

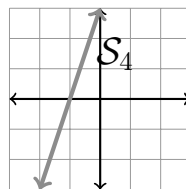
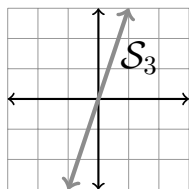
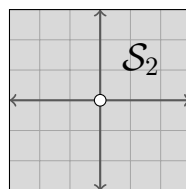
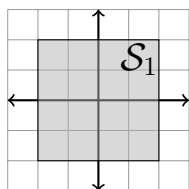
Instructions. This packet is due on Gradescope no later than **5pm on Monday, September 28**. Please complete your work directly on this packet. We will spend time together during lecture working on most of this packet together. You are responsible for completing all portions of this packet, including lecture activities not discussed in class, and completing the definitions, result statements, and result proofs included in the packet. Solutions will be posted to the course website after the assignment due date.

Pre-class Reading and Suggested Resources

1. Read the Chapter 3 preamble (one paragraph on p. 55) of the [course lecture notes](#).
2. Watch and/or read the 3Blue1Brown resource: [Linear combinations, span, and basis vectors](#). Note that we'll use the notation $\vec{e}_1$ for $\hat{i}$ and the notation $\vec{e}_2$ for $\hat{j}$.
3. Read Appendix A.3 of the [course lecture notes](#).

Chapter 3 Lecture Activities: Vectors Subspace of $\mathbb{R}^n$

Lecture Activity 3.1. Determine which of the following subsets of $\mathbb{R}^2$ drawn below are spans of a set of vectors in $\mathbb{R}^2$. Note that the set S_2 in the second image is meant to extend infinitely in all directions, and we describing these sets with the geometric shorthand introduced in Example ??.



Lecture Activity 3.2. Find a generating set for the following vector spaces.

P1. $V = \left\{ \begin{pmatrix} 2x \\ 0 \end{pmatrix} : x \in \mathbb{R} \right\}$

P2. $W = \left\{ \begin{pmatrix} x - y \\ x + y + 2z \\ y + z \end{pmatrix} : x, y, z \in \mathbb{R} \right\}$

Lecture Activity 3.3. Determine which of the following sets are bases for the vector space $V = \mathbb{R}^3$.

P1. $\mathcal{B}_1 = \left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$

P2. $\mathcal{B}_2 = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} \right\}$

$$\text{P3. } \mathcal{B}_3 = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}$$

$$\text{P4. } \mathcal{B}_4 = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

Lecture Activity 3.4. Show that the standard basis $\mathcal{E}$ is a basis for $\mathbb{R}^n$. Conclude that the vector space $\mathbb{R}^n$ has dimension n .

Lecture Activity 3.5. In this problem we'll find a basis for the vector space W from Lecture Activity 3.2, defined by

$$W = \left\{ \begin{pmatrix} x - y \\ x + y + 2z \\ y + z \end{pmatrix} : x, y, z \in \mathbb{R} \right\}.$$

Recall that $W = \text{Span}(\vec{u}, \vec{v}, \vec{w})$ where

$$\vec{u} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \vec{v} = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}, \vec{w} = \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}.$$

Letting $A = (\vec{u} \ \vec{v} \ \vec{w})$ we can calculate

$$\text{rref}(A) = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Using **only** the reduced row echelon form of A , show the following.

P1. Show that the set $\{\vec{u}, \vec{v}, \vec{w}\}$ is linearly dependent, and hence is not a basis for W .

P2. Show that $\vec{w} \in \text{Span}(\vec{u}, \vec{v})$.

P3. Use P2 to show that $W = \text{Span}(\vec{u}, \vec{v})$.

P4. Use $\text{rref}(A)$ to show that $\{\vec{u}, \vec{v}\}$ is linearly independent. Conclude that this set forms a basis for W and state $\dim(W)$.

Lecture Activity 3.6. Find a basis for the following vector spaces, and state their dimension.

P1. $V = \text{Span} \left(\begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right)$

P2. $W = \text{Span} \left(\begin{pmatrix} 1 \\ 0 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 0 \\ 6 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 7 \\ 2 \\ 12 \\ 7 \end{pmatrix} \right)$

Chapter 3 Key Definitions.

