

Instructions. This packet is due on Gradescope no later than **5pm on Monday, September 21**. Please complete your work directly on this packet. We will spend time together during lecture working on most of this packet together. You are responsible for completing all portions of this packet, including lecture activities not discussed in class, and completing the definitions, result statements, and result proofs included in the packet. Solutions will be posted to the course website after the assignment due date.

Pre-class Reading and Suggested Resources

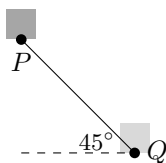
1. Read the Chapter 2 preamble (p. 33) of the [course lecture notes](#).
2. Watch and/or read the 3Blue1Brown resource: [Vectors, what even are they?](#).
3. Read Appendix A.1 and A.2 of the [course lecture notes](#).

Chapter 2 Lecture Activities: Vectors in Euclidean Space

Lecture Activity 2.1. Find the standard coordinates of the following displacement vectors.

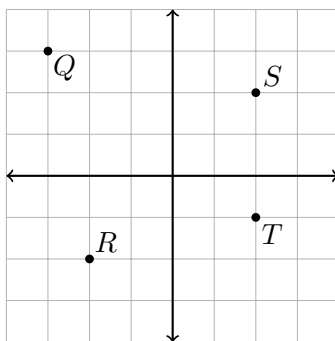
- P1. To walk from point A to point B you travel 3 kilometers west and 1 kilometer north. Sketch a picture of the displacement vector $\overrightarrow{AB}$ and find its standard coordinates $[\overrightarrow{AB}]$.

- P2. A box slides down a ramp from point P to point Q as drawn below. Sketch a picture of the displacement vector $\overrightarrow{PQ}$ and find its standard coordinates $[\overrightarrow{PQ}]$.



If the box slides down a ramp of length 2 meters, describe the displacement vector $\overrightarrow{PQ}$.

- P3. For the points Q , R , S , and T drawn below, draw the displacement vectors $\overrightarrow{QR}$, $\overrightarrow{RS}$, and $\overrightarrow{ST}$ and find their standard coordinates.



Lecture Activity 2.2. You are piloting an airplane equipped with two fixed-direction thrusters that assist in maneuvering. Each thruster provides thrust in a specific, constant direction and can be fired forward or in reverse for any number of seconds. In this scenario, we assume the airplane is already at cruising altitude, and the two thrusters only affect horizontal position.

- Firing **Thruster A** for one second in the forward direction moves the airplane 17 meters East and 7 meters North; firing Thrusters A for one second in the backward direction moves the airplane 17 meters West and 7 meters South.
- Firing **Thruster B** for one second in the forward direction moves the airplane 5 meters East and 18 meters North; firing Thruster B for one second in the backward direction moves the airplane 5 meters West and 18 meters South.

P1. Find a vector $\vec{a}$ which describes the displacement of the airplane after firing Thruster A in the forward direction for one second.

P2. Find a vector $\vec{b}$ which describes the displacement of the airplane after firing Thruster B in the forward direction for one second.

P3. Find a vector $\vec{x}$ which describes the displacement of the airplane after firing Thruster A in the forward direction by two seconds.

P4. Find a vector $\vec{y}$ which describes the displacement of the airplane after firing Thruster B in the backward direction by one second.

P5. Find a vector $\vec{z}$ which describes the displacement of the airplane after firing Thruster A in the forward direction by two seconds and then firing Thruster B in the backward direction by one second.

Lecture Activity 2.3. You are piloting an airplane equipped with two fixed-direction thrusters that assist in maneuvering. Each thruster provides thrust in a specific, constant direction and can be fired forward or in reverse for any number of seconds. In this scenario, we assume the airplane is already at cruising altitude, and the two thrusters only affect horizontal position.

- Firing **Thruster A** for one second in the forward direction moves the airplane 17 meters East and 7 meters North; firing Thrusters A for one second in the backward direction moves the airplane 17 meters West and 7 meters South.
- Firing **Thruster B** for one second in the forward direction moves the airplane 5 meters East and 18 meters North; firing Thruster B for one second in the backward direction moves the airplane 5 meters West and 18 meters South.

In this problem, we explore what these thruster directions imply about the airplane's maneuverability at its current fixed altitude.

- P1. Suppose you're instructed to reach a waypoint located 235 meters East and 33 meters North of your current position. Can you reach the waypoint using only Thruster A *or* only Thruster B? If yes, determine how many seconds you need to fire the thruster to reach the waypoint. If not, explain why not.

- P2. Can you reach the waypoint from P1 using a combination of both thrusters? If yes, determine how many seconds you need to fire each thruster to reach the waypoint. If not, explain why not.

