably well with the orbit of Mercury.

What if we had a point mass centered at $r = 0$? Recall that g is undefined at $r = 0$, and this is *not* simply a defect a coordinate. Perhaps $r \to 0$ means that gravity is "blowing up" as we approach the centre. All light and matter that enters the $r = 2M$ ball cannot escape. This is a black hole! In this context, the Schwarzschild radius is called the *event horizon*.

Since 1916, there have been many solutions found (in 4 and other dimensions). For example,

- Reissner-Nordström (models gravity near a charged mass)
- Taub-NUT (models gravity near a massive magnetic monopole[2])
- Gödel (contains closed distance-minimizing curves, i.e. particles that revisit moments they have already experienced!)
- Mixmaster (models the end state of a chaotic universe)
- Kerr (models gravity near a rotating mass or black hole — featured in *Interstellar*!)

Up until now we have only been posing the $R(g) = \alpha g$ question for Lorentzian metrics. This is really only a physical restriction: could we find Riemannian solutions?

Consider the Euclidean metric $g = \sum_j da_j \otimes da_j$. if we map $a_1 \to ia_1$, then g becomes

$$d(ia_1) \otimes d(ia_1) + \sum_{j=2}^{n} da_j \otimes da_j = -da_1 \otimes da_1 + \sum_{j=2}^{n} da_j \otimes da_j \tag{54}$$

This is known as the *Wick transformation*. In the late 1960's, it was realized that Riemannian metrics were potentially useful for unifying gravity. (M,g), with g Riemannian (and some conditions on its decay: $g = \text{Euclidean} + O(r^{-4})$) is called a *gravitational instanton*). The takeaway here is that Riemannian metrics and complex manifolds (because of the Wick transformation) became interesting to physics!

We will not explicitly emphasize Riemannian over Lorentzian (or vice-versa) in what follows, but most examples that we consider will be Riemannian.

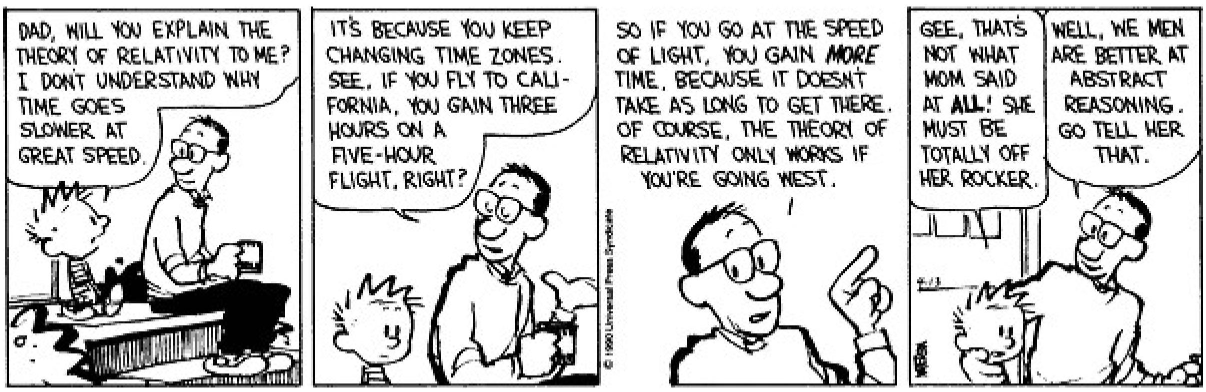


Figure 2: Calvin and Hobbes by Bill Watterson (not precisely the type of relativity which we are talking about, but still entertaining).

[2] A magnetic monopole, a particle with a single unit of "magnetic charge", has never been observed in nature. On the contrary, electric monopoles — which we usually call electrons — are observed all the time. This is an interesting asymmetry in electromagnetic theory.

3.5 The Idea of Unification

At the intersection of general relativity and geometry, we have an object of interest, namely Einstein manifolds, which we recall are manifolds M with metric g such that $R(g) = \alpha g$. Lorentzian Einstein manifolds were of chief intereset after 1930. In parallel, the following were occuring:

- The formalization of classical mechanics (position and momentum). In the mid 1800's , Hamilton's study of this lead to symplectic geometry (M with a symplectic structure on its tangent spaces (along with an extra condition)) as its own field.
- Riemann pioneered complex manifolds (through his study of Riemann surfaces), leading to a formal definition around 1950.

One of the themes here is the idea of "unificiation": For examples of unification in physics, we have

- Force and its resulting acceleration are equivalent ($F = ma$). (Newton, 1687)
- The electric and magnetic forces are equivalent. (Maxwell, 1865)
- The weak and strong and electromagnetic forces are equivalent. (Glashow-Salam-Weinberg, 1979)
- Gravity and the other forces are equivalent. (string theory (?))

On the other hand, for examples of unification in mathematics, we have

- The algebraification of geometry. That is, the Cartesian plane and equations for common shapes. (Descartes, 1600's)
- The Fundamental Theorem of Calculus: this is the idea that area can be viewed as a difference of antiderivatives. (Newton, 1687)
- The Nullstellensatz ("theorem of zeroes") which provides a link between algebraic objects known as ideals and geometric objects called varieties. (Hilbert, 1900)
- The unification of complex, symplectic, and Riemannian geometry: the idea that a complex structure and a symplectic form induce a metric $g = \omega \circ J$. (Eric Kähler, 1932)
- The Langlands Program (a link between algebra, geometry and representation theory)

By around 1950, mathematics was better formalized than it was previously, largely due to the efforts of David Hilbert and Emmy Noether. At this point, Kähler manifolds (manifolds which have (ω, J, g) all compatible) were garnering attention for the unification that they represented. To formally define these, we need to define complex and symplectic manifolds.

4 Calabi-Yau Manifolds

4.1 Complex Manifolds

There are two natural definitions of a complex manifold, based on what we have done so far:

Definition. $(\mathbf{1})$ A *complex manifold* is a smooth manifold M with a complex structure on each tangent space T_pM. That is, a linear map $J_p : T_pM \to T_pM$ such that $J_p^2 = -\mathrm{Id}$ for each $p \in M$. We refer to the J_p collectively as J.

Note that this definition requires $\mathrm{rk}(TM)$ to be even (equivalently, $\dim(M)$ even).

Definition. $(\mathbf{2})$ A *complex manifold* is a set M with a covering by subsets U_α and charts $\varphi_\alpha : U_\alpha \hookrightarrow \mathbb{C}^n$ such that $\varphi_\alpha(U_\alpha)$ is an open ball in $\mathbb{C}^n$. Further, we must have for each α, β that the transition functions $\varphi_{\alpha\beta} = \varphi_\beta \circ \varphi_\alpha^{-1} :$ $\varphi_\alpha(U_\alpha \cap U_\beta) \to \varphi_\beta(U_\alpha cap U_\beta)$ are holomorphic.

Here, *holomorphic* means that $\phi_{\alpha\beta}$ satisfies the Cauchy-Riemann Equations, which in turn is equivalent to the following: For each $\vec{a} \in \varphi_\alpha(U_\alpha \cap U_\beta) \subset \mathbb{C}^n \cong \mathbb{R}^{2n}$, $J_{\vec{a}}\varphi_{\alpha\beta}$ is a $2n \times 2n$ matrix, interpreted as a linear transformation $T^\alpha_{\vec{a}}M \to T^\beta_{\varphi_{\alpha\beta}(\vec{a})}M$. The fibre $T^\alpha_{\vec{a}}M \cong \mathbb{R}^{2n}$ has a linear transformation $J^2_{\vec{a}} = -\mathrm{Id}$ (this is the complex structure that was making $\mathbb{R}^{2n}$ into $\mathbb{C}^n$). Likewise, $T^\alpha_{\vec{b}}M \cong \mathbb{R}^{2n}$ has $J^2_{\vec{b}} = -\mathrm{Id}$. The condition $\varphi_{\alpha\beta}$ holomorphic means that $J_\beta \circ J_\alpha \varphi_{\alpha\beta} = J_\alpha \varphi_{\alpha\beta} \circ J_\alpha$.

