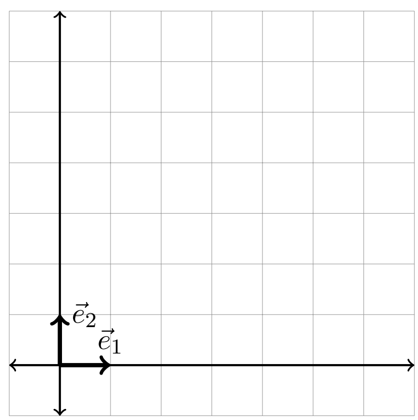
Instructions. This packet is due on Quercus no later than **11:59pm on Monday, March 16th**. Please complete your work directly on this packet. We will spend time together during lecture working on most or all of the activities in this packet. You are responsible for completing all portions of this packet, including lecture activities not discussed in class, and completing the definitions included in the packet. Solutions will be posted to the course website after the assignmeFnt due date.

Lecture Activity 9.1. The standard coordinate grid for $\mathbb{R}^2$ is drawn below. Explain how we can use the standard basis $\mathcal{E} = \{\vec{e}_1, \vec{e}_2\}$ to draw the grid lines below. Then, draw the vectors

$$\vec{u} = 4\vec{e}_1 + \vec{e}_2 \text{ and } \vec{v} = -\vec{e}_1 + 5\vec{e}_2$$

on the coordinate grid below.

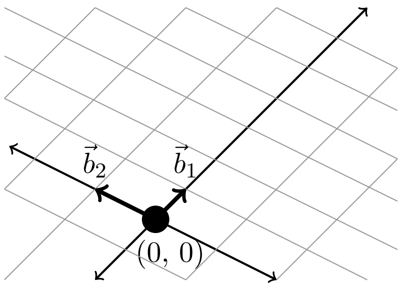


Lecture Activity 9.2. Let $\mathcal{B} = \{\vec{b}_1, \vec{b}_2\}$ be the basis for $\mathbb{R}^2$ with

$$\vec{b}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \text{ and } \vec{b}_2 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}.$$

The image below shows how we can use $\mathcal{B}$ to create a "nonstandard" coordinate grid for $\mathbb{R}^2$.

P1. Explain how the vectors $\vec{b}_1, \vec{b}_2$ can be used to draw the grid lines below.

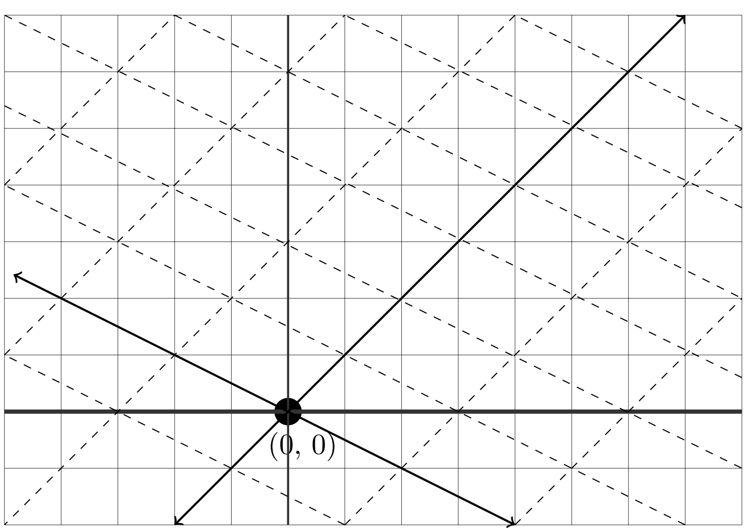


P2. The following graph includes the standard coordinate grid defined by the standard basis $\mathcal{E}$ (drawn with solid lines) and the "nonstandard" coordinate grid defined by the basis $\mathcal{B}$ (drawn with dashed lines). Draw the following vectors on the graph below

$$\vec{u} = 4\vec{e}_1 + \vec{e}_2, \;\; \vec{v} = -\vec{e}_1 + 5\vec{e}_2$$

$$\vec{w} = 2\vec{b}_1 - \vec{b}_2, \;\; \vec{z} = 3\vec{b}_1 + 2\vec{b}_2$$

What do you notice?



P3. Consider the vector $\vec{x} = 3\vec{e}_1 + 6\vec{e}_2$. Use the graph from P2 to find real numbers x_1, x_2 so that $\vec{x} = x_1\vec{b}_1 + x_2\vec{b}_2$.