- P3. Can you reach every possible waypoint at your current elevation (i.e., in the same horizontal plane) using a combination of Thrusters A and B? Explain your answer.

Lecture Activity 2.4. Find the augmented matrix for the system of linear equations that has the same solution set as the following vector equations. Then, find all solutions to the vector equation.

P1. $x \begin{pmatrix} 1 \\ 2 \end{pmatrix} + y \begin{pmatrix} -1/2 \\ 5 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$

$$\text{P2. } x \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + y \begin{pmatrix} 3 \\ 4 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 5 \\ 1 \end{pmatrix}$$

P3. $x \begin{pmatrix} 2 \\ 1 \end{pmatrix} + y \begin{pmatrix} 5 \\ 3 \end{pmatrix} + z \begin{pmatrix} -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \end{pmatrix}$

Lecture Activity 2.5. Give a geometric description of the following spans.

P1. $\text{Span} \left(\begin{pmatrix} 1 \\ 1 \end{pmatrix} \right)$

P2. $\text{Span} \left(\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right)$

P3. $\text{Span} \left(\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \end{pmatrix} \right)$

P4. $\text{Span} \left(\begin{pmatrix} 0 \\ 0 \end{pmatrix} \right)$

P5. If S is any set of vectors in $\mathbb{R}^2$, what are the possible geometric objects that $\text{Span}(S)$ could be?

Lecture Activity 2.6 (Flight Navigation II). You are piloting an airplane equipped with four fixed-direction thrusters that assist in maneuvering. Each thruster provides thrust in a specific, constant direction and can be fired forward or in reverse for any number of seconds.

- Firing **Thruster A** for one second in the forward direction moves the airplane 10 meters East, 9 meters North, and 2 meters Up. Firing Thruster A for one second in the backward direction has the opposite effect; that is, moves the airplane 10 meters West, 9 meters South, and 2 meters Down.
- Firing **Thruster B** for one second in the forward direction moves the airplane 4 meters East, 7 meters North, and 1 meters Up. Firing B in reverse has the opposite effect.
- Firing **Thruster C** for one second in the forward direction moves the airplane 0 meters East, 2 meter North, and 3 meter Up. Firing C in reverse has the opposite effect.
- Firing **Thruster D** for one second in the forward direction moves the airplane 2 meter East, 5 meters South, and 0 meters Up. Firing D in reverse has the opposite effect.

In this problem, we use geometric reasoning to explore what these thruster directions imply about the airplane's maneuverability in 3 dimensional space. To complete this activity, we suggest following [this Desmos link](#). Note that you do not need to provide formal justification for your answers. The purpose of this problem is to build geometric intuition.

P1. Use geometric reasoning to argue that you can reach any waypoint using all four thrusters.

P2. Is it possible to reach any waypoint using only Thrusters A, B, and C? Use geometric reasoning to justify your answer.

P3. Is it possible to reach any way point using only Thrusters B, C, and D? Use geometric reasoning to justify your answer.

P4. Do you think it's possible to reach any waypoint using only two thrusters? What about only one thruster?

Lecture Activity 2.7. Use the definition of linear dependence to determine which of the sets are linearly dependent and which are linearly independent. For the sets that are linearly dependent, demonstrate how to write one of the vectors as a linear combination of the others.

$$\text{P1. } S = \left\{ \begin{pmatrix} -1 \\ 4 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix} \right\}$$

$$\text{P2. } T = \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix} \right\}.$$

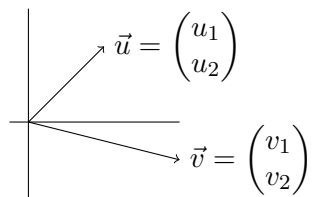
Lecture Activity 2.8. Use the Linear Dependence Theorem (Theorem 2.27) to determine which of the following sets are linearly dependent and which are linearly independent.

P1. $S = \left\{ \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix}, \begin{pmatrix} -1 \\ 4 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix} \right\}$

P2. $T = \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix} \right\}$

$$\text{P3. } U = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix} \right\}$$

Lecture Activity 2.9. Let $\vec{u}$ and $\vec{v}$ be the vectors drawn below.



P1. Find a formula for the length of $\vec{u}$ and a formula for the length of $\vec{v}$.

P2. Find a formula for the distance between $\vec{u}$ and $\vec{v}$. That is, find the distance between the points (u_1, u_2) and (v_1, v_2) .

Lecture Activity 2.10. Let $\vec{u}$ and $\vec{v}$ be vectors in $\mathbb{R}^2$, and define the following notation

$$\|\vec{u}\| := \sqrt{\vec{u} \cdot \vec{u}}.$$

Use your work from Lecture Activity 2.9 to show the following.