The big question here is now the question of whether these two definitions are equivalent. In fact, they are different as stated. Definition 2 implies definition 1, but definition 1 is not strong enough to imply definitition 2. We must add a condition to definition 1 to make them agree, and that condition is the following: For all v, w vector fields on M and for each $p \in M$, we ask that J_p satisy

$$N_{J_p}(v_p, w_p) = 0 \tag{55}$$

where

$$N_{J_p}(v_p, w_p) = [J_p v_p, J_p w_p] - J_p[J_p v_p, w_p] - J_p[v_p, J_p w_p] - [v_p, w_p] \tag{56}$$

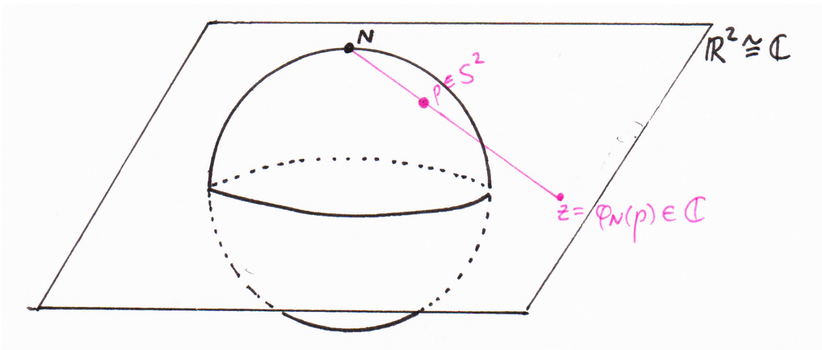
where $[-,-]$ is the Lie bracket defined by

$$[v_p, w_p](f) = v_p(w_p(f)) - w_p(v_p(f)) \tag{57}$$

where v_p and w_p are differentiating $f : M \to \mathbb{R}$ by interpreting v_p as $\sum k_i(p) \frac{d}{dx_i}\big|_p$ (and similarly for w_p). N_J is called the *Nijenhuis tensor*. With this extra condition, the two definitions are equivalent (this is the Newlander-Nirenberg Theorem (1957)).

The vanishing of N_J captures the notion of the J_p's varying holomorphically from tangent space to tangent space. *Without* the vanishing of N_J, J is called an *almost complex structure*. Having both definitions is very useful! The first is easier to work with in terms of Kähler structure, while the second is easier to construct examples from.

Example. Our basic example of a comploex manifold will be the the *projective line*, denoted $\mathbb{P}^1$. This is actually S^2 with a complex structure obtained as follows: cover S^2 with U_N, U_S and choose charts φ_n, φ_S that map into $\mathbb{C}$ instead of $\mathbb{R}^2$. The transition function here is $\varphi_{NS}(z) = \frac{1}{z}$ (the complex analogue of the S^1 transition function!). So, $\varphi_{NS} : \varphi_N(U_N \cap U_S) \to \varphi_S(U_N \cap U_S)$ is a map from CC^* to $\mathbb{C}^*$. Further, one can check that $\frac{1}{z}$ is holomorphic on $\mathbb{C}^*$, so we have constructed a 1-dimensional complex manifold.



This is called the projective line; why? This is because is a complex "line" ($\mathbb{C}^1$) together with an extra point, to which all real lines in $\mathbb{C}$ converge. This extra point is the point at "infinity" that φ_N sends $p = N$ to. The adjective "projective" refers to the fact that parallel lines converge at infinity. As a set, $\mathbb{P}^1$ is S^2, and in terms of holomorphic coordinates it has a coordinate $z \in \mathbb{C}$ (z is the north-pole coordinate) as well as an extra point $z = \infty$.

Example. Every 2-dimensional (over $\mathbb{R}$) smooth surface of genus g can be turned into a 1-dimensional complex manifold: These are called *curves* since they are 1-dimensional over $\mathbb{C}$, but are unhelpfully collectively referred to as *Riemann surfaces*.

genus	smooth	complex
0	the 2-sphere S^2	the projective line $\mathbb{P}^1$
1	the torus T^2	the elliptic curve
2	the 2-holed pretzel P^2	the genus 2 curve
$\vdots$	$\vdots$	$\vdots$

So, a *complex curve* is 1-dimensional over $\mathbb{C}$ and 2-dimensional over $\mathbb{R}$, a *complex surface* is 2-dimensional over $\mathbb{C}$ and 4-dimensional over $\mathbb{R}$, a 3-*fold* is 3-dimensional over $\mathbb{C}$ and 6-dimensional over $\mathbb{R}$, et cetera. Note the following convention: an *n-manifold* is an n-dimensional smooth manifold over $\mathbb{R}$, while an *n-fold* is an n-dimensional complex manifold ($2n$-dimensional over $\mathbb{R}$).

We can gain some intuition about complex manifolds by studying Riemann surfaces. Here, M is 1-dimensional over $\mathbb{C}$, so TM has rank 1 (over $\mathbb{C}$). That is, the tangent bundle of a Riemann surface is a holomorphic line bundle. Section of TM are called *holomorphic vector fields*. These look like $v = k\frac{d}{dz}$, where k is a single holomorphic function $k^\alpha : \varphi_\alpha(U_\alpha) \to \mathbb{C}$. The functions k^α (recall that the superscript α denotes the chart) collectively vanish finitely many times over M. This is a result from complex analysis, owing to the fact that M is closed and compact.

Let $M = \mathbb{P}^1$ and let v be a holomorphic vector filed on M. the data of v is really $v^N = k^N\frac{d}{dz}$ and $v^S = k^S\frac{d}{dz}$, where k^N is a function of z and k^S is a function of $\frac{1}{z}$. The transition function relates k^N and k^S:

$$\frac{-1}{z^2}k^n(z) = k^S(\frac{1}{z}). \tag{58}$$

Equivalently, $k^N(z) = z^2k^S(\frac{1}{z})$ (we absorb the -1 into k^S). Recall that holomorphic imples complex analytic, which means that we can expand these functions as Taylor series'. Thus, we have

$$k_0 + k_1z + k_2z^2 + k_3z^3 + \cdots = z^2(l_0 + l_1\frac{1}{z} + l_2\frac{1}{z^2} + \ldots) \tag{59}$$

which simplifies to

$$k_0 + k_1z + k_2z^2 + k_3z^3 + \cdots = l_0z^2 + l_1z + l_2 + l_3\frac{1}{z} + l_4\frac{1}{z^2} + \ldots \tag{60}$$

which tells us that there are only three possible nonzero coefficients on each side; $k_0 = l_2$, $k_1 = l_1$ and $k_2 = l_0$. In particular, $k^N(z) = k_0 + k_1z + k_2z^2 = z^2k^S(\frac{1}{z})$. At any $p \in \mathbb{P}^1$, $v_p^N = k^N(p)\frac{d}{dz}\big|_p = (k_0 + k_1(p) + k_2(p^2))\frac{d}{dz}\big|_{z=p}$.

Hence; we have proven that holomorphic vector fields on $\mathbb{P}^1$ are represented by degree 2 polynomials (in the coordinates of one chart) and so the space of holomorphic vector fields on $\mathbb{P}^1$ is $\mathbb{C}^3$.

