

**Instructions.** This packet is due on Gradescope no later than **5pm on Monday, September 14**. Please complete your work directly on this packet. We will spend time together during lecture working on most of this packet together. You are responsible for completing all portions of this packet, including lecture activities not discussed in class, and completing the definitions, result statements, and result proofs included in the packet. Solutions will be posted to the course website after the assignment due date.

**Pre-class Reading and Suggested Resources**

1. Read the [course syllabus](#).
2. Watch the [course introduction videos](#) on Quercus.
3. Read the preface from the [course lecture notes](#).

## Chapter 1 Lecture Activities: Systems of Linear Equations

**Lecture Activity 1.1.** Find a system of linear equations that models each of the problems below. Note that you will solve these systems in the Chapter Exercises. You should only setup the systems in this activity.

- P1. *Blending Problems* arise in contexts where several inputs are combined to produce a final product with specific desired properties. Given the characteristics of each input, the goal is to determine how much of each input is needed to meet the target requirements.

You are planning a meal using three foods. Each unit of food contains a different combination of protein, carbohydrates, and fat, as shown below:

| Food   | Protein (g) | Carbs (g) | Fat (g) |
|--------|-------------|-----------|---------|
| Food A | 2           | 3         | 1       |
| Food B | 1           | 1         | 2       |
| Food C | 4           | 2         | 3       |

Suppose that you want to construct a meal out of these foods that contains exactly 30 grams of protein, 25 grams of carbs, and 20 grams of fat. Letting  $x$ ,  $y$ , and  $z$  represent the quantities of Foods A, B, and C, respectively, find a system of linear equations that the quantities  $x$ ,  $y$ , and  $z$  must satisfy in order for the meal to meet these requirements.

**Solution.** Using the table above, we need to solve the system

$$\begin{cases} 2x + y + 4z = 30 \\ 3x + y + 2z = 25 \\ x + 2y + 3z = 20 \end{cases}$$

- P2. Given a collection of data, it is often useful to find a mathematical model that fits the data exactly. This process, called *interpolation*, allows us to estimate missing values and understand the underlying pattern of our data.

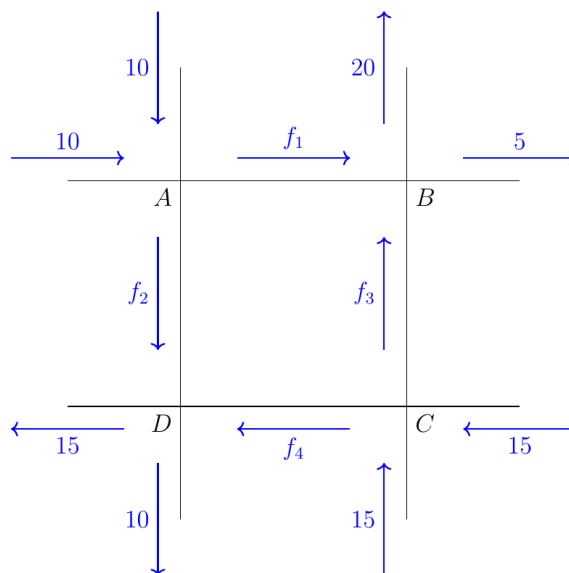
Suppose that we want to model some data using a cubic function  $F(x) = ax^3 + bx^2 + cx + d$ . Find a system of linear equations that the coefficients  $a$ ,  $b$ ,  $c$ , and  $d$  must satisfy in order for the graph of  $F$  to pass through all of the points  $(0, -2)$ ,  $(1, 1)$ ,  $(2, 6)$ , and  $(3, 10)$ .

**Solution.** Plugging each of the given points into the function  $F$  yields the system of linear equations

$$\begin{cases} d = -2 \\ a + b + c + d = 1 \\ 8a + 4b + 2c + d = 6 \\ 27a + 9b + 3c + d = 10 \end{cases}$$

- P3. *Flow conservation* problems arise in systems where something moves through a network so that at each junction, the total amount flowing in must equal the total amount flowing out.

The downtown core of a city consists of one-way streets, and the traffic flow has been measured at each intersection. For the city block shown, the numbers represent the average numbers of vehicles per minute entering and leaving intersections A, B, C, and D during business hours.



