The Geometry of Twisted Cyclic Quiver Varieties

Evan J. A. Sundbo

Department of Mathematics and Statistics
University of Saskatchewan

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Joint work with Steven Rayan
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Higgs bundles

Fix a Riemann surface X of genus $g \geq 0$ and a line bundle L of degree t.

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An *L-twisted Higgs bundle* on X is a pair (E, Φ) where E is a holomorphic vector bundle on X and $\Phi : E \to E \otimes L$.

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Definition

Two Higgs bundles (E, Φ) and (E', Φ') are *equivalent* if E and E' are isomorphic as vector bundles and $\Phi = \Psi \Phi' \Psi^{-1}$ for some $\Psi \in H^0(X, \mathrm{Aut}(E))$.

The Hitchin system

We denote the moduli space of L-twisted Higgs bundles by $\mathcal{M}_{X,L}(r,d)$.

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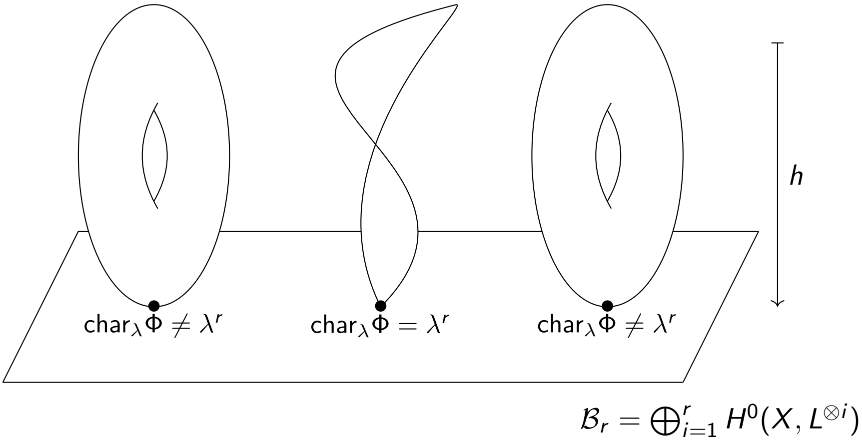

Non-abelian Hodge theory

$$\begin{tikzcd}
\text{Representations of } \pi_1(X) \text{ in } G \arrow[rr, leftrightarrow] \arrow[dr, leftrightarrow] & & \text{Irreducible flat connections on } X \arrow[dl, leftrightarrow] \\
& \text{Stable } \omega_X\text{-twisted } G\text{-Higgs bundles on } X &
\end{tikzcd}$$

The Hitchin section

Let $g \geq 2$, $L = \omega_X$, and consider (E, Φ) with

$$E = \omega_X^{\frac{1}{2}} \oplus \omega_X^{\frac{-1}{2}}, \quad \Phi = \begin{pmatrix} 0 & q \\ 1 & 0 \end{pmatrix}$$

where $q : \omega_X^{\frac{-1}{2}} \to \omega_X^{\frac{1}{2}} \otimes \omega_X$.

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where $q : \omega_X^{\frac{-1}{2}} \to \omega_X^{\frac{1}{2}} \otimes \omega_X$. This is section of $\mathcal{M}_{X,\omega_X}(2,0)$.

Cyclic Higgs bundles

Definition

An *L-twisted cyclic Higgs bundle* on X is a pair (E, Φ) of the form

$$E = U_1 \oplus \cdots \oplus U_n, \qquad \Phi = \begin{pmatrix} 0 & \cdots & & \phi_n \\ \phi_1 & \ddots & & \\ & \ddots & & \\ 0 & & \phi_{n-1} & 0 \end{pmatrix}$$

where U_i are holomorphic line bundles on X and $\phi_i : U_i \to U_{i+1} \otimes L$.

Quivers

Definition

A *quiver* is a directed graph. A *type* $(r_1, \ldots, r_n)$ *cyclic quiver* is one of the form

$$\begin{tikzcd} \bullet_{r_1,d_1} \arrow[r] & \bullet_{r_2,d_2} \arrow[r] & \cdots \arrow[r] & \bullet_{r_n,d_n} \arrow[lll, bend right] \end{tikzcd}$$

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Consider representations of a quiver Q in $\mathrm{Bun}(X, L)$. The moduli space of such representations is denoted $\mathcal{M}_{X,L}(Q)$.

