Section B.

INSTRUCTIONS:

1. Each problem in this section is worth **three points**.
2. Problems with multiple parts will be worth one point each. Otherwise, no partial credit will be given.
3. You do not need to show your work or provide justification on any problem in Section B.
4. **Your answer must be placed in the answer box provided.**
5. We have provided extra space for your scratch work on each problem, but nothing outside of the answer box will be considered toward your score on the Section B problems.

B1. (3 points) For the systems of linear equations described below, determine whether the system has no solution, exactly one solution, or infinitely many solutions.

a) A system whose coefficient matrix is invertible. RC , e.g. $\left(\begin{array}{cc|c} 1 & 0 & * \\ 0 & 1 & * \end{array}\right)$

Answer: The system of linear equations has

○ No solutions ● Exactly one solution ○ Infinitely many solutions

b) A system whose augmented matrix is invertible. pivot in last col, e.g. $\left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right)$

Answer: The system of linear equations has

● No solutions ○ Exactly one solution ○ Infinitely many solutions

c) The system with augmented matrix A^T, where A is the augmented matrix representing the system in part (a).

Answer: The system of linear equations must have

● No solutions ○ Exactly one solution ○ Infinitely many solutions

Given $A = (C \mid \vec{b})$ where C inv $\Rightarrow C^T$ is invertible (since $\det(C^T) = \det(C) \neq 0$)

$\Rightarrow A^T = \begin{pmatrix} C^T \\ \vec{b} \end{pmatrix}$

Since rref(C^T) has pivot in last col, rref(A^T) has pivot in last col $\underset{RC}{\Longrightarrow}$ no sols

B2. (3 points) For each linear transformation defined below, determine whether the reduced row echelon form of its defining matrix has a pivot in every row, every column, both, or neither.

a) $F : \mathbb{R}^4 \to \mathbb{R}^3$ satisfying that $\{F(\vec{e}_1), F(\vec{e}_3), F(\vec{e}_4)\}$ is linearly independent.

Answer:

○ Pivots in every row and every column ● Pivots in every row but not every column

○ Pivots in every column but not every row ○ None of the above

$$A_F = \begin{pmatrix} F(\vec{e}_1) & F(\vec{e}_2) & \underbrace{F(\vec{e}_3)}_{\text{no pivot!}} & F(\vec{e}_4) \end{pmatrix} \quad 3 \times 4$$

pivots pivot

w/ 3 pivots

b) $G = T_Q$, where Q is an $n \times n$ orthogonal matrix.

Answer:

● Pivots in every row and every column ○ Pivots in every row but not every column

○ Pivots in every column but not every row ○ None of the above

$Q = (\vec{v}_1 \cdots \vec{v}_n)$ where $\{\vec{v}_1, \dots, \vec{v}_n\}$ orthonormal basis

& square

c) $H : \mathbb{R}^2 \to \mathbb{R}^4$ with $H(\vec{e}_1) \neq H(\vec{e}_2) \neq \vec{0}$.

Answer:

○ Pivots in every row and every column ○ Pivots in every row but not every column

○ Pivots in every column but not every row ● None of the above

$$A_H = \begin{pmatrix} H(\vec{e}_1) & H(\vec{e}_2) \end{pmatrix}$$

4×2

$\begin{pmatrix} 1 & 2 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$ no pivot in 2nd col

$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$ pivot in evry col

Could be either!

B3. (3 points) Calculate the following determinants.

a) $\det(AB)$ where $A = \begin{pmatrix} 3 & 40 & -1 \\ 0 & 7 & 3 \\ 0 & 0 & -3 \end{pmatrix}$ and $B = A^T$, the transpose of A.

$\det(AB) = 3969$

$\det(A) = 3 \cdot 7 (-3) = -63,$

since A upper Δ

$\det(A A^T) = \det(A) \det(A^T)$

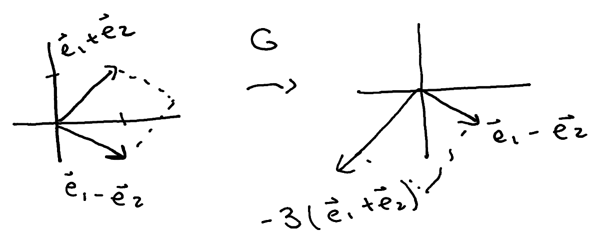
$= \det(A) \det(A)$

$= 63^2 = (60+3)(60+3) = 3600 + 2 \cdot (180) + 9$

$= 3600 + 369$

b) $\det(F)$, where F is the inverse of the function G where $G \colon \mathbb{R}^2 \to \mathbb{R}^2$ is the linear transformation which stretches vectors in the $\vec{e}_1 + \vec{e}_2$ direction by -3 and leaves the $\vec{e}_1 - \vec{e}_2$ direction unchanged.