The fundamental rule governing flow through a network is conservation of flow: at each node, the flow in equals the flow out. Let  $f_1, f_2, f_3, f_4$  represent the unknown traffic flows shown in the diagram. Use conservation of flow to find a system of linear equations that the flows  $f_1, f_2, f_3, f_4$  must satisfy.

**Solution.** Using conservation of flow at each intersection gives the system of linear equations

$$\begin{cases} f_1 + f_2 = 20, \\ f_1 + f_3 = 25, \\ f_3 + f_4 = 30, \\ f_2 + f_4 = 25. \end{cases}$$

**Lecture Activity 1.2.** For each of the following matrices: (i) write down the system of linear equations that they represent as augmented matrices, and (ii) find all solutions to the system.

P1.  $A = \begin{pmatrix} 1 & 2 & 2 \\ 0 & 1 & 1 \end{pmatrix}$

**Solution.** The system of linear equations with  $A$  as its augmented matrix is given by

$$\begin{cases} x + 2y = 2 \\ y = 1 \end{cases}$$

Plugging  $y = 1$  into the first equation gives

$$x + 2 = 2 \Rightarrow x = 0.$$

So, this system has one solution given by  $(x, y) = (0, 1)$ .

P2.  $B = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 2 & 2 \end{pmatrix}$

**Solution.** The system of linear equations with  $B$  as its augmented matrix is given by

$$\begin{cases} x + y + z = 1 \\ y - z = 1 \\ 2z = 2 \end{cases}$$

From the last equation, we can solve  $z = 1$ . We can then plug this into the previous equation to find that

$$y - 1 = 1 \Rightarrow y = 2.$$

Finally, we can plug our values for  $y, z$  into the first equation to get

$$x + 2 + 1 = 1 \Rightarrow x = -2.$$

So, this system has one solution given by  $(x, y, z) = (-2, 2, 1)$ .

P3.  $C = \begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 4 \end{pmatrix}$

**Solution.** The system of linear equations with  $C$  as its augmented matrix is given by

$$\begin{cases} x = 2 \\ y = 3 \\ z = 4 \end{cases}$$

Since all of our variables are already determined, we see that this system has one solution given by  $(x, y, z) = (2, 3, 4)$ .

P4.  $D = \begin{pmatrix} 2 & 2 & -1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

**Solution.** The system of linear equations with  $C$  as its augmented matrix is given by

$$\begin{cases} 2x + 2y - z = 1 \\ 0 = 1 \end{cases}$$

Note that the last equation could have been written as

$$0x + 0y + 0z = 1.$$

Since there are no values of  $x, y, z$  that make this equality true, this system has no solutions.

P5.  $E = \begin{pmatrix} 1 & 2 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{pmatrix}$

**Solution.** The system of linear equations with  $E$  as its augmented matrix is given by

$$\begin{cases} x + 2y = 2 \\ z = 3 \end{cases}$$

Observe that there are infinitely many solutions to this system. Indeed, any triple of real numbers  $(x, y, 3)$  so that  $x + 2y = 2$  would be a solution. Later on in this chapter, we'll discuss how to describe solution sets of this type.

**Lecture Activity 1.3.** Determine which of the following matrices are in row echelon form, reduced row echelon form, or neither.

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 0 & 1 \\ 3 & 4 & -1/2 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & 2 & 1 \end{pmatrix},$$

$$D = \begin{pmatrix} 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad E = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad F = \begin{pmatrix} 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

**Solution.**

$A$  is in neither row echelon or reduced row echelon form

$B$  is in reduced row echelon form

$C$  is in row echelon form but not reduced row echelon form

$D$  is in row echelon form but not reduced row echelon form

$E$  is in neither row echelon or reduced row echelon form

$F$  is in reduced row echelon form

**Lecture Activity 1.4.** Find examples of matrices which are **in reduced row echelon form** and satisfy the given conditions.

P1. A  $3 \times 4$  matrix with exactly two pivots and exactly two entries which are not equal to 0 or 1.

**Solution.** One example of a matrix satisfying these conditions is given by

$$\begin{pmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

P2. A  $3 \times 3$  matrix which does not contain a row of zeros.

