

Instructions. This packet is due on Gradescope no later than **5pm on Monday, October 5**. Please complete your work directly on this packet. We will spend time together during lecture working on most of this packet together. You are responsible for completing all portions of this packet, including lecture activities not discussed in class, and completing the definitions, result statements, and result proofs included in the packet. Solutions will be posted to the course website after the assignment due date.

Pre-class Reading and Suggested Resources

1. Read the Chapter 4 preamble (one paragraph on p. 61) of the [course lecture notes](#).
2. Watch and/or read the 3Blue1Brown resource: [Linear transformations and matrices](#).

Chapter 4 Lecture Activities: Linear Transformations

Lecture Activity 4.1. Consider the matrices

$$A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \\ -1 & 1 \end{pmatrix} \text{ and } B = \begin{pmatrix} 3 & -1 & 1 \\ 2 & 0 & 1 \end{pmatrix}.$$

P1. Calculate the matrix-vector product $A\vec{x}$ where $\vec{x} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$.

P2. Calculate the matrix-vector product $B\vec{y}$ where $\vec{y} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$.

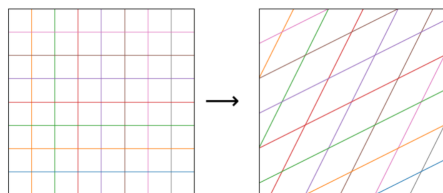
P3. Let $\vec{x}$ and $\vec{y}$ be as in the previous problems. Explain why the matrix-vector products $A\vec{y}$ and $B\vec{x}$ are not defined.

P4. Let $\vec{z}$ be a vector in $\mathbb{R}^2$. How many components does the vector $A\vec{z}$ have?

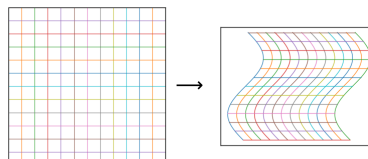
P5. Let $\vec{w}$ be a vector in $\mathbb{R}^3$. How many components does the vector $B\vec{w}$ have?

Lecture Activity 4.2. In the images below, we've plotted where the indicated function sends the standard coordinate grid for $\mathbb{R}^2$. What do you notice?

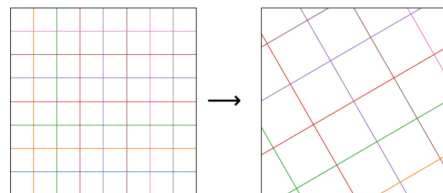
$$T_A \text{ where } A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$



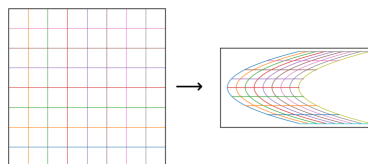
$$F\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} x + \sin(y) \\ y \end{pmatrix}$$



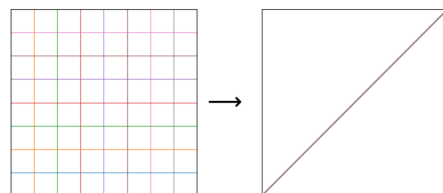
$$T_B \text{ where } B = \begin{pmatrix} \sqrt{2} & -\sqrt{2} \\ \sqrt{2} & \sqrt{2} \end{pmatrix}$$



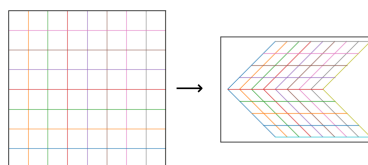
$$G\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} x + y^2 \\ y \end{pmatrix}$$



$$T_C \text{ where } C = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$



$$H\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} x + |y| \\ y \end{pmatrix}$$



Lecture Activity 4.3. Determine which of the following are linear transformations. Give a formal justification for your answer by showing that the function does or does not satisfy the conditions of Definition 4.4.