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Consider representations of a quiver Q in $\mathrm{Bun}(X, L)$. The moduli space of such representations is denoted $\mathcal{M}_{X,L}(Q)$.

$$\begin{tikzcd} U_1 \arrow[r, "\phi_1"'] & U_2 \arrow[r, "\phi_2"'] & \cdots \arrow[r, "\phi_{n-1}"'] & U_n \arrow[lll, bend right, "\phi_n"'] \end{tikzcd}$$

Let $g \geq 2$, $L = \omega_X$, and Q be the quiver

$$\begin{tikzcd} \bullet_{1,g-1} \arrow[r, bend right] & \bullet_{1,1-g} \arrow[l, bend right] \end{tikzcd}$$

The Hitchin section revisited

Let $g \geq 2$, $L = \omega_X$, and Q be the quiver

$$\begin{tikzcd} \bullet_{1,g-1} \arrow[r, bend right] & \bullet_{1,1-g} \arrow[l, bend right] \end{tikzcd}$$

The Hitchin section lies inside $\mathcal{M}_{X,L}(Q)$.

The Hitchin section revisited

Let $X = \mathbb{P}^1$, $L = \mathcal{O}(2)$, and Q be the quiver

$$\begin{tikzcd} \bullet_{1,1} \arrow[r, bend right] & \bullet_{1,-1} \arrow[l, bend right] \end{tikzcd}$$

Let $X = \mathbb{P}^1$, $L = \mathcal{O}(2)$, and Q be the quiver

$$\begin{tikzcd} \bullet_{1,1} \arrow[r, bend right] & \bullet_{1,-1} \arrow[l, bend right] \end{tikzcd}$$

Here, $\mathcal{M}_{\mathbb{P}^1,\mathcal{O}(2)}(Q)$ is a section of the moduli space $\mathcal{M}_{\mathbb{P}^1,\mathcal{O}(2)}(2,0)$.

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A-type quiver varieties

Definition

An A-type quiver is a quiver of the form $\bullet \longrightarrow \cdots \longrightarrow \bullet$.

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Lemma

Given a $(1, \ldots, 1)$ cyclic quiver Q with stable representations, there is a unique underlying $(1, \ldots, 1)$ A-type quiver which admits stable representations, which we call Q^A.

A-type quiver varieties

A representation of Q looks like

$$
\begin{tikzcd}
U_1 \arrow[r, "\phi_1"'] & U_2 \arrow[r, "\phi_2"'] & \cdots \arrow[r, "\phi_{n-1}"'] & U_n \arrow[lll, bend right, "\phi_n"']
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Lemma

Let Q be a $(1, \ldots, 1)$ cyclic quiver. Then $\mathcal{M}_{X,L}(Q^A)$ is an r^{2g}-fold covering of $\prod_{i=1}^{n-1} \mathrm{Sym}^{d_{i+1}-d_i+t}(X)$.

The Hitchin map

For a representation (U_i, ϕ_i) of a cyclic quiver Q,

$$h((U_i, \phi_i)) = \phi_1 \dots \phi_n \in H^0(X, L^{\otimes n})$$

The Hitchin map

For a representation (U_i, ϕ_i) of a cyclic quiver Q,

$$h((U_i, \phi_i)) = \phi_1 \dots \phi_n \in H^0(X, L^{\otimes n})$$

So

$$\mathcal{M}_{X,L}(Q)\big|_{h^{-1}(0)} = \mathcal{M}_{X,L}(Q^A)$$

Cyclic quiver varieties

Theorem (Rayan, S.)

$$\mathcal{M}_{X,L}(Q)\Big|_{h^{-1}(\gamma)} \cong \Big\{(U_i;\phi_i) \in \mathcal{M}_{X,L}(Q^A) : (\phi_1 \dots \phi_{n-1}) \subseteq (\gamma)\Big\}$$

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Cyclic quiver varieties on $\mathbb{P}^1$

On $\mathbb{P}^1$, a representation of Q looks like

$$\begin{tikzcd}
\mathcal{O}(d_1) \arrow[r, "\phi_1"'] & \mathcal{O}(d_2) \arrow[r, "\phi_2"'] & \cdots \arrow[r, "\phi_{n-1}"'] & \mathcal{O}(d_n) \arrow[lll, bend right, "\phi_n"']
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and

$$\mathrm{Rep}(Q) \cong \prod_{i=1}^{n-1} \left(H^0(\mathbb{P}^1, \mathcal{O}(d_{i+1} - d_i + t)) \setminus \{0\} \right) \times H^0(\mathbb{P}^1, \mathcal{O}(d_1 - d_n + t))$$