$\det(F) = -1/3$



$\det(F) = \det(G^{-1})$

$= \det(G)^{-1}$

$= -1/3$

volume changes by factor of 3,

orientation reverses

$\Rightarrow \det(G) = -3$

c) Let C be standard defining matrix of F from part (b). Is it possible that C similar to the matrix AB from part (a)?

○ Yes, it is possible ● No, it is not possible

$\det(AB) \neq \det(C)$

B4. (3 points) Determine which of the following matrices are invertible. If there is not enough information to determine whether the matrix is invertible or not invertible, select "could be either".

a) A 3×3 matrix N satisfying that N^3 is the zero matrix.

- ○ Is invertible
- ● Is not invertible
- ○ Could be either

If N inv & $N^3 = 0 \leftarrow 0$ matrix

$\Rightarrow N^{-1}N^{-1}N^3 = N^{-1}N^{-1}0$

$\Rightarrow N = 0 \not\to$ (0 matrix is not invertible!)

b) A symmetric matrix.

- ○ Is invertible
- ○ Is not invertible
- ● Could be either

$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

c) The defining matrix of the linear transformation in $\mathbb{R}^3$ that rotates counterclockwise around the z-axis by an angle of $\frac{\pi}{4}$.

- ● Is invertible
- ○ Is not invertible
- ○ Could be either

inverse = rotate clockwise around z-axis by $\pi/4$

B5. (3 points) [redacted]

Let A be a 3×3 matrix with eigenvalues $0, 1, 2$. Find the eigenvalues of the following matrices:

a) The matrix A^2.

$\boxed{0, 1, 4}$

b) The matrix $A - I_3$.

$\boxed{-1, 0, 1}$

c) The matrix $3(A^T)^2$.

$\boxed{0, 3, 12}$

Obs A is diagonalizable, so

(a) v1 $\quad A = P \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} P^{-1}$

$\Rightarrow A^2 = P \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{pmatrix} P^{-1}$

(a) v2

iff if A invertible (but it's not!)

$A\vec{x} = \lambda \vec{x} \Rightarrow A^2 \vec{x} = A \cdot A\vec{x} = A(\lambda \vec{x})$
$= \lambda A\vec{x} = \lambda^2 \vec{x}$

$\Rightarrow 0^2, 1^2, 2^2$ eigenvalues

& there aren't any more since A is 3×3

(c) $\quad 3(A^T)^2 = 3\left[\left(P \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} P^{-1}\right)^T\right]^2$

$= 3\left((P^{-1})^T \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} P^T\right)^2$

$= 3 \quad (P^{-2})^T \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{pmatrix} (P^2)^T = \tilde{P} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 12 \end{pmatrix} \tilde{P}^{-1}$

eigenvalues 0, 3, 6

(b) λ eigenvalue of $A \iff \det(A - \lambda I_3) = 0$
$\iff \det(A - I_3 - (\lambda - 1) I_3) = 0$

$\Rightarrow$ eigenvalues

$0 - 1$
$1 - 1$
$2 - 1$

B6. (3 points) Let $\mathcal{E}$ be the standard basis of $\mathbb{R}^3$. Consider the basis $\mathcal{B} = \left\{ \begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix} \right\}$.

a) Find $[\vec{v}]_{\mathcal{E}}$ given that $[\vec{v}]_{\mathcal{B}} = \begin{pmatrix}2\\3\\1\end{pmatrix}$.

$$[\vec{v}]_{\mathcal{E}} = \begin{pmatrix}\boxed{5}\\\boxed{6}\\\boxed{2}\end{pmatrix}$$

$$[\vec{v}]_{\mathcal{E}} = 2\begin{pmatrix}1\\1\\1\end{pmatrix} + 3\begin{pmatrix}1\\1\\0\end{pmatrix} + 1\begin{pmatrix}0\\1\\0\end{pmatrix}$$

b) Find the change of basis matrix from $\mathcal{B}$ to $\mathcal{E}$.