P1. $F : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ defined by

$$F \left(\begin{pmatrix} x \\ y \end{pmatrix} \right) = \begin{pmatrix} x^2 \\ y^2 \end{pmatrix}$$

P2. $G : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ defined by

$$G \left(\begin{pmatrix} x \\ y \end{pmatrix} \right) = \begin{pmatrix} x + y \\ x \end{pmatrix}.$$

Lecture Activity 4.4. Suppose that $F : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is a linear transformation satisfying

$$F(\vec{e}_1) = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \text{ and } F(\vec{e}_2) = \begin{pmatrix} 1 \\ 2 \end{pmatrix},$$

where we recall that $\vec{e}_1, \vec{e}_2$ are the standard basis vectors defined in Definition ?? . That is,

$$\vec{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \vec{e}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

P1. Write the vectors

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} \text{ and } \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

as a linear combination of the standard basis vectors $\vec{e}_1$ and $\vec{e}_2$.

P2. Using your work in P1 and the fact that F is linear, find $F\left(\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right)$ and $F\left(\begin{pmatrix} 2 \\ 3 \end{pmatrix}\right)$.

P3. Using a strategy similar to P2, find a formula for $F\left(\begin{pmatrix} x \\ y \end{pmatrix}\right)$.

P4. Find a 2×2 matrix M so that $F(\vec{x}) = M\vec{x}$ for all vectors $\vec{x} \in \mathbb{R}^2$.

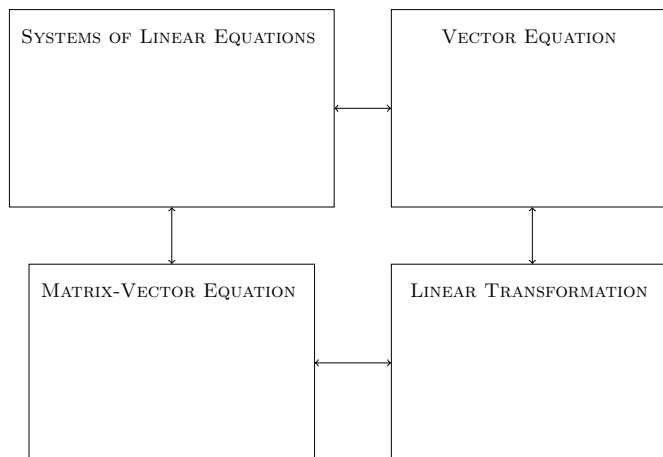
Lecture Activity 4.5. Let $F : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be the transformation which rotates every vector θ° counterclockwise about the origin.

P1. Use geometric reasoning to argue that F is a linear transformation.

P2. Find the defining matrix M_F when $\theta = 90^\circ$.

P3. Find the defining matrix M_F for any value of θ . Note that your matrix will depend on the unknown angle θ .

Lecture Activity 4.6. Let's call the diagram below a “translation square”. For each of the equations given below, fill in all components of a translation square.



P1.
$$\begin{cases} 2x + 3y = 4 \\ 5x + \frac{1}{2}y = 4 \end{cases}$$

P2.
$$\begin{pmatrix} 1 & 2 & 1/2 \\ 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$$

P3. $T_A \left(\begin{pmatrix} x \\ y \end{pmatrix} \right) = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$, where $A = \begin{pmatrix} 1 & -1 \\ 2 & 2 \\ 3 & 2 \end{pmatrix}$ and $T_A : \mathbb{R}^2 \rightarrow \mathbb{R}^3$.

P4. $x \begin{pmatrix} 1 \\ 1 \end{pmatrix} + y \begin{pmatrix} 2 \\ 3 \end{pmatrix} + z \begin{pmatrix} 0 \\ 1 \end{pmatrix} + w \begin{pmatrix} 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

Lecture Activity 4.7. Consider the matrices

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}, B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}, \text{ and } C = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

and their associated linear transformations $T_A : \mathbb{R}^2 \rightarrow \mathbb{R}^3$, $T_B : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ and $T_C : \mathbb{R}^2 \rightarrow \mathbb{R}^2$.