Cyclic quiver varieties on $\mathbb{P}^1$

The action of the automorphisms of $\mathcal{O}(d_i)$ on Φ is equivalent to the action of $(\mathbb{C}^*)^{n-1}$

$$(\lambda_1, \ldots, \lambda_{n-1}) \cdot \Phi = \begin{pmatrix} 0 & \cdots & & (\lambda_1^{-1} \ldots \lambda_{n-1}^{-1})\phi_n \\ \lambda_1\phi_1 & \ddots & & \\ & \ddots & & \\ 0 & & \lambda_{n-1}\phi_{n-1} & 0 \end{pmatrix}$$

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Now

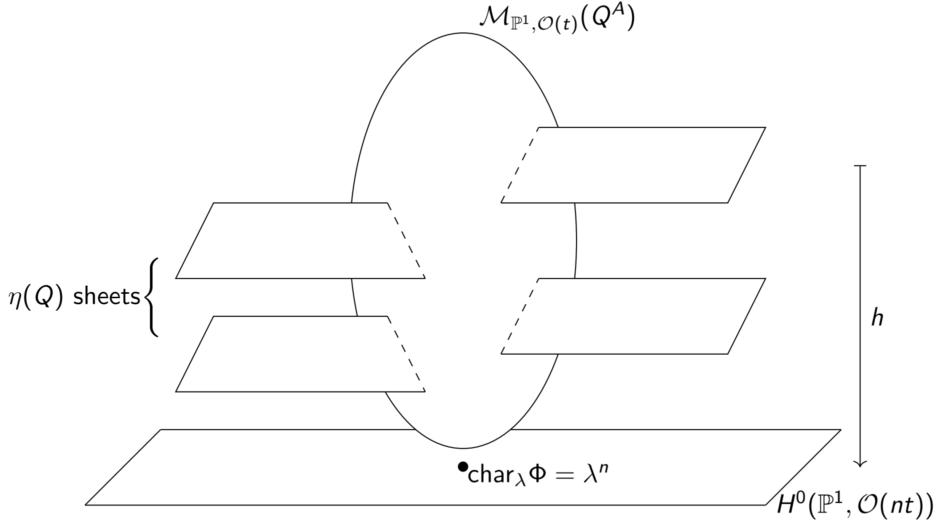
$$\mathcal{M}_{\mathbb{P}^1, \mathcal{O}(t)}(Q) \cong \frac{\prod_{i=1}^{n-1} \left(\mathbb{C}^{d_{i+1} - d_i + t + 1} \setminus \{0\} \right) \times \mathbb{C}^{d_1 - d_n + t + 1}}{(\mathbb{C}^*)^{n-1}}.$$

Cyclic quiver varieties on $\mathbb{P}^1$

Theorem part A (Rayan, S.)

The moduli space of representations of a $(1, \ldots, 1)$ cyclic quiver Q in the category $\mathrm{Bun}(\mathbb{P}^1, \mathcal{O}(t))$ is a finite covering of $H^0(\mathbb{P}^1, \mathcal{O}(nt)) \setminus \{0\}$ which branches over points with roots of multiplicity greater than one, and whose sheets intersect over $0 \in H^0(\mathbb{P}^1, \mathcal{O}(nt))$ as $\prod_{i=1}^{n-1} \mathbb{P}^{d_{i+1}-d_i+t}$.

$\mathcal{M}_{\mathbb{P}^1,\mathcal{O}(t)}(Q)$ as a cover



$\mathcal{M}_{\mathbb{P}^1,\mathcal{O}(t)}(Q)$ as a cover

Recall

$$(\lambda_1, \ldots, \lambda_{n-1}) \cdot \Phi = \begin{pmatrix} 0 & \cdots & & (\lambda_1^{-1} \ldots \lambda_{n-1}^{-1})\phi_n \\ \lambda_1\phi_1 & \ddots & & \\ & \ddots & & \\ 0 & & \lambda_{n-1}\phi_{n-1} & 0 \end{pmatrix}$$

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Fix $\gamma = \phi_1 \ldots \phi_n \neq 0$, then there are $\eta(Q) = \binom{nt}{d_2 - d_1 + t, \ldots, d_n - d_{n-1} + t, d_1 - d_n + t}$ ways to distribute the zeroes.

$\mathcal{M}_{\mathbb{P}^1,\mathcal{O}(t)}(Q)$ as a cover

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Fix $\gamma = \phi_1 \ldots \phi_n \neq 0$, then there are $\eta(Q) = \binom{nt}{d_2-d_1+t, \ldots, d_n-d_{n-1}+t, d_1-d_n+t}$ ways to distribute the zeroes.