**Solution.** There is only one matrix in reduced row echelon form which satisfies this condition, and is given by

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

P3. The augmented matrix for a system of 3 linear equations in 3 variables which contains exactly one pivot and three entries which are not equal to 0 or 1.

**Solution.** One example of a matrix satisfying these conditions is given by

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

**Lecture Activity 1.5.** Find the reduced row echelon forms of the following matrices.

P1.  $A = \begin{pmatrix} 1 & 1 \\ 1 & 3 \end{pmatrix}$

**Solution.** We have

$$\begin{aligned} A &= \begin{pmatrix} 1 & 1 \\ 1 & 3 \end{pmatrix} \\ &\sim \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}, \text{ via } R_2 - R_1 \\ &\sim \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \text{ via } \frac{1}{2}R_2 \\ &\sim \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \text{ via } R_1 - R_2 \end{aligned}$$

which is in reduced row echelon form, and so

$$\text{rref}(A) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

P2.  $B = \begin{pmatrix} 1 & 1 & 3 \\ -2 & 0 & -2 \\ 2 & 1 & 4 \end{pmatrix}$

**Solution.** We have

$$\begin{aligned} B &= \begin{pmatrix} 1 & 1 & 3 \\ -2 & 0 & -2 \\ 2 & 1 & 4 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 3 \\ -2 & 0 & -2 \\ 0 & 1 & 2 \end{pmatrix}, \text{ via } R_3 + R_2 \\ &\sim \begin{pmatrix} 1 & 1 & 3 \\ 0 & 2 & 4 \\ 0 & 1 & 2 \end{pmatrix}, \text{ via } R_2 + 2R_1 \\ &\sim \begin{pmatrix} 1 & 1 & 3 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{pmatrix}, \text{ via } \frac{1}{2}R_2 \\ &\sim \begin{pmatrix} 1 & 1 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}, \text{ via } R_3 - R_2 \\ &\sim \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}, \text{ via } R_1 - R_2 \end{aligned}$$

which is in reduced row echelon form, and so

$$\text{rref}(B) = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}.$$

P3.  $C = \begin{pmatrix} 1 & 1 & 1 & 7 & -1 \\ 0 & 1 & 1 & 4 & -1 \\ -1 & 2 & 2 & 5 & -2 \end{pmatrix}$

**Solution.** We have

$$\begin{aligned} C &= \begin{pmatrix} 1 & 1 & 1 & 7 & -1 \\ 0 & 1 & 1 & 4 & -1 \\ -1 & 2 & 2 & 5 & -2 \end{pmatrix} \\ &\sim \begin{pmatrix} 1 & 1 & 1 & 7 & -1 \\ 0 & 1 & 1 & 4 & -1 \\ 0 & 3 & 3 & 12 & -3 \end{pmatrix}, \text{ via } R_3 + R_1 \\ &\sim \begin{pmatrix} 1 & 1 & 1 & 7 & -1 \\ 0 & 1 & 1 & 4 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \text{ via } R_3 - 3R_2 \\ &\sim \begin{pmatrix} 1 & 0 & 0 & 3 & 0 \\ 0 & 1 & 1 & 4 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \text{ via } R_1 - R_2 \end{aligned}$$

which is in reduced row echelon form, and so

$$\text{rref}(C) = \begin{pmatrix} 1 & 0 & 0 & 3 & 0 \\ 0 & 1 & 1 & 4 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

**Lecture Activity 1.6.** Use Gauss-Jordan elimination to find all solutions to the system of linear equations

$$\begin{cases} 2x + y + z = 7 \\ x + 2y + z = 8 \\ x + y + z = 5 \end{cases}$$

**Solution.** This system has augmented matrix

$$A = \begin{pmatrix} 2 & 1 & 1 & 7 \\ 1 & 2 & 1 & 8 \\ 1 & 1 & 1 & 5 \end{pmatrix}.$$

We have

$$\begin{aligned} A &= \begin{pmatrix} 2 & 1 & 1 & 7 \\ 1 & 2 & 1 & 8 \\ 1 & 1 & 1 & 5 \end{pmatrix} \\ &= \begin{pmatrix} 2 & 1 & 1 & 7 \\ 0 & 1 & 0 & 3 \\ 1 & 1 & 1 & 5 \end{pmatrix}, \text{ via } R_2 - R_3 \\ &= \begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 3 \\ 1 & 1 & 1 & 5 \end{pmatrix}, \text{ via } R_1 - R_3 \\ &= \begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 1 & 1 & 3 \end{pmatrix}, \text{ via } R_3 - R_1 \\ &= \begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \text{ via } R_3 - R_2. \end{aligned}$$