Definition 9.1. Let $\mathcal{B} = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ be an ordered basis for a vector space V. Recall that every vector $\vec{x}$ in V can be written in the form

$$\vec{x} = x_1\vec{v}_1 + \cdots + x_n\vec{v}_n.$$

The $\mathcal{B}$-COORDINATES of $\vec{x}$ is the vector in $\mathbb{R}^n$ given by

Lecture Activity 9.3. Let $\mathcal{B} = \{\vec{b}_1, \vec{b}_2, \vec{b}_3\}$ be the ordered basis for $\mathbb{R}^3$ where

$$\vec{b}_1 = \begin{pmatrix}1\\1\\0\end{pmatrix}, \vec{b}_2 = \begin{pmatrix}2\\1\\-1\end{pmatrix}, \text{ and } \vec{b}_3 = \begin{pmatrix}0\\1\\3\end{pmatrix}.$$

Find the $\mathcal{B}$-coordinates of the vector $\vec{v} = \begin{pmatrix}1\\3\\4\end{pmatrix}$.

Definition 9.4. Let $\mathcal{C}$ and $\mathcal{B}$ be bases for a vector space V. Then, the CHANGE OF BASIS matrix $M_{\mathcal{C}\leftarrow\mathcal{B}}$ is the matrix satisfying

for every vector $\vec{x}$ in V.

Lecture Activity 9.4. Consider the basis $\mathcal{B} = \{\vec{b}_1, \vec{b}_2\}$ for $\mathbb{R}^2$ where

$$\vec{b}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \text{ and } \vec{b}_2 = \begin{pmatrix} 5 \\ 3 \end{pmatrix}.$$

P1. Find $M_{\mathcal{E}\leftarrow\mathcal{B}}$.

P2. Show that $M_{\mathcal{E}\leftarrow\mathcal{B}}$ is invertible with $M_{\mathcal{B}\leftarrow\mathcal{E}} = M_{\mathcal{E}\leftarrow\mathcal{B}}^{-1}$.

P3. Use your work in P2 to find $M_{\mathcal{B}\leftarrow\mathcal{E}}$.

Lemma 9.6. Let $\mathcal{C}$ be a basis for a vector space V. Then, for any $\vec{x}, \vec{y} \in V$ and scalar $k \in \mathcal{R}$ ...

Lemma 9.7. Let V be a vector space with basis $\mathcal{C}$. Then, a subset $\{\vec{b}_1, \ldots, \vec{b}_n\}$ of V is linearly independent if and only if ...

Theorem 9.8. Let V be a vector space with basis $\mathcal{C} = \{\vec{c}_1, \ldots, \vec{c}_n\}$. Then, a subset $\mathcal{B} = \{\vec{b}_1, \ldots, \vec{b}_n\}$ of V is a basis for V if and only if $\begin{pmatrix} [\vec{b}_1]_{\mathcal{C}} & \cdots & [\vec{b}_n]_{\mathcal{C}} \end{pmatrix}$ is invertible. In this case, we have

$$M_{\mathcal{C}\leftarrow\mathcal{B}} = \begin{pmatrix} [\vec{b}_1]_{\mathcal{C}} & \cdots & [\vec{b}_n]_{\mathcal{C}} \end{pmatrix}.$$

and furthermore $= M_{\mathcal{B}\leftarrow\mathcal{C}} = M^{-1}_{\mathcal{C}\leftarrow\mathcal{B}}$.

Proof. Suppose first that $\mathcal{B}$ is a basis. For a vector $\vec{x}$ in V, let $[\vec{x}]_{\mathcal{B}} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$.

Complete the proof: Use Lemma 9.6 to show that $M_{\mathcal{C}\leftarrow\mathcal{B}} = \begin{pmatrix} [\vec{b}_1]_{\mathcal{C}} & \cdots & [\vec{b}_n]_{\mathcal{C}} \end{pmatrix}$

Now, by Lemma 9.7, we know that the columns of $M_{\mathcal{C}\leftarrow\mathcal{B}}$ are linearly independent.

Complete the proof: show that $M_{\mathcal{C}\leftarrow\mathcal{B}}$ is invertible, and that $M_{\mathcal{C}\leftarrow\mathcal{B}}^{-1} = M_{\mathcal{B}\leftarrow\mathcal{C}}$.

Conversely, suppose that the matrix $