$$M_{\mathcal{E}\leftarrow\mathcal{B}} = \begin{pmatrix}\boxed{1}&\boxed{1}&\boxed{0}\\\boxed{1}&\boxed{1}&\boxed{1}\\\boxed{1}&\boxed{0}&\boxed{0}\end{pmatrix}$$

$$M_{\mathcal{B}\leftarrow\mathcal{E}} = M_{\mathcal{E}\leftarrow\mathcal{B}}^{-1} \ldots$$

$$\left(\begin{array}{ccc|ccc}1&1&0&1&0&0\\1&1&1&0&1&0\\1&0&0&0&0&1\end{array}\right)$$

$$\overset{R_2-R_1}{\sim}\left(\begin{array}{ccc|ccc}1&1&0&1&0&0\\0&0&1&-1&1&0\\1&0&0&0&0&1\end{array}\right)$$

$$\overset{R_1-R_3}{\sim}\left(\begin{array}{ccc|ccc}0&1&0&1&0&-1\\0&0&1&-1&1&0\\1&0&0&0&0&1\end{array}\right)$$

$$\overset{\text{swap rows}}{\sim}\left(\begin{array}{ccc|ccc}1&0&0&0&0&1\\0&1&0&1&0&-1\\0&0&1&-1&1&0\end{array}\right)$$

c) Find the change of basis matrix from $\mathcal{E}$ to $\mathcal{B}$.

$$M_{\mathcal{B}\leftarrow\mathcal{E}} = \begin{pmatrix}\boxed{0}&\boxed{0}&\boxed{1}\\\boxed{1}&\boxed{0}&\boxed{-1}\\\boxed{-1}&\boxed{1}&\boxed{0}\end{pmatrix}$$

CHECK:

$$\begin{pmatrix}1&1&0\\1&1&1\\1&0&0\end{pmatrix}\begin{pmatrix}0&0&1\\1&0&-1\\-1&1&0\end{pmatrix} = \begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix} ✓$$

B7. (3 points) Let

$$A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{pmatrix}$$

.

$$\det \begin{pmatrix} -x & 1 & 1 \\ 1 & -x & -1 \\ 1 & -1 & -x \end{pmatrix} =$$

a) Find the characteristic polynomial of A.

Answer: $\chi_A(x) = -(x-1)^2(x+2)$

$$-x \det \begin{pmatrix} -x & -1 \\ -1 & -x \end{pmatrix} - \det \begin{pmatrix} 1 & -1 \\ 1 & -x \end{pmatrix} + \det \cancel{\begin{pmatrix} 1 & -x \\ 1 & 1 \end{pmatrix}}$$

$$= -x(x^2-1) - (-x+1) + (-1+x)$$

$$= -x(x+1)(x-1) + 2(x-1) = (x-1)(-x(x+1)+2)$$

$$= (x-1)(-x^2-x+2)$$

$$= -(x-1)(x^2+x-2)$$

$$= -(x-1)(x-1)(x+2)$$

b) Find the dimension of the 1-eigenspace E_1.

Answer: $\dim E_1 = 2$

$$E_1 = \mathrm{Nul} \begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & -1 \\ 1 & -1 & -1 \end{pmatrix} \quad R_3 - R_2$$

$$= \mathrm{Nul} \begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \quad R_2 + R_1$$

$$= \mathrm{Nul} \begin{pmatrix} -1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & \underbrace{0 \quad 0}_{\text{2 free vars}} \end{pmatrix}$$

c) Find an invertible matrix C so that $C^{-1}AC$ is a diagonal matrix.

$$C = \begin{pmatrix} 1 & 1 & -1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

$$E_1 = \mathrm{Nul} \begin{pmatrix} -1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} : -x+y+z=0 \right\} = \left\{ \begin{pmatrix} y+z \\ y \\ z \end{pmatrix} : y,z \in \mathbb{R} \right\}$$

$$= \left\{ \begin{pmatrix} y \\ y \\ 0 \end{pmatrix} + \begin{pmatrix} z \\ 0 \\ z \end{pmatrix} : y,z \in \mathbb{R} \right\}$$

$$= \mathrm{Span}\left(\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right)$$

$E_{-2} = \dots$ (see additional pages)

B8. (3 points) Determine which of the following statements are always true and which are always false. If there's not enough information to determine whether a statement is always true or always false, select "could be true or false".

a) If $\vec{v}$ and $\vec{w}$ are two vectors in $\mathbb{R}^n$ such that $\vec{v} \cdot \vec{w} = 0$, then $\{\vec{v}, \vec{w}\}$ is a linearly independent set.