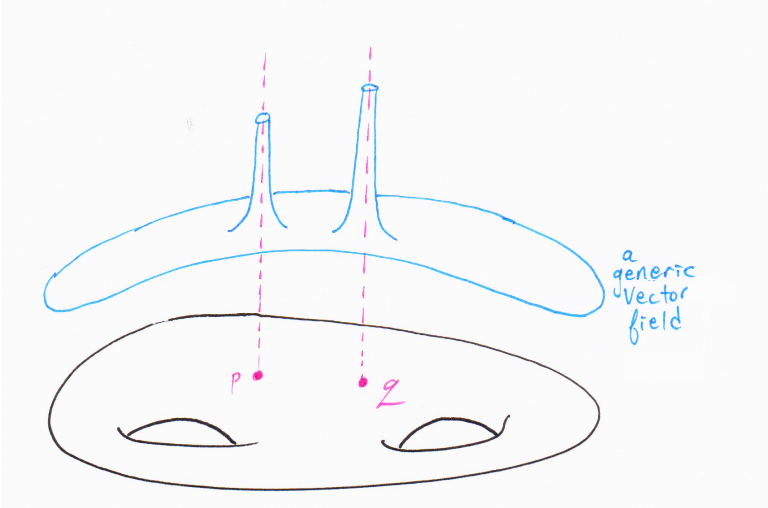
As a corollary, every nonzero holomorphic vector field on $\mathbb{P}^1$ vanishes at 2 points on $\mathbb{P}^1$. Note that if a vector field is constant on one chart, it vanishes ar 2 points on the other chart. This is reminescent of the so-called hairy ball theorem. Every holomorphic vector field is a smooth vector field, and smooth vector fields on S^2 must vanish twice. Moreover, the topological Euler characteristic of the sphere is also 2. Thus, our result about the vanishing of holomorphic vector fields is an example of complex/algebraic geometry detecting a topological property.

Recall that the elliptic curve ($g = 1$) is really just the fundamental domain in $\mathbb{C}$. That is, $\mathbb{C}/D = M$. Thus the tangent bundle is trivial ($TM = M \times \mathbb{C}$) and $v = k\frac{d}{dz}$ on a single chart. The function k is a holomorphic function $\varphi(M) \to \mathbb{C}$, and since $\varphi(M)$ is compact in $\mathbb{C}$, k is constant and thus the space of holomorphic vector fields on the elliptic curve is just $\mathbb{C}$. Thus the generic ($k \neq 0$) vector field on M is nowhere vanishing.

We have the facts

genus	the number of times a generic vector field vanishes
0	2
1	0
$\vdots$	$\vdots$
g	$2-2g$
$\vdots$	$\vdots$

The fact that when $g \geq 2$, $2-2g < 0$ seems strange. What this means is that a holomorphic vector fields on a genus 2 (or higher) surface has $|2-2g| = 2g-2$ more "poles" (places where $k^\alpha \to \infty$) than zeroes. These poles are two-dimensional asymptotes. In the following, $2-2g = -2$, and p, q are the poles of the vector field.



As a corollary: on a genus $g \geq 2$ surface, there are no nonzero holomorphic vector fields.

Definition. The number $2-2g$ is called the *degree* of the tangent bundle a Riemann surface M of genus g. This is also called the *first chern class of* TM.

$$\deg(TM) = c_1(TM) = 2-2g \tag{61}$$

Chern classes were introduced in 1946 by Shiing-Shen Chern, one of the fathers of complex geometry. We will see a different interpretation of them later on.

Let us focus on the interpretation of $\deg(TM) = 2-2g$.

- $M = \mathbb{P}^1$: We have previously computed $\deg(T\mathbb{P}^1) = 2$ (from $v^N(z) = (k_0 + k_1 z + k_2 z)\frac{d}{dz}$), and so there is a $\mathbb{C}^3$'s worth of vector fields on $\mathbb{P}^1$.

- For M the elliptic curve, M has a one-dimensional space of holomorphic vector fields $v^\alpha(z) = kddz^\alpha$. The transitions for TM mean that k is the same on each chart, and each such v is nonvanishing (except when $k=0$ and we have the zero vector field. This is consistent with the idea that on an elliptic curve, $TM = M \times \mathbb{C}$

- When $g \geq 2$, $2-2g \leq 0$ and so every nonzero vector field has $2-2g$ more poles than zeroes.

By duality, $\deg(T^*M) = 2g-2$. This implies that $\mathbb{P}^1$ has no nonzero holomorphic one-forms (they all have exactly two more poles than zeroes), the elliptic curve has constant one-forms, and a curve with $g \geq 2$ as plenty of holomorphic one-forms, vanishing at $2g-2$ points.

4.2 Symplectic Manifolds

Now, what about symplectic geometry?

Definition. A *symplectic manifold* is a smooth manifold with a family ω of symplectic structures ω_p on each tangent space T_pM (that is, $\omega \in \wedge^2(T_p^*M)$) such that ω_p is nondegenerate at each point $p \in M$ and $d\omega = 0$ (when $d\omega = 0$, we say ω is *closed*).

Note that this definition again necessitates that $\mathrm{rk}(TM) = \dim(M)$ is even over $\mathbb{R}$. As a section of $\wedge^2(T^*M)$, ω can be written in a basis of one-forms as $\omega = \sum_{i,j} \omega_{ij} dx_i \wedge dx_j$.

Definition. Elements of $\wedge^k(T^*M)$ are called *k-forms*. The function d is the *exterior derivative*,

$$d : \wedge^k(T^*M) \to \wedge^{k+1}(T^*M)$$

defined by the following:

$$d(\alpha \wedge \beta) = d\alpha \wedge \beta + \alpha \wedge d\beta \;\text{ and }\; d(f dx) = df \wedge dx \tag{62}$$

where $(df)_p(v_p) = (D_p f)(v_p)$. These equations also imply $d^2 x = d(dx) = 0$

From this definition, we can write

$$\begin{aligned} d\omega &= d\left(\sum_{i,j} \omega_{ij} dx_i \wedge dx_j\right) \\ &= \sum_{i,j} d\left(\omega_{ij} dx_i \wedge dx_j\right) \\ &= \sum_{i,j} d(\omega_{ij} dx_i) \wedge dx_j + \omega_{ij} dx_i \wedge d^2 x_j \\ &= \sum_{i,j} d\omega_{ij} \wedge dx_i \wedge dx_j, \end{aligned} \tag{63}$$

a 3-form which we ask to be identically zero. Why is this? From a physics point of view, if a system of differential equations is Hamiltonian (i.e. has an energy function H) then there exists a Poisson bracket. If this bracket is everywhere nondegenerate, then the bracket becomes ω, and the closure condition $d\omega = 0$ corresponds to conservation of energy H along flows of the system. From a mathematical point of view, $d\omega = 0$ gives Darboux's Theorem on manifolds: that is, around each point $p \in M$, there exists an open (in the manifold topology) set U such that ω can be written $\omega = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$ at each point in U. In particular, ω is locally constant.

The conditions $N_J = 0$ and $d\omega = 0$ are referred to as *integrability conditions*.

4.3 Kähler Manifolds and the Calabi Conjecture

Now suppose that a manifold M is both complex and symplectic. Then by the Newlander-Nirenberg Theorem, there exists a family J of complex structures on tangent spaces. We say that ω is *compatible* with J if $\omega(Ju, Jv) = \omega(u, v)$ for all smooth vector fields u, v on M. Technically, we should say $\omega_p(J_p u_p, J_p v_p) = \omega(u_p, v_p)$ for all $p \in M$. Then $(\omega \circ J)(u, v) = \omega(u, Jv)$ is a metric on M (a nondegenerate section of $\mathcal{S}^2(T^*M)$). Alternatively, if M is a complex manifold with metric g, and $g(Ju, Jv) = g(u, v)$ for all smooth vector fields u, v, then $(g \circ J)(u, v) = g(Ju, v)$ is an (almost) symplectic form on M (only 'almost' due to the fact that $d\omega = 0$ is not guaranteed in this direction).

Definition. If (M, J, ω) has ω compatible with J, then M is called a *Kähler manifold*.

Alternatively,

Definition. If (M, J, g) has g compatible with J and $d(g \circ J) = 0$, then M is called a *Kähler manifold*.

The form ω is called the *Kähler form* and g is called the *Kähler metric*. It is an important fact that all compact Riemann surfaces are Kähler. The following is sometimes called the *Kähler package*:

Structure	J	ω	g
Integrability	$N_J = 0$	$d\omega = 0$	?

It seems that there is no integrability condition on g. Should there be? If so, what should it be? Calabi was the first to notice this asymmetry, and proposed that we consider $R(g) = 0$. That is we could say g integrable if and only if g is a Ricci-flat Einstein metric.