Since this matrix is in reduced row echelon form, we have

$$\text{rref}(A) = \begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

which represents the equivalent system

$$\begin{cases} x = 2 \\ y = 3 \\ z = 0 \end{cases}$$

So, our system of linear equations has one solution given by

$$(x, y, z) = (2, 3, 0).$$

**Lecture Activity 1.7.** Let  $A$  be the augmented matrix for a system of linear equations in variables  $x, y, z$  with

$$\text{rref}(A) = \begin{pmatrix} 1 & 0 & 2 & 2 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

P1. Identify the pivot variables and the nonpivot variables of the system.

**Solution.** The pivot variables are  $x$  and  $y$ , and the nonpivot variable is  $z$ .

P2. Parameterize the pivot variables in terms of the nonpivot variables.

**Solution.** Observe that  $\text{rref}(A)$  is the augmented matrix for the system

$$\begin{cases} x + 2z = 2 \\ y + z = 3 \end{cases}$$

Solving for the pivot variables  $x$  and  $y$  in terms of the nonpivot variable  $z$  gives  $x = 2 - 2z$  and  $y = 3 - z$ .

P3. Give a parametric description of the solution set for this system by allowing the nonpivot variable(s) to be free.

**Solution.** Letting  $z$  be free produces all solutions to the system. Thus, the solution set to this system can be described parametrically as

$$\{(2 - 2z, 3 - z, z) : z \in \mathbb{R}\}.$$

**Lecture Activity 1.8.** Let  $B$  be the augmented matrix for a system of linear equations in variables  $x_1, x_2, x_3, x_4, x_5$  with

$$\text{rref}(B) = \begin{pmatrix} 1 & 2 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{pmatrix}.$$

P1. Identify the pivot variables and the nonpivot variables of the system.

**Solution.** The pivot variables are  $x_1, x_3, x_5$ , and the nonpivot variables are  $x_2, x_4$ .

P2. Parameterize the nonpivot variables in terms of the pivot variables.

**Solution.** Observe that  $\text{rref}(B)$  is the augmented matrix for the system

$$\begin{cases} x_1 + 2x_2 = 1 \\ x_3 + x_4 = 0 \\ x_5 = 2. \end{cases}$$

Solving for the pivot variables in terms of the nonpivot variables gives  $x_1 = 1 - 2x_2$ ,  $x_3 = -x_4$ , and  $x_5 = 2$ .

P3. Describe the solution set for this system of linear equations by allowing the nonpivot variables to be free.

**Solution.** Letting  $x_2$  and  $x_4$  be free produces all solutions to the system. Thus, the solution set to this system can be described parametrically as

$$\{(1 - 2x_2, x_2, -x_4, x_4, 2) : x_2, x_4 \in \mathbb{R}\}.$$

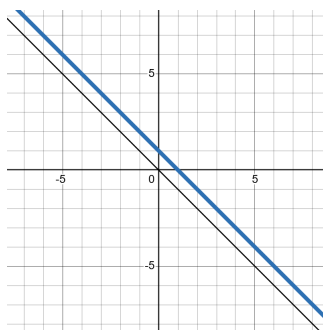
**Lecture Activity 1.9.** Consider the system of linear equations

$$\begin{cases} x + y = 0 \\ ax + by = c \end{cases}$$

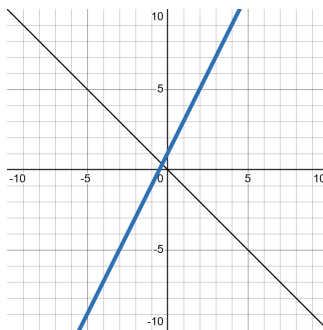
Use [this Desmos Example](#), along with geometric reasoning, to answer the following questions.

- P1. Using the sliders on the Desmos example linked above, find values of  $a$ ,  $b$ , and  $c$  so that this system has (i) no solutions, (ii) exactly one solution, and (iii) infinitely many solutions.