● Always true ○ Always false ○ Could be true or false

b) If a linear transformation $F : \mathbb{R}^3 \to \mathbb{R}^3$ preserves the angle between every pair of vectors, then its defining matrix A is orthogonal.

○ Always true ○ Always false ● Could be true or false

Only if also preserves lengths ! For ex, $A = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$ preserves angles but not lengths

c) Let V be a vector subspace of $\mathbb{R}^n$. Every basis of V has n elements.

○ Always true ○ Always false ● Could be true or false

Section C.

Instructions:

1. Each problem in this section is worth **5 points**.
2. You must provide justification for all of your answers in Section C.
3. Points will be awarded based on the rubric below. Note that half points may be awarded, and further rubric items may be added to cover potential cases not outlined below.

Points	**Rubric**
5	Solution is presented with clear justification that is logically complete and correct. May include minor typos and computational errors if they do not majorly impact the argument. No important steps are missing or assumed. All assumptions and special cases have been covered. All suggestions for improvement come under the category of "improvements for clarity" rather than "correcting logical errors". Omission of details will be judged depending on context of the material, with simpler steps being acceptable for omission when covering more advanced topics.
4	Solution is close to full and complete, but contains either a computational error or an error in reasoning that majorly impacts the argument. This score is also appropriate for solutions that are mathematically sound but confusingly written.
3	Solution is incorrect, but understanding of the problem was demonstrated and student provided a clear outline of a potential approach with information about where they got stuck **-or-** solution is correct but justification is insufficient or so confusingly written that it cannot be followed with a reasonable amount of effort.
2	Solution is incorrect, but student demonstrated understanding of the problem **-or-** solution is correct and student did not provide justification for their answer.
1	Solution is incorrect and student did not demonstrate understanding of the problem, but did demonstrate some knowledge of relevant material.
0	Solution is incorrect or incomplete, and there was no demonstration of knowledge of relevant material.

C1. (5 points) Let A be an $m \times (n+1)$ matrix, and suppose that the system of linear equations in n variables with augmented matrix A has at least one solution. Show that the homogeneous system of linear equations in $n+1$ variables with coefficient matrix A has infinitely many solutions

Proof.

Sps A is augm matrix for syst w/ at least one sol. By Roucé-Capelli, rref(A) does not have pivot in last col.

Now, consider the homogeneous syst w/ aug matrix $(A|\vec{0})$. This syst is consistent, & the rref of its coeff matrix A has a col w/o pivot. So, by Roucé-Capelli, this syst has ∞ sols.

$\boxed{K}$

C2. (5 points) Let $B = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_6\}$ be a set of vectors in $\mathbb{R}^4$. Then B cannot be a basis of $\mathbb{R}^4$. **If true, provide a proof. If false, provide a counterexample, and justify why this is one.**

● True ○ False

Proof or Counterexample:

v1:

Let $A = (\vec{v}_1\ \vec{v}_2\ \cdots\ \vec{v}_6)$. Then, A is 4×6

& so rref(A) has at most 4 pivots

$\Rightarrow$ rref(A) has a column w/o pivot,

& so by $\{\vec{v}_1, -, \vec{v}_6\}$ is lin'ly dependent

& hence cannot be a basis for $\mathbb{R}^4$ ☒

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v2:

If B were a basis for $\mathbb{R}^4$, then

$\dim(\mathbb{R}^4) = 6$,

but we know $\dim(\mathbb{R}^4) = 4$. So, B

cannot be a basis ☒

C3. (5 points) Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation and $\mathcal{B}$ a basis for $\mathbb{R}^n$. Show that if the defining matrix A_F is invertible, then $A_{F,\mathcal{B}}$ is also invertible.

Proof.

Let $\mathcal{B} = \{\vec{b}_1, -, \vec{b}_n\}$ be basis for $\mathbb{R}^n$ & $F : \mathbb{R}^n \to \mathbb{R}^n$ be invertible. Note that F inj $\Rightarrow \ker F = \{\vec{0}\}$.

<u>OBS</u>: $\{F(\vec{b}_1), \ldots, F(\vec{b}_n)\}$ is lin'ly ind.