The following list contains the key vocabulary from this chapter and will be used throughout the course. For each term listed below, a successful student should be able to (i) state its definition formally as it appears in the lecture notes from memory, (ii) produce at least one example and one nonexample, and (iii) demonstrate a firm conceptual understanding of its meaning.

On course assessments, definitions will scaffolded precisely as they are below. Students will be expected to (1) fill in the missing details of the definitions listed below and (2) provide one example and one nonexample of each definition.

Definition 3.3. A VECTOR SUBSPACE of $\mathbb{R}^n$ is any set of n -dimensional vectors that satisfies all of the following properties:

In

this case, we refer to $\mathbb{R}^n$ as the AMBIENT SPACE.

Example:

Nonexample:

Definition 3.5. Let V be a vector subspace of $\mathbb{R}^n$. A GENERATING SET for V is ...

Example:

Nonexample:

Definition 3.6. A BASIS for a vector space V is ...

Example:

Nonexample:

Definition 3.10. Let V be a nonzero vector subspace of $\mathbb{R}^n$. Then, the DIMENSION of V , denoted $\dim V$, is ...

Example:

Nonexample:

Definition 3.11. The STANDARD BASIS for $\mathbb{R}^n$ is the set $\mathcal{E} := \{\vec{e}_1, \vec{e}_2, \dots, \vec{e}_n\}$ where ...

Example:

Nonexample:

Definition 3.16. The TRIVIAL VECTOR SUBSPACE of $\mathbb{R}^n$ is ...

When the ambient space $\mathbb{R}^n$ is understood to be fixed, we just call $\{\vec{0}\}$ the TRIVIAL SPACE.

Example:

Nonexample:

Chapter 3 Key Results

The following list contains the key results from this chapter. A successful student should be able to (i) state each result precisely as it appears in the lecture notes from memory, and (ii) recognize when and how to apply each result to solve different types of problems. Additionally, for all results listed in **bold-faced font**, a successful student should be able to reproduce a formal, logically correct proof of the result from memory.

On course assessments, students will be expected to fill in the missing details of the required proofs below, which will be scaffolded precisely as they appear in this packet. Additionally, students may use any of the theorems included below to solve problems.

Theorem 3.1 (Span-Vector Space Equivalence Theorem)

Proof (testable). Suppose that V is a vector subspace of $\mathbb{R}^n$. Let $B = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_m\}$ be a linearly independent subset of V . By Chapter Exercise P3.1, we must have $m \leq n$. So, we may choose such a set B to be *maximal* with respect to linear independence.

Complete the proof: explain what we meant by B being “maximal”.

We will show that $V = \text{Span}(B)$. First, observe that $\text{Span}(\vec{v}_1, \vec{v}_2, \dots, \vec{v}_m) \subset V$, since V is a vector space. For the opposite set inclusion, take any $\vec{v} \in V$.

Complete the proof: show that if $\vec{v} \in B$ then $\vec{v} \in \text{Span}(\vec{v}_1, \dots, \vec{v}_m)$.

So, suppose that $\vec{v} \notin B$.

Complete the proof: explain why the set $\{\vec{v}, \vec{v}_1, \dots, \vec{v}_m\}$ is linearly dependent in this case.

So, by the Linear Dependence Theorem (Theorem 2.27) there exists a nontrivial solution $(c_1, \dots, c_m, c_{m+1})$ to the vector equation

$$x_1\vec{v}_1 + \dots + x_m\vec{v}_m + x_{m+1}\vec{v} = \vec{0}. \quad (3.1)$$

Complete the proof: show that $\vec{v} \in \text{Span}(\vec{v}_1, \dots, \vec{v}_m)$.

Hence, $V = \text{Span}(\vec{v}_1, \dots, \vec{v}_m)$, as needed. Observe that the converse follows by Lemmas 3.1-3.2. $\square$

Theorem 3.9 (Dimension Theorem).

Proof (not testable).

□

Lemma 3.12 (Pivot Column Lemma).

Proof (testable).

Prove Lemma 3.12.

□

Theorem 3.13 (Finding Bases).

Theorem 3.14 (Basis Characterization).

Lemma 3.15 (Subspace Dimension Lemma).