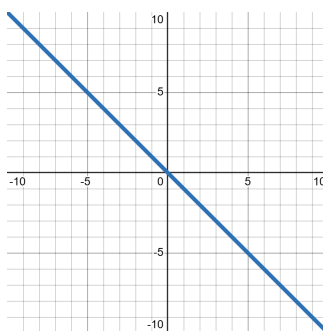
**Solution.** Setting  $a = b = c = 1$  yields two lines with no points of intersection, and so the system has no solutions in this case



Setting  $a = -2, b = 1, c = 1$  yields two lines with exactly one point of intersection, and so the system has exactly one solution in this case



Setting  $a = 2, b = 2, c = 0$  yields two identical lines, meaning that every point on both lines is a point of intersection, and so the system has infinitely many solutions in this case



P2. Is it possible for this system to have exactly two solutions?

**Solution.** No, it is not possible. Since lines cannot curve, two lines cannot have exactly two points of intersection.

P3. If the system has more than one solution, how many solutions must it have?

**Solution.** If two lines intersect at two or more points, they must be exactly the same line. So, if the system has more than one solution, it must have infinitely many solutions.

P4. Using your work above, how many solutions do you think a system of linear equations in two variables could have?

**Solution.** Note that two lines can either not intersect, intersect at exactly one point, or intersect at more than one point. So, from our work above, we reason that a system of linear equations in two variables can have no solutions, exactly one solution, or infinitely many solutions.

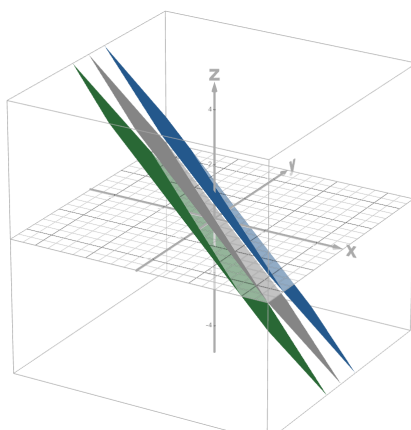
**Lecture Activity 1.10.** Consider the system of linear equations

$$\begin{cases} x + y + z = 0 \\ a_1x + b_1y + c_1z = d_1 \\ a_2x + b_2y + c_2z = d_2 \end{cases}$$

Use [this Desmos Example](#), along with geometric reasoning, to answer the following questions.

- P1. Using the sliders on the Desmos example linked above, find values coefficients  $a_1, b_1, c_1$  and  $a_2, b_2, c_2$  so that this system has (i) no solutions, (ii) exactly one solution, and (iii) infinitely many solutions.

**Solution.** Setting all coefficients equal to 1 yields three planes with no points of intersection, and so the system has no solutions in this case

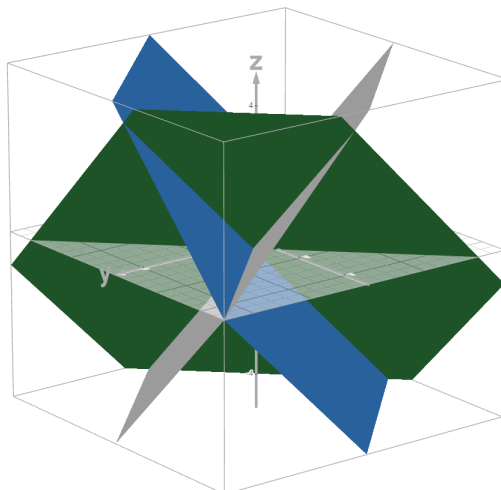


Setting

$$a_1 = 1, b_1 = -1, c_1 = 1, d_1 = -1$$

$$a_2 = 1, b_2 = 1, c_2 = -1, d_2 = 0$$

yields three planes which intersect at exactly one point, meaning that every point on both lines is a point of intersection, and so the system has infinitely many solutions in this case