Note that if we start with g and form ω as $\omega \circ J$, we can do the following:

$$\begin{array}{ccc} g & \xrightarrow{\text{curvature}} & R \\ \Big\downarrow \text{antisymmetrize} & & \Big\downarrow \text{antisymmetrize} \\ (g \circ J)(u, v) = g(Ju, v) & \xrightarrow{\text{curvature}} & (R \circ J)(u, v) = R(Ju, v) \end{array} \tag{64}$$

This defines $R(\omega)$, the *Ricci form* or *Ricci curvature of* ω. The Ricci tensor $R(g)$ can be thought of as the symmetric curvature, and the Ricci form $R(\omega)$ as the antisymmetric curvature. These contain the same information and can be obtained from one another by composing with J.

Definition. If (M, J, ω) is Kähler, and $g = \omega \circ J$ satisfies $R(g) = 0$ (equivalently, if $R(\omega) = 0$) then M is said to be *Kähler-Einstein*. The equation $R(\omega) = 0$ is the *Kähler-Einstein equation* (which is a special case of $R(\omega) = \alpha\omega$). Equivalently, M is *Kähler-Einstein* if the Kähler data is tri-integrable: $N_J = 0$, $d\omega = 0$, and $R(g) = 0$.

The Kähler-Einstein Problem is the following: given (M, J, ω) a Kähler manifold, when is (M, J, ω) Kähler-Einstein? We will attempt to gain insight into this problem by studying Riemann surfaces.

Instead of defining $c_1(TM)$ as $\deg(TM)$ a priori, we can define $c_1(TM) = \int_M R(\omega)$. The fact that $R(\omega)$ is a nondegenerate 2-form means that it can be integrated over the 2-dimensional real surface M. Moreover, the fact that M is compact implies that the integral converges.

Theorem 4. *(The Gauss-Bonnet-Riemann-Roch-Chern Theorem):*

$$\frac{1}{2\pi} c_1(TM) = \frac{1}{2\pi} \int_M R(\omega) = \chi(M) = 2 - 2g = deg(TM) \tag{65}$$

This theorem can be proven using the so-called "$d\bar{d}$-lemma" for Kähler manifolds, which we will state shortly. In addition, it has the following consequences. It does not matter which (compatible) ω is used to define $c_1(TM)$: it is $2\pi(2g-2)$ regardless. Thus $c_1(TM)$, up to a factor of 2π, it is equal to the Euler characteristic (which is a purely topological invariant). The topological invariants g and χ are obtainable from one another in a one-to-one way. We usually redefine $c_1(TM)$ as $\frac{1}{2\pi}\int_M R(\omega)$ so that $c_1(TM) \in \mathbb{Z}$. We can define the *first Chern class of a manifold* $c_1(M) = c_1(TM)$.

It is possible to say similar things in higher dimension but for complex dimension ≥ 2, $\deg(TM)$ is still a number but it is not canonical. Also, $c_1(M)$ is no longer a number (but can be turned into one again by making some choices). Unless, of course, $c_1(M)$ is a zero object, in which case $c_1(M)$ *is* a number: it is 0.

Let M be any Kähler manifold (M, J, ω). If $R(\omega) = 0$, then $c_1(TM) = \frac{1}{2\pi}\int_M R(\omega) = 0$. That is, a Kähler-Einstein structure imposes $c_1(M) = 0$. Immediately, if M is a Riemann surface with Kähler-Einstein structure, then $c_1(M) = 0 = 2 - 2g$, which implies $g = 1$ and thus the only Riemann surface which can have a Kähler-Einstein structure is the elliptic curve. Consider the other direction: is $c_1(M) = 0$ enough to guarentee the existence of Kähler-Einstein structures? For example, let M be the elliptic curve with any compatible ω and compute $R(\omega)$. Calabi was not able to find any explicit compatible ω with $R(\omega) = 0$. He did, however, pose the following:

The Calabi Conjecture (1957): If a compact Kähler manifold (M, J, ω) has $c_1(M) = 0$, then there exists a smooth function $\phi : M \to \mathbb{R}$ such that $R(\omega + d\bar{d}\phi) = 0$.

The term $d\bar{d}$ is the complex Laplacian, which is the antiholomorphic exterior derivative followed by the holomorphic one, giving a complex 2-form. So what we are doing is taking ω and disturbing it a bit: that is, disturbing (M, J, ω) to $(M, J, \omega + d\bar{d}\phi)$. However a sequence of ϕ such that $R(\omega + d\bar{d}\phi) = 0$ was unable to be found. We say that $\omega + d\bar{d}\phi$ is in the same Kähler class as ω since $d(\omega + d\bar{d}\phi) = d\omega + d^2\bar{d}\phi = 0$. The function ϕ is called a *Kähler potential.*

Lemma 5. *(The $d\bar{d}$-Lemma): ω is compatible with J if and only if $\omega = d\bar{d}f$ for some funciton $f : M \to \mathbb{R}$, where d and $\bar{d}$ are exterior derivatives coming from J.*

Recall that the Kähler cone of J is $\wedge^2_J(T^*M)$, the set of all nondegenerate closed ω's compatible with J. So if we start with a compatible symplectic form ω, then $\omega + d\bar{d}\phi = d\bar{d}f + d\bar{d}\phi = d\bar{d}(f + \phi)$, and so $\omega + d\bar{d}\phi \in \wedge^2_J(T^*M)$. This tells us that compatibility is not an issue when we deform ω in this way.

Example. We could consider $M = \mathbb{R}^2$ with J given by $C =$ and ω given by $B =$. This is Kähler and hence a Kähler manifold. Recall that $\omega \circ J$ is the metric given by I_2. That is, the Kähler metric is the Euclidean metric. ω constant implies that $R(\omega) = 0$, and hence Euclidean space is Kähler-Einstein. However this is not the sort of solution that Calabi was looking for: he was looking for noncompact examples, such as on a Riemann surface.

Physicists did not necessarily see immediate value in these constructions, even if $\omega \circ J$ was Lorentzian. Compatibility is a strong condition: why should the Einstein metric g have to be compatible with ω via some complex structure J? Moreover, a Kähler-Einstein structure imposes $c_1(M) = 0$, and there was no evidence that the first chern class of the universe was zero. What they wanted was exact solutions to the original Einstein equations, not some even more difficult equations! Mathematicians thought there was a potential reward: if $c_1(M) = 0$ implies the existence of a Kähler-Einstein structure, then Kähler-Einstein metrics are purely topological in nature, and we would know they exist simply from the topology of a space (no equations required). In the 1960's, the Kähler-Einstein problem was identified by the leading geometers of the day (Calabi, Chern, Atiyah and others) to be a major problem in mathematics, a problem bridging topology and geometry in need of an analytical solution. In the late 1970's Shing-Tung Yau, a student of Chern, solved this problem.

4.4 A Sketch of Yau's Proof of the Calabi Conjecture

Shing-Tung Yau proved the Calabi conjecture to be true. The following is a rough sketch of his revolutionary proof.

We have $c_1(M) = 0$, and thus $\int_M R(\omega) = 0$. One way that this could happen is if $R(\omega) = 0$. Another possibility is that $R(\omega) = d\bar{d}f$ for some f, in which case

$$\int_M R(\omega) = \int_M d\bar{d}f = \int_V d^2\bar{d}f = \int_V 0 = 0 \tag{66}$$

where V is the volume contained in M. This is in fact always the case, by another application of the $d\bar{d}$-Lemma. So $R(\omega) - d\bar{d}f = 0$. What we wnat next is a function $\phi : M \to \mathbb{R}$ such that $R(\omega + d\bar{d}\phi) = R(\omega) - d\bar{d}f$. Calabi found that this equation is equivalent to

$$\begin{aligned} -d\bar{d}\log\det(\omega + d\bar{d}\phi) &= -d\bar{d}\log\det(\omega) - d\bar{d}f \\ d\bar{d}\log\left(\frac{\det(\omega + d\bar{d}\phi)}{\det(\omega)}\right) &= d\bar{d}f \\ \det(\omega + d\bar{d}\phi) &= e^f\det(\omega) \end{aligned} \tag{67}$$

which is a nonlinear partial differential equation in variables $z_i, \bar{z}_i$, in particular is a type of Monge-Ampére equation. The question becomes: does there exist a solution ϕ for each f? Note that there is a particular normalization at $R(\omega) = d\bar{d}f$ that ensures that two different ϕ cannto be solutions for the same f. Calabi showed that *if* a solution ϕ exists for f, then it is unique (up to addition of a constant). The next point is to realize that there is at least one solution: for $f = 0$ there is a trivial solution $\phi = 0$.