P1. Show that T_A is injective but not surjective.

P2. Show that T_B is surjective but not injective.

P3. Show that T_C is bijective.

Lecture Activity 4.8. Use the Pivot Characterization Theorem for Linear Transformations (Theorem 4.13) to determine which of the following functions are injective, surjective, bijective, or neither.

P1. $F : \mathbb{R}^2 \rightarrow \mathbb{R}^3, \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x \\ y \\ 0 \end{pmatrix}$

P2. $G : \mathbb{R}^3 \rightarrow \mathbb{R}^3, \begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} x - y \\ y + z \\ x + z \end{pmatrix}$

P3. $H : \mathbb{R}^3 \rightarrow \mathbb{R}^3, \begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} x - y \\ y + z \\ z \end{pmatrix}$

Chapter 4 Key Definitions.

The following list contains the key vocabulary from this chapter and will be used throughout the course. For each term listed below, a successful student should be able to (i) state its definition formally as it appears in the lecture notes from memory, (ii) produce at least one example and one nonexample, and (iii) demonstrate a firm conceptual understanding of its meaning.

On course assessments, definitions will scaffolded precisely as they are below. Students will be expected to (1) fill in the missing details of the definitions listed below and (2) provide one example and one nonexample of each definition.

Definition 4.1. Let A be an $m \times n$ matrix with column vectors $A = (\vec{v}_1 \ \vec{v}_2 \ \cdots \ \vec{v}_n)$. Then, for a vector $\vec{x}$ in $\mathbb{R}^n$ the MATRIX-VECTOR PRODUCT of A and $\vec{x}$ is the vector in $\mathbb{R}^m$ defined by ...

where the $c_1, c_2, \dots, c_n$ are scalars called the COEFFICIENTS of the linear combination.

Example:

Definition 4.2. Let A be an $m \times n$ matrix. Then, the MATRIX TRANSFORMATION associated to A is the function ...

Example:

Nonexample:

Definition 4.9. Let $F : \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a linear transformation. Then, the DEFINING MATRIX of F is the $m \times n$ matrix M satisfying

for all vectors $\vec{x}$ in $\mathbb{R}^n$, and is denoted by $M = M_F$.

Example:

Definition 4.11. Let $f : X \rightarrow Y$ be a function between sets X and Y .

1. f is called INJECTIVE (or ONE-TO-ONE) if the following property holds:

2. f is called SURJECTIVE (or ONTO) if the following property holds:

3. f is called BIJECTIVE if the following property holds:



Example of a function that's injective but not surjective:

Example of a function that's surjective but not injective:

Example of a bijective function:

Chapter 4 Key Results

The following list contains the key results from this chapter. A successful student should be able to (i) state each result precisely as it appears in the lecture notes from memory, and (ii) recognize when and how to apply each result to solve different types of problems. Additionally, for all results listed in **bold-faced font**, a successful student should be able to reproduce a formal, logically correct proof of the result from memory.

On course assessments, students will be expected to fill in the missing details of the required proofs below, which will be scaffolded precisely as they appear in this packet. Additionally, students may use any of the theorems included below to solve problems.

Proposition 4.6 (Linearity of Matrix Transformations).

Proof (not testable).

□

Theorem 4.8 (Defining Matrix Theorem).

Proof (testable). Suppose that $F : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is a linear transformation. Since $\{\vec{e}_1, \vec{e}_2, \dots, \vec{e}_n\}$ forms a basis for $\mathbb{R}^n$ we can write $\vec{x} = x_1\vec{e}_1 + x_2\vec{e}_2 + \dots + x_n\vec{e}_n$.

Use the fact that F is linear to complete the proof.

□

Theorem 4.13 (Pivot Characterization Theorem).