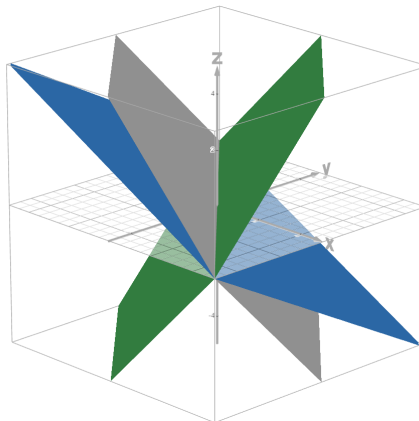


Setting

$$a_1 = 1, b_1 = 1, c_1 = -1, d_1 = 0$$

$$a_2 = 1, b_2 = 1, c_2 = 2, d_2 = 0$$

yields three planes which intersect at a line. Since every point on this line is a point of intersection, the system has infinitely many solutions in this case



P2. Is it possible for this system to have exactly two solutions?

**Solution.** No, it is not possible. Since planes cannot curve, planes cannot have exactly two points of intersection.

P3. If the system has more than one solution, how many solutions must it have?

**Solution.** Observe that if planes intersect at more than one point, they intersect at a line, or are identical planes. In either case, the system would have infinitely many solutions.

P4. Using your work above, how many solutions do you think a system of linear equations in three variables could have?

**Solution.** Since planes can either not intersect, intersect at exactly one point, or intersect at more than one point, we reason that systems of linear equations in three variables can have no solutions, exactly one solution, or infinitely many solutions.

**Solutions will not be included for the rest of this packet.** Please use the course lecture notes to check your work.

## Chapter 1 Key Definitions.

The following list contains the key vocabulary from this chapter and will be used throughout the course. For each term listed below, a successful student should be able to (i) state its definition formally as it appears in the lecture notes from memory, (ii) produce at least one example and one nonexample, and (iii) demonstrate a firm conceptual understanding of its meaning.

On course assessments, definitions will scaffolded precisely as they are below. Students will be expected to (1) fill in the missing details of the definitions listed below and (2) provide one example and one nonexample of each definition.

**Definition 1.16.** The PIVOT of a row in a matrix is ...

**Example:**

**Nonexample:**

**Definition 1.18.** A matrix is in ROW ECHELON FORM if ...

**Example:**

**Nonexample:**

**Definition 1.19.** A matrix is in REDUCED ROW ECHELON FORM if ...

**Example:**

**Nonexample:**

**Definition 1.26.** Let  $C$  be the coefficient matrix for a system of linear equations in variables  $x_1, \dots, x_n$ . If the system of linear equations has at least one solution, we define the following.

1. We say that  $x_i$  is a PIVOT VARIABLE for the system if ...

2. We say that  $x_i$  is a NONPIVOT VARIABLE of the system if ...

**Example of pivot variable:**

**Example of nonpivot variable:**

## Chapter 1 Key Results

The following list contains the key results from this chapter. A successful student should be able to (i) state each result precisely as it appears in the lecture notes from memory, and (ii) recognize when and how to apply each result to solve different types of problems. Additionally, for all results listed in **bold-faced font**, a successful student should be able to reproduce a formal, logically correct proof of the result from memory.

On course assessments, students will be expected to fill in the missing details of the required proofs below, which will be scaffolded precisely as they appear in this packet. Additionally, students may use any of the theorems included below to solve problems.

Theorem 1.15 (Row Equivalence).

Theorem 1.21 (Gauss-Jordan).

Lemma 1.27 (Pivot Parameterization).

**Theorem 1.30 (Rouché-Capelli).**

*Proof of Rouché-Capelli (testable).* Consider a system of  $m$  linear equations in  $n$  variables with augmented matrix  $A$  and coefficient matrix  $C$ . Let's first prove the forward directions first. For part (1), suppose that  $\text{rref}(A)$  has a pivot in the last column. Then...

*Complete the proof: show that the system has no solutions.*

For part (2), suppose that  $\text{rref}(A)$  does not have a pivot in the last column, and every column of  $\text{rref}(C)$  has a pivot. By a chapter exercise we must have  $m \geq n$ , and so ...

*Complete the proof: show that the system has exactly one solution.*

For part (3), suppose that  $\text{rref}(A)$  does not have a pivot in the last column and that  $\text{rref}(C)$  has a column without a pivot.

*Complete the proof: show that the system has exactly one solution.*

To prove the converse of each part, we only need to use logical reasoning. We leave this step as an exercise to the reader.

□