Next, we want to consider the map $\Phi : \phi \mapsto f$ which takes solutions ϕ of $\det(\omega + d\bar{d}\phi) = e^f\det(\omega)$ to the f for which they are solutions. Let the set of solutions to 67 be $\mathcal{S} \subseteq C^\infty(M)$. Thus $\Phi : \mathcal{S}/\mathbb{R} \to C^\infty(M)$ ($\mathcal{S}/\mathbb{R}$ is $\mathcal{S}$ modding out by translation by $\mathbb{R}$). So $\mathrm{Im}(\Phi)$ is the set of f with solutions. The fact that if ϕ exists for f then it is unique implies that Φ is injective. We also know that $\mathrm{Im}(\Phi)$ is nonempty. Next, Calabi showed that $d\bar{d}$ is locally invertible: if we have a solution ϕ_0 for f_0 then there exists a set of solutions for f's near to f_0. Thus $\mathrm{Im}(\Phi)$ is a nonempty open subset of $C^\infty(M)$.

The following is the key to the proof, due to Yau. Consider $\overline{\mathrm{Im}(\Phi)} \subseteq C^\infty(M)$. Now suppose $F \in \overline{\mathrm{Im}(\Phi)} \setminus \mathrm{Im}(\Phi)$ and suppose that there exists a sequence $\phi_1, \phi_2, \ldots$ of solutions such that $f_1, f_2, \ldots$ approaches F. We want to show that $\{\phi_i\}$ converges to a ϕ_∞ that is a solution for 67 for F. Yau provided *a priori* bounds on the ϕ_i and their derivatives in terms of the derivatives of $\log(f_i)$. The goal was to show that the sequence $\{\phi_i\}$ lies in a compact subset of $C^\infty(M)/\mathbb{R}$. This would imply that the ϕ_i converge to a (finite) ϕ_∞ that is a solution for F. Then $\overline{\mathrm{Im}(\Phi)} = \mathrm{Im}(\Phi)$, so $\mathrm{Im}(\Phi)$ is open and closed, as well as nonempty. Thus $\mathrm{Im}(\Phi) = C^\infty(M)$ and so every f in $C^\infty(M)$ has a solution ϕ, and so the Calabi Conjecture is true. Most such bounds are not initially good enough to conclude compactness: they require careful incremental improvements (about a dozen in this case), each requiring different analytical tricks and estimates. Yau was able to do this this for the equation 67 with the condition $\det(\omega + d\bar{d}\phi) > 0$. This method of proof is now called the *continuity method*. Yau was awarded the 1982 Fields Medal for this work.

This solves the Kähler-Einstein problem in the affirmative:

Theorem 6. *(The Calabi-Yau Theorem): If (M, J, ω) is Kähler with $c_1(M) = 0$ then M admits a Ricci-flat Kähler structure within its Kähler class (that is, ω is deformed to $\omega + d\bar{d}\phi$)).*

Such manifolds are now called Calabi-Yau manifolds (Kähler-Einstein more generally refers to Kähler manifolds with $R(\omega) = \alpha\omega$ for some α, and Calabi-Yau refers specifically to the $\alpha = 0$ case).

4.5 Calabi-Yau Geometry and Hodge Numbers

We have the following different types of Kähler-Einstein manifolds:

- $\alpha = 0$: Calabi-Yau.
- $\alpha > 0$: Fano (or del Pezzo).
- $\alpha < 0$: General type. The name "general type" comes from the fact that when M is of any dimension, α plays the role of $2-2g$. Recall that most Riemann surfaces have $2-2g < 0$, so it can be thought of as the "most common".

For $\alpha > 0$, the Kähler-Einstein problem is solved n the positive: there exists ϕ such that $R(\omega + d\bar{d}\phi) = \alpha\omega$. This is known as the Aubin-Yau Theorem, and was proved earlier than the $\alpha = 0$ case.

For $\alpha < 0$, the Kähler-Einstein problem is negative. The Riemann surface $M = \mathbb{P}^1$ with Kähler structure given by the complex analogue of the round metric (this is known as the Fubini-Study metric) is Kähler-Einstein for $\alpha = 1$: $R(\omega) = \omega$. Thus $\mathbb{P}^1$ is an example of a Fano Kähler-Einstein manifold. However there are Kähler manifolds with $c_1(M)$ a positive class for which $R(\omega + d\bar{d}\phi) = \alpha\omega$ has no solution ϕ for any $\alpha > 0$ (the first examples of such manifolds were found by Tian). So when exactly is a Fano Kähler manifold Einstein? A condition was found in 2012, by Donaldson/Tian/et al.

Note that Yau's proof for $\alpha = 0$ is a proof of existence and uniqueness, but it is not constructive. In particular, *no one* has ever written down an explicit Calabi-Yau metric on a compact Calabi-Yau manifold.

One could ask for a rough classification of Calabi-Yau manifolds:

complex dimension	Calabi-Yau manifolds
1	The elliptic curve E
2	$K3$ surfaces
3	Calabi-Yau 3-folds
$\vdots$	$\vdots$
n	Calabi-Yau n-folds
$\vdots$	$\vdots$

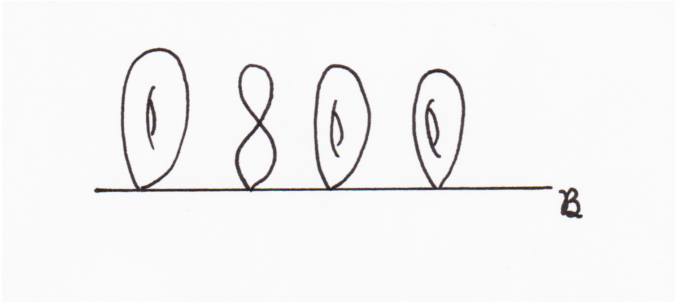
The $K3$ surfaces include $E \times E$ as well as other nontrivial elliptic fibrations. There is ongoing interest in Calabi-Yau 3-folds.

There is a small issue with the definition of Calabi-Yau manifolds that we use. What if M were noncompact but not just $\mathbb{R}^{2n}$ (that is, noncompact but interesting). Certain aspects of our arguements break down in the absence of compactness. We could replace the condition $c_1(M) = 0$ with $\det(T^*M) = M \times \mathbb{C}$. The determinant of the vector bundle T^*M is defined in the following way: the transition functions of T^*M are elements of $\mathrm{GL}(n, \mathbb{C})$ (i.e. they are invertible, complex-valued Jacobians) where $n = \dim_{\mathbb{C}}(M)$. The determinants of these are nonzero complex numbers (i.e. elements of $\mathbb{C}^* = \mathrm{GL}(1, \mathbb{C})$) which can be used to define a line bundle on M. Now note that $\det(T^*M) = M \times \mathbb{C}$ certainly implies that $c_1(M) = 0$, but not the converse. So this is a stronger condition and we are thus restricting our class a bit. The line bundle $\det(T^*M)$ is called the *canonical line bundle* of M.

A complex manifold M is called *projective* if there is a complex embedding $M \hookrightarrow \mathbb{P}^N$. A projective manifold is always Kähler (by restricting the Fubini-Study Kähler structure from $\mathbb{P}^N$ to M). We have two different definitions of Calabi-Yau manifolds: a Kähler manifold with $c_1(M) = 0$ and $R(\omega) = 0$ can be thought of as the differential geometer's or physicist's definition of Calabi-Yau manifolds, while a projective complex manifold with $\det(T^*M) = M \times \mathbb{C}$ can be thought of as the algebraic geometer's definition.