P1. Show that the length of $\vec{u}$ is equal to $\|\vec{u}\|$.

P2. Show that the distance between $\vec{u}$ and $\vec{v}$ is equal to $\|\vec{u} - \vec{v}\|$

P3. Show that the angle between $\vec{u}$ and $\vec{v}$ is equal to

$$\theta = \arccos \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\| \|\vec{v}\|}.$$

Conclude that $\vec{u}$ and $\vec{v}$ are perpendicular if and only if $\vec{u} \cdot \vec{v} = 0$. (*Hint: use law of cosines, along with Lemma 2.31*).

P4. Use P3 to show that $\vec{u}$ and $\vec{v}$ are perpendicular if and only if $\vec{u} \cdot \vec{v} = 0$.

Lecture Activity 2.11. Determine which of the following pairs of vectors $\vec{u}$ and $\vec{v}$ are orthogonal.

P1. $\vec{u} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ and $\vec{v} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$

P2. $\vec{u} = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$ and $\vec{v} = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$

P3. $\vec{u} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 1 \end{pmatrix}$ and $\vec{v} = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 1 \end{pmatrix}$

Chapter 2 Key Definitions.

The following list contains the key vocabulary from this chapter and will be used throughout the course. For each term listed below, a successful student should be able to (i) state its definition formally as it appears in the lecture notes from memory, (ii) produce at least one example and one nonexample, and (iii) demonstrate a firm conceptual understanding of its meaning.

On course assessments, definitions will scaffolded precisely as they are below. Students will be expected to (1) fill in the missing details of the definitions listed below and (2) provide one example and one nonexample of each definition.

Definition 2.21. A LINEAR COMBINATION of vectors $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$ in $\mathbb{R}^m$ is a vector of the form ...

where the $c_1, c_2, \dots, c_n$ are scalars called the COEFFICIENTS of the linear combination.

Example:

Nonexample:

Definition 2.23. The SPAN of vectors $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$ in $\mathbb{R}^m$ is the set ...

Example:

Nonexample:

Definition 2.26. A set of vectors $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$ in $\mathbb{R}^m$ is called LINEARLY DEPENDENT if ...

Otherwise, the set of vectors is called LINEARLY INDEPENDENT.

Example of linearly dependent set:

Example of linearly independent set:

Definition 2.28. Let $\vec{u}$ and $\vec{v}$ be n -dimensional vectors with coordinates

$$\vec{u} = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix} \text{ and } \vec{v} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}.$$

The DOT PRODUCT of $\vec{u}$ and $\vec{v}$ is the scalar ...

Example:

Nonexample:

Definition 2.32. Let $\vec{u}$ and $\vec{v}$ be vectors in $\mathbb{R}^n$.

1. The NORM of $\vec{u}$ is defined by ...

2. The DISTANCE between vectors $\vec{u}$ and $\vec{v}$ is defined by ...

3. We say that $\vec{u}$ and $\vec{v}$ are ORTHOGONAL if ...

Examples:

Nonexample (skip for norm and distance):

Chapter 2 Key Results

The following list contains the key results from this chapter. A successful student should be able to (i) state each result precisely as it appears in the lecture notes from memory, and (ii) recognize when and how to apply each result to solve different types of problems. Additionally, for all results listed in **bold-faced font**, a successful student should be able to reproduce a formal, logically correct proof of the result from memory.

On course assessments, students will be expected to fill in the missing details of the required proofs below, which will be scaffolded precisely as they appear in this packet. Additionally, students may use any of the theorems included below to solve problems.

Proposition 2.19 (Vector Equations and Systems).

Proof (not testable).

□

Theorem 2.27 (Linear Dependence Theorem).

Proof (testable). Suppose that the set $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$ is linearly dependent. By relabeling, we may assume that $\vec{v}_1 \in \text{Span}(\vec{v}_2, \dots, \vec{v}_n)$. So, there are real numbers $c_2, \dots, c_n$ so that $\vec{v}_1 = c_2\vec{v}_2 + \dots + c_n\vec{v}_n$.

Complete the proof: show that the system has a nontrivial solution.

Conversely, suppose that the vector equation $x_1\vec{v}_1 + x_2\vec{v}_2 + \cdots + x_n\vec{v}_n = \vec{0}$ has a nontrivial solution $(c_1, c_2, \dots, c_n)$. Then, one of the c_i is nonzero. By relabeling, we may assume that $c_1 \neq 0$.

Complete the proof: show that the set $\{\vec{v}_1, \dots, \vec{v}_n\}$ is linearly dependent.

□

Lemma 2.31 (Properties of the Dot Product).