Indeed, we have

$$x_1 F(\vec{b}_1) + \cdots + x_n F(\vec{b}_n) = \vec{0}$$
$$\Leftrightarrow F(x_1\vec{b}_1 + \cdots + x_n\vec{b}_n) = \vec{0}$$
$$\Leftrightarrow x_1\vec{b}_1 + \cdots + x_n\vec{b}_n = \vec{0}, \text{ since } \ker F = \{\vec{0}\}$$
$$\Leftrightarrow x_1 = \cdots = x_n = 0$$

So, by result from LNs we know that

$$\{[F(\vec{b}_1)]_{\mathcal{B}}, \ldots, [F(\vec{b}_n)]_{\mathcal{B}}\}$$

is lin'ly ind.

Indeed, $x_1[F(\vec{b}_1)]_{\mathcal{B}} + \cdots + x_n[F(\vec{b}_n)]_{\mathcal{B}} = \vec{0}$

$$\Leftrightarrow [x_1 F(\vec{b}_1) + \cdots + x_n F(\vec{b}_n)]_{\mathcal{B}} = \vec{0}$$
$$\Leftrightarrow x_1 F(\vec{b}_1) + \cdots + x_n F(\vec{b}_n) = \vec{0}$$
$$\Leftrightarrow x_1 = \cdots = x_n = 0$$

So, by IMT, $A_{F,\mathcal{B}} = ([F(\vec{b}_1)]_{\mathcal{B}} \cdots [F(\vec{b}_n)]_{\mathcal{B}})$ is invertible. $\boxed{\text{M}}$

since $A_{F,\mathcal{B}}$ has lin'ly ind cols!

C4. (5 points) Let A be a 6 × 7 matrix. Is it possible that the nullity of A equals the nullity of its transpose A^T? **If yes, find an example and prove that it is an example. If no, prove it.**

Proof or *Example.*

No, this is not possible.

we didn't have time to prove this in Ch. 4! Pf is included below but you don't need this for the final!

Lm: $\text{rank}(A) = \text{rank}(A^T)$.

By rank-nullity we have

$$\text{nullity}(A) = 7 - \text{rank}(A)$$

but, $\text{nullity}(A^T) = 6 - \text{rank}(A^T)$

$$= 6 - \text{rank}(A)$$

$$\neq \text{nullity}(A)$$

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pf of Lm: Recall that $\text{Col}(A^T) =: \text{Row}(A)$.

Obs that $\text{Row}(A) = \text{Row}(\text{rref}(A))$

(b/c rows of rref(A) are lin combos of rows of A & vice versa)

Also obs the dimension of $\text{Row}(\text{rref}(A))$ is equal to the # of nonzero rows of $\text{rref}(A)$, which is precisely the # pivots of $\text{rref}(A)$

$\Rightarrow \underbrace{\dim \text{Row}(A)}_{\dim \text{Col}(A^T)} = \dim \text{Col}(A)$ & so $\text{rank}(A) = \text{rank}(A^T)$, as needed $\boxed{4}$

YOU MUST SUBMIT THIS PAGE.

If you would like work on this page scored, then clearly indicate to which question the work belongs and indicate on the page containing the original question that there is work on this page to score.

B7 (c), cont...

$$E_{-2} = \mathrm{Nul}\begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix} \xrightarrow{R_1 - 2R_2}$$

$$= \mathrm{Nul}\begin{pmatrix} 0 & -3 & 3 \\ 1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix} \xrightarrow{R_2 - R_3}$$

$$= \mathrm{Nul}\begin{pmatrix} 0 & -3 & 3 \\ 0 & 3 & -3 \\ 1 & -1 & 2 \end{pmatrix} \xrightarrow{R_2 - R_1 \text{ + swap rows}}$$

$$= \mathrm{Nul}\begin{pmatrix} 1 & -1 & 2 \\ 0 & 3 & -3 \\ 0 & 0 & 0 \end{pmatrix} \xrightarrow{\frac{1}{3}R_2}$$

$$= \mathrm{Nul}\begin{pmatrix} 1 & -1 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \xrightarrow{R_2 + R_1}$$

$$= \mathrm{Nul}\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$= \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} : \begin{matrix} x + z = 0 \\ \& \ y - z = 0 \end{matrix} \right\}$$

$$= \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} : \begin{matrix} x = -z \\ y = z \end{matrix} \right\}$$

$$= \left\{ \begin{pmatrix} -z \\ z \\ z \end{pmatrix} : z \in \mathbb{R} \right\} = \mathrm{Span}\left(\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} \right).$$