The Strominger-Yau-Zaslow Conjecture (1996): Every Calabi-Yau manifold (either definition) is a torus fibration (possibly degenerate).

That is, the conjecture is that there exists a manifold $\mathcal{B}$ and a smooth map π such that $\pi : M \to \mathcal{B}$ is surjective and $\pi^{-1}(b)$ is homeomorphic to $S^1 \times \cdots \times S^1$ (except possibly at some $b \in \mathcal{B}$, where the fibre is some degenerate torus). The compactness of M is controlled by $\mathcal{B}$. That is, if M is compact then $\mathcal{B}$ is compact and if M is noncompact then $\mathcal{B}$ is noncompact (but the fibres are always compact). Note that $\mathcal{B}$ need not be a complex manifold.



Example. The elliptic curve E is a torus fibration in two different ways: $\mathcal{B} \times E$ where $\mathcal{B}$ is a single point, as well as $S^1 \times S^1$.

Around 1985, Calabi-Yau manifolds gained attention in physics due to a paper by Candelas, Horowitz, Strominger, and Witten called "Vacuum configurations for superstrings".

Figure 3: https://smbc-comics.com/index.php?id=4130

A natural question arose: are there equations that integrate string theory into gravitational theory? This was important because of the belief that string theory might be the correct forum for unifying gravity with other forces and quantum theory. The following set of equations, which mimic the Einstein equation in some sense, accomplishes this:

$$R(g) + \delta(T) + 2\nabla^g(d\Phi) = 0 \tag{68}$$

$$\delta(T) = \nabla^{\text{grad}}(\Phi) \bullet T$$

$$\nabla^g \psi = 0$$

$$(2d\Phi - T) \bullet \psi = 0$$

These are equations on (M, g, Φ, T, ψ), where Φ is a function (called the *dilaton*), T is a 3-form, ψ is a "spinor field" (which can be thought of as the string), and ∇^g is the Levi-Civita connection of the metric. Under the assumption of "supersymmetry", these equations were found to have a 10-dimensional space of solutions, the idea being that the strings themselves should generate spacetime and its properties, at least locally. So, the freedom of the strings should equal the dimension of spacetime.

This seems to say that our universe M should be (at least locally)

$$\mathbb{R}^{3,1} \times X,$$

where $\mathbb{R}^{3,1}$ is just 4-dimensional Minkowski space and X is 6-dimensional over $\mathbb{R}$. We only experience 4 dimensions, so X should be compact and small enough so that we cannot detect it ($\mathrm{diam}_g(X) \leq 10^{-17}$cm).

If we have a compact X with g and T, then T can be reduced to a 2-form ω_T (by evaluating one of its slots). We can then ask that $g = \omega_T \circ J$ for some complex structure J. This can be thought of as an additional equation, one that makes g and T have a relationship to one another other than equation 68. This is important because, if we take Φ to be constant (a common assumption in physics), then the original relationship between g and T coming from equation 68 is lost. In fact, under the "constant dilaton" assumption, equation 68 reduces to simply

$$R(g) = 0.$$

Since we have a complex structure J, a symplectic form ω_T, and a metric g (all related by $g = \omega_T \circ J$) with $R(g) = 0$, we have that X is a Calabi-Yau manifold, by Yau's result!

That is to say, the compactification of the 6-dimensional non-Minkowski part of superstring theory is a Calabi-Yau 3-fold. From here on, what we know (or *think* we know) about Calabi-Yau manifolds is largely due to the efforts of physicists. String theory has lead to some serious controversy in physics. An experimentalist could ask whether string theory leads to testable predictions, and a theorist would reply that indeed they do, although these are such high energy predictions that we will never see such experiments [3]. The latest developments are

That is not to say that theorists are completely satisfied with the theory; there are some questions. For example, our simplest nontrivial solution to the Einstein equation $R(g) = 0$ was the Schwazchild metric, which assumes $M = \mathbb{R} \times \mathbb{R}_{>0} \times S^2$. We need a topology before we can solve the equation! Another question is that of how many Calabi-Yau 3-folds X the string equations could be solved on. That is, from the point of string theory, which X could we use? Are there many reasonable choices or only a few? This is known as the *problem of moduli* (or *vacua*). Moreover, when are two Calabi-Yau manifolds isomorphic and when are they different? We are going to introduce more invariants which will help us answer this question.

Recall that a complex manifold M comes with operators d and $\overline{d}$, as we have seen in the context of Yau's proof. We will now define a new kind of form, having so-called "mixed type".

Definition. Let M be a complex manifold. A (p,q)*-form* is a $p+q$-form defined by

$$\theta = \sum f_{i_1,\dots,i_p,j_1,\dots,j_q} dz_{i_1} \wedge \cdots \wedge dz_{i_p} \wedge d\bar{z}_{j_1} \wedge \cdots \wedge d\bar{z}_{j_q}. \tag{69}$$

We denote by $\Lambda^{p,q}(T^*M)$ the set (in fact, the vector space) of all (p,q)-forms on M.

Example. We have already seen examples of such forms in our study of Kähler structures: $\omega = d\overline{d}f$ is a $(1,1)$-form.

Definition. We define $\overline{d} : \Lambda^{p,q}(T^*M) \to \Lambda^{p,q+1}(T^*M)$ by

$$\overline{d}\theta = \left(J_{(\bar{z}_1,\dots\bar{z}_n)} f\right) \begin{pmatrix} d\bar{z}_1 \\ \vdots \\ d\bar{z}_n \end{pmatrix} \wedge dz_{i_1} \wedge \cdots \wedge dz_{i_p} \wedge d\bar{z}_{j_1} \wedge \cdots \wedge d\bar{z}_{j_q}. \tag{70}$$

One can calculate that $\overline{d}^2 = 0$.

Definition. The *Dolbeault cohomology groups* $H^{p,q}(M)$ are defined by

$$H^{p,q}(M) = \frac{\ker\left(\overline{d} : \Lambda^{p,q}(T^*M) \to \Lambda^{p,q+1}(T^*M)\right)}{\mathrm{im}\left(\overline{d} : \Lambda^{p,q-1}(T^*M) \to \Lambda^{p,q}(T^*M)\right)}. \tag{71}$$

A form θ is *closed* if $\overline{d}\theta = 0$. A form ψ is *exact* if $\psi = \overline{d}\eta$ for some η.

[3] Do two wrongs make a right? Possibly!

If ψ is exact, then $\overline{d}\psi = \overline{d}^2\eta = 0$, so all exact (p,q)-forms are closed. The Dolbeault cohomology groups are basically closed (p,q)-forms modulo exact (p,q)-forms.

Definition. The *Hodge numbers* of M are $h^{p,q} = \dim_{\mathbb{C}} H^{p,q}(M)$.

When M is smooth, compact and Kähler, the Hodge numbers are topological invariants. Note that $H^{p,q}(M) = 0$ when either p or q is less than zero, and moreover $H^{p,q}(M) = 0$ when either p or q is greater than the complex dimension of M, which we denote by n. Now recall that we had redefined Calabi-Yau manifolds to be projective manifolds with $\det(T^*M) = M \times \mathbb{C}$. Projectivity of the manifold implies $H^{p,q} \cong H^{n-p,n-q}$. Finally, being Calabi-Yau imposes another condition: $H^{p,0} \cong H^{n-p,0}$.

The Hodge numbers can be viewed as forming a diamond:

Example. Consider a Riemann surface, so $\dim_{\mathbb{C}}(M) = n = 1$. The Hodge numbers can be organized as

$$\begin{array}{ccc} & h^{0,0} & \\ h^{1,0} & & h^{0,1} \\ & h^{1,1} & \end{array} \tag{72}$$

where projectivity implies $h^{1,0} = h^{0,1}$ and $h^{1,1} = h^{0,0}$.

Definition. The *k-th Betti number* of M is $b_k = \sum_{p+q=k} h^{p,q}$.

The Betti numbers are sums of the rows of the Hodge diamond. So continuing our example above, we have $b_0 = h^{0,0}$, $b_1 = h^{1,0} + h^{0,1} = 2h^{1,0}$, and $b_2 = h^{1,1}$.