Also, $\mathcal{M}_{\mathbb{P}^1,\mathcal{O}(t)}(Q)\big|_{h^{-1}(0)} = \mathcal{M}_{\mathbb{P}^1,\mathcal{O}(t)}(Q^A)$.

Cyclic quiver varieties on $\mathbb{P}^1$

Theorem part B (Rayan, S.)

The moduli space of representations of a $(1, \ldots, 1)$ cyclic quiver Q in the category $\mathrm{Bun}(\mathbb{P}^1, \mathcal{O}(t))$ is the total space of

$$\mathcal{O}_{\prod_{i=1}^{n-1} \mathbb{P}^{d_{i+1}-d_i+t}}(-1, \ldots, -1)^{\oplus d_1 - d_n + t + 1},$$

where

$$\mathcal{O}_{\prod_{i=1}^{n-1} \mathbb{P}^{d_{i+1}-d_i+t}}(-1, \ldots, -1) = \bigotimes_{i=1}^{n-1} p_i^* \mathcal{O}_{\mathbb{P}^{d_{i+1}-d_i+t}}(-1)$$

and p_i is the natural projection onto the i-th factor.

$\mathcal{M}_{\mathbb{P}^1,\mathcal{O}(t)}(Q)$ as a vector bundle

Note that

$$\begin{pmatrix} 0 & \cdots & & c\phi_n \\ \phi_1 & \ddots & & \\ & \ddots & & \\ 0 & & \phi_{n-1} & 0 \end{pmatrix}$$

goes to the same point p in $\mathcal{M}_{\mathbb{P}^1,\mathcal{O}(t)}(Q^A)$ as $c \to 0$, regardless of ϕ_n.

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That is, we have a fibration by $\mathbb{C}^{d_1-d_n+t+1}$.

$\mathcal{M}_{\mathbb{P}^1,\mathcal{O}(t)}(Q)$ as a vector bundle

Using the fact that

$$\frac{\mathbb{C}^{a+1} \setminus \{0\} \times \mathbb{C}}{\mathbb{C}^*} \cong \mathcal{O}_{\mathbb{P}^a}(-1)$$

$\mathcal{M}_{\mathbb{P}^1,\mathcal{O}(t)}(Q)$ as a vector bundle

Using the fact that

$$\frac{\mathbb{C}^{a+1} \setminus \{0\} \times \mathbb{C}}{\mathbb{C}^*} \cong \mathcal{O}_{\mathbb{P}^a}(-1)$$

and the description

$$\mathcal{M}_{\mathbb{P}^1,\mathcal{O}(t)}(Q) \cong \frac{\prod_{i=1}^{n-1} \left(\mathbb{C}^{d_{i+1}-d_i+t+1} \setminus \{0\}\right) \times \mathbb{C}^{d_1-d_n+t+1}}{(\mathbb{C}^*)^{n-1}}$$

we can realize this fibration as a holomorphic vector bundle embedded in $\mathcal{M}_{\mathbb{P}^1,\mathcal{O}(t)}(r,d)$.

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$(k, 1)$ Cyclic quiver varieties on $\mathbb{P}^1$

Let Q be the quiver

$$Q = \begin{tikzcd} \bullet_{k,d_1} \arrow[r, bend right] & \bullet_{1,d_2} \arrow[l, bend right] \end{tikzcd}$$

$(k, 1)$ Cyclic quiver varieties on $\mathbb{P}^1$

Let Q be the quiver

$$Q = \begin{tikzcd} \bullet_{k,d_1} \arrow[r, bend right] & \bullet_{1,d_2} \arrow[l, bend right] \end{tikzcd}$$

Assume the first bundle splits into line bundles of mutually distinct degrees.

$$\begin{tikzcd}
\mathcal{O}(a_1) \arrow[dr, bend left, "\phi_1"] & \\
\vdots & \mathcal{O}(d_2) \arrow[ul, bend left, "\phi_2"] \arrow[dl, bend left, "\phi_{2k}"] \\
\mathcal{O}(a_k) \arrow[ur, bend left, "\phi_{2k-1}"] &
\end{tikzcd}$$

$(k,1)$ Cyclic quiver varieties on $\mathbb{P}^1$

The moduli space decomposes as

$$\mathcal{M}_{\mathbb{P}^1,\mathcal{O}(t)}(Q;\mathbf{a}) \cong \mathcal{M}_{\mathbb{P}^1,\mathcal{O}(t)}(Q_1) \times \prod_{i=2}^{k} \mathcal{M}_{\mathbb{P}^1,\mathcal{O}(t)}\left(Q_i^{-\sum_{j=1}^{i-1}(a_j - a_i + 1)}\right).$$

The End

Thank you!