The Betti number b_k has a topological interpretation as the number of deformation classes of closed topological submanifolds a real dimension k.

Example. Let $M = S^1$ (this is not a complex manifold, but Betti numbers are purely topological so we can still look at them). Up to deformation, there is only one point on S^1 (as any point can be rotated to any other) and there is only one circle, S^1 itself. Hence $b_0 = 1$ and $b_1 = 1$.

Example. Consider the 2-sphere S^2. Again $b_0 = 1$, and now $b_2 = 1$ since there is only one sphere in S^2, itself. $b_1 = 0$, since all closed loops on S^2 can be contracted to points.

Example. Now consider the torus, $M = T^2$. Again $b_0 = 1$ and $b_2 = 1$, and here $b_1 = 2$, since there are two distinct incompressible loops on T^2 (these could be thought of as the two copies of S^1 that generate T^2 as $S^1 \times S^1$).

For any Riemann surface M, $b_0 = 1$, $b_1 = 2g$, and $b_2 = 1$. So the Hodge diamond of any Riemann surface is

$$\begin{array}{ccc} & 1 & \\ g & & g \\ & 1 & \end{array} \tag{73}$$

We can write directly

$$h^{0,0} = \dim_{\mathbb{C}} \left(\ker \left(\overline{d} : \Lambda^{0,0} \to \Lambda^{0,1} \right) \right). \tag{74}$$

Note that $\Lambda^{0,0}$ is the space of functions $f : M \to \mathbb{C}$, so we are asking for functions which have $\overline{z}_i$ derivates 0. That is, $H^{0,0}$ is the space of holomorphic functions $f : M \to \mathbb{C}$. Since M is compact, all holomporphic functions are constant, so $H^{0,0} \cong \mathbb{C}$.

In addition,

$$\begin{aligned} h^{1,1} &= \dim_{\mathbb{C}}\left(\frac{\ker\left(\overline{d}:\Lambda^{1,1}\to\Lambda^{1,2}\right)}{\operatorname{im}\left(\overline{d}:\Lambda^{1,0}\to\Lambda^{1,1}\right)}\right) \\ &= T_\omega \Lambda^2_J(T^*M) \end{aligned} \tag{75}$$

where ω is our symplectic form and $\Lambda^2_J(T^*M)$ is the Kähler cone of J. This tells us that $h^{1,1}$ is the dimension of the Kähler cone of J, which is 1 for the elliptic curve.

Recall that the elliptic curve can be realized by $\mathbb{C}/\Gamma$, where Γ is some lattice structure on $\mathbb{C}$:

$$\begin{aligned} \Gamma &= \{mw_1 + nw_2 | m,n \in \mathbb{Z}\} \\ &\cong \{m + n\frac{w_2}{w_1} | m,n \in \mathbb{Z}\} \end{aligned} \tag{76}$$

Thus $\tau := \frac{w_2}{w_1} \in \mathbb{C}^*$ determines Γ and hence the elliptic curve. $H^{0,0} = T_J\mathbb{C}^*$ (the complex cone of the torus) and so $h^{0,0} = 1$, which is consistent with the dimension of $\mathbb{C}^*$. Thus for M the elliptic curve we have

$$\begin{matrix} & 1 & \\ 1 & & 1 \\ & 1 & \end{matrix} \tag{77}$$

where the first row accounts for "complex deformations" of M, the second row comes from $g = 1$, and the bottom row accounts for "symplectic deformations". We must fine-tune two real numbers (the *complex modulus* and symplectic modulus) of the elliptic curve to set the background for the string equations.

Now let us turn our attention to $K3$ surfaces: here, the Hodge diamond is

$$\begin{matrix} & & h^{0,0} & & \\ & h^{1,0} & & h^{0,1} & \\ h^{2,0} & & h^{1,1} & & h^{0,2} \\ & h^{2,1} & & h^{1,2} & \\ & & h^{2,2} & & \end{matrix} \tag{78}$$

with some symmetries induced by the conditions on the Hodge numbers. In fact, for any $K3$ surface, one can calculate that the Hodge diamond is

$$\begin{matrix} & & 1 & & \\ & 0 & & 0 & \\ 1 & & 20 & & 1 \\ & 0 & & 0 & \\ & & 1 & & \end{matrix} \tag{79}$$

The fact that $h^{1,1} = 20$ implies that there are 40 real parameters required to fix a particular $K3$ surface. Equivalently, therea re 40 real parameters required to fix the background fro $K3$ string theory.

Finally, we can consider Calabi-Yau 3-folds, which have Hodge diamonds calculated as

$$
\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & 0 & & 0 & & \\
 & 0 & & h^{1,1} & & 0 & \\
1 & & h^{2,1} & & h^{1,2} & & 1 \\
 & 0 & & h^{2,2} & & 0 & \\
 & & 0 & & 0 & & \\
 & & & 1 & & &
\end{array}
\tag{80}
$$

Now we see that not all of the terms are constant, although we do have $h^{1,1} = h^{2,2}$ (the symplectic dimension) and $h^{1,2} = h^{2,1}$ (the complex dimension).

4.6 Mirror Symmetry and a Look Forward

Let us take a look at a particular CY 3-fold:

Example. The Fermat quintic M is a Calabi-Yau submanifold of $\mathbb{P}^4$ defined by the equation $x_1^5 + x_2^5 + x_3^5 + x_4^5 + x_5^5 = 0$, and has Hodge diamond

$$
\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & 0 & & 0 & & \\
 & 0 & & 1 & & 0 & \\
1 & & 101 & & 101 & & 1 \\
 & 0 & & 1 & & 0 & \\
 & & 0 & & 0 & & \\
 & & & 1 & & &
\end{array}
\tag{81}
$$

This says that there are 202 real numbers required to fix a complex structure on M. This is certainly not ideal for physicists, who wish to do calculations!

In 1990, Greene and Plesser found a different Calabi-Yau 3-fold $\check{M}$ with Hodge diamond

$$
\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & 0 & & 0 & & \\
 & 0 & & 101 & & 0 & \\
1 & & 1 & & 1 & & 1 \\
 & 0 & & 101 & & 0 & \\
 & & 0 & & 0 & & \\
 & & & 1 & & &
\end{array}
\tag{82}
$$

The 3-fold $\check{M}$ can be obtained from M via quotient by a finte group action. The important fact is that physics is the same on M and $\check{M}$, and so problems on one could be solved on the other if they are simpler there (and indeed this turned out to be the case). In this case, the idea would be to work symplectically in M and complexly in $\check{M}$.

The idea of mirror symmetry is that there are pairs of Calabi-Yau 3-folds in which the interior diamond is rotated. If M solves the type II string equations (known as the B-model), then $\check{M}$ solves the type I string equations (known as the A-model). Whether the universe is type A or B is indistinguishable by phsyics, so these are equally valid.

For example, counting degree n curves on the Fermat quintic is a natural problem. Candelas and de la Ossa, who were physicists, published a result on this topic using the idea that counting curves on M is the the same as counting solutions to (II) to on M, which is the same as counting solutions to (I) on $\check{M}$. Around the same time, Ellingsrud and Strømme (who were mathematicians) published another solution using classical methods. But these two results were different! It turned out to be the mathematicians who were mistaken, and this dustup re-interested many mathematicians in Calabi-Yau manifolds and got them working on mirror symmetry.

Computations have found many examples of mirror pairs, but mathematically this seems to be a coincidence: why is this happening? What is the truth?

In 1994, Kontsevich proposed a rigorous interperetation of mirror symmetry, the understanding of which involves the following ideas: Note that complex manifolds form a category, which we call Cpx. The morphisms of Cpx are complex analytic maps between manifolds. Symplectic manifolds also form a category, Symp. The morphisms of Symp are *symplectomorphisms*, which are defined in the following way. If $(M,\omega),(M',\omega') \in \mathrm{Ob}(\mathrm{Symp})$, $f : M \to M'$ is a symplectomorphism if $\omega'(f_*u, f_*v) = \omega(u,v)$ (f_* is the induced map on the tangent space).

The Homological Mirror Symmetry Conjecture: Mirror symmetry is a pair of functors $\mathrm{Cpx} \xrightarrow{\mathrm{MS}} \mathrm{Symp}$ and $\mathrm{Symp} \xrightarrow{\mathrm{SM}} \mathrm{Cpx}$ such that $\mathrm{MS} \circ \mathrm{SM} = \mathrm{Id}$.

Note that this conjecture has no reference to Calabi-Yau geometry, it is more general than that. The idea is that complex structure can be transformed to symlectic structure and vice-versa. If (M,J,ω) is CY, then the mirror of (M,J,ω) should be $(\check{M}, \mathrm{MS}(J), \mathrm{SM}(\omega) = (\check{M}, \check{\omega}, \check{J})$.

The queston of how to define these functors in general is open.

Example. Recalling our discussion of symplectic and complex structures on vector spaces, $\mathbb{R}^2$ has $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $\omega = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. That is, on vector spaces these structures are truly the same (they are equivalent up to a change of basis), and when we move on to the study of such structures on manifolds, we lose sight of this equivalence. Mirror symmetry suggests that we could still cross this gap on manifolds!

For $\mathbb{R}^2$, the mirror symmetry functor is nothing but the transpose:

$$\mathrm{MS}(\mathbb{R}^2, J) = (\mathbb{R}^2, J^T). \tag{83}$$

Example. Considering the elliptic curve E , both the Kähler and complex cones are $\mathbb{C}^*$, which suggests that MS should be a map from $\mathbb{C}^*$ to $\mathbb{C}^*$. Indeed,

$$\mathrm{MS}(E,\tau) = (\check{E}, d\bar{d}e^{2\pi i\tau z\bar{z}}) \tag{84}$$

where τ is the lattice parameter of E. This is a Fourier transform. If $E = S^1 \times S^1$ with radii r_1 and r_2, then $\check{E} = S^1 \times S^1$ with radii $\frac{1}{r_1}$ and $\frac{1}{r_2}$. This is known as *T-duality*.

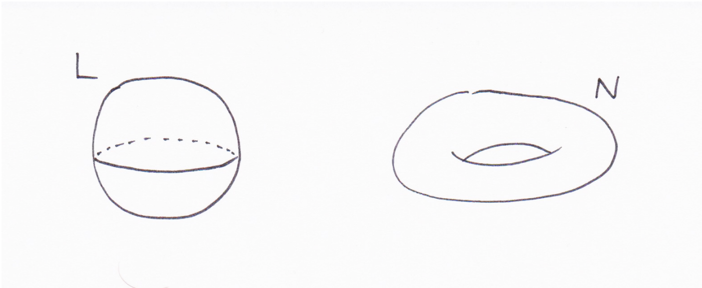
A further part of the conjecture says that the mirror of a torus fibration is realized by dualizing the fibres. In this context, the question is what is is happening on the singular locus (the bad fibres).

Kontsevich actually went even further, and conjectured that mirror symmetry takes each holomorphic bundle V on (M,J) to a submanifold $L \subset \check{M}$ with $\check{\omega}\big|_L = 0$ (this is what is known as a *Lagrangian submanifold*). Such and L is necessarily half the dimension of $\check{M}$, for reasons coming from linear algebra.

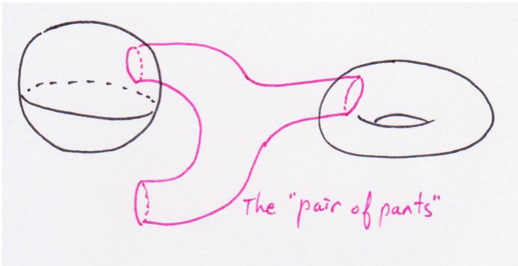
Vector bundles have an operation on them: they can be tensored together. If U has transition functions $E_{\alpha\beta} \in \mathrm{GL}(m,\mathbb{C})$ and V has transition functions $F_{\alpha\beta} \in \mathrm{GL}(n,\mathbb{C})$, then the tensor product $U \otimes V$ is defined as the bundle with transition functions $E_{\alpha\beta}F_{\alpha\beta}$ (the tensor product of matrices):

$$E_{\alpha\beta}F_{\alpha\beta} = \begin{pmatrix} (E_{\alpha\beta})_{11}F_{\alpha\beta} & (E_{\alpha\beta})_{12}F_{\alpha\beta} & \cdots \\ (E_{\alpha\beta})_{21}F_{\alpha\beta} & \ddots & \\ \vdots & & \\ s_{n1} & & \end{pmatrix}. \tag{85}$$

So $\mathrm{rk}(U \otimes V) = mn$. Kontsevich conjectured that if mirror symmetry takes U to L and V to N, then $U \otimes V$ should go to $L \otimes N$. The problem here is, what is the tensor product of two manifolds? Kontsevich defined $L \otimes N$ to be the so-called *"pair of pants" product* (or *quantum product*), inspired by complex dimension 2: For $K3$ surfaces, Lagrangian submanifolds have dimension $\frac{1}{2}2 = 1$, so they are Riemann surfaces. If L and N are Riemann surfaces



then $L \otimes N$ is the pair of pants product



and this is still Lagrangian. The part of the conjecture that is missing is what this product should be in higher dimension.

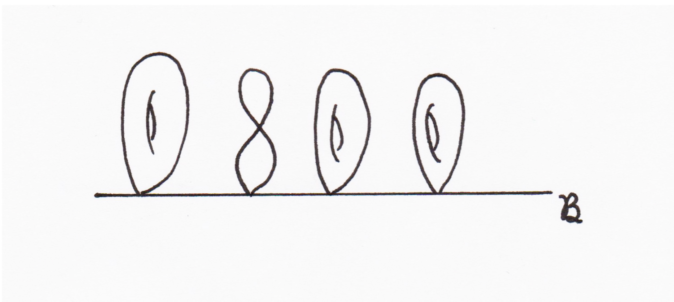
Now let's consider other directions. In 1987, Nigel Hitchin produced and interesting manifold by solving the Yang-Mills equations on a Riemann surface M of genus g:

$$\begin{aligned} R(V) + \phi \wedge \phi &= 0 \\ \overline{d}\phi &= 0 \end{aligned} \tag{86}$$

where $R(V)$ is a skew-symmetric form on V. Solutions to these equations are pairs (V, ϕ) in which V is a holomorhpic vectro bundle on M and ϕ is a holomorphic map $V \to V \times T^*M$. If we label $\mu_1(V,\phi) = R(V) + \phi \wedge \phi$ and $\mu_2(V,\phi) = \overline{d}\phi$, then these are moment maps! The space of solutions is, up to equivalence,

$$\frac{\mu_1^{-1}(0) \cap \mu_2^{-1}(0)}{\mathrm{GL}(r, \mathbb{C})}. \tag{87}$$

This is a hyperkähler quotient , thus the solution space is (noncompact) Calabi-Yau, and when $r = 2$ it has dimension $6g - 6$. So for $g = 2$, this sopace is 6 complex dimensional. This is known as the *moduli space of Higgs bundles*, and it has a very natural torus fibration



where a point in a fibre is (V, ϕ), and there is a map h that takes a point in the fibre to the characteristic polynomial of ϕ, so the base $\mathcal{B}$ is the space of characteristic polynomials. This base is exactly 3 complex dimensional, so the fibres are as well. Here the tori are T^6 (boring Calabi-Yau 3-folds) but the singular fibres are very interesting. Thaddeus and Hausel considered dualizing the fibres: The tori are now Lagrangian submanifolds, but what should the mirror of the bad fibres be? Moreover, is this mirror a moduli space of something?

If we ask for the transition funcitons of V to be $E_{\alpha\beta} \in \mathrm{SL}(2, \mathbb{C})$, then V with transition functions in $\mathrm{PGL}(2, \mathbb{C})$ generate the mirror.

There is a conjecture that the nilpotent cone (one of the bad fibres) is the X that we were looking for before: it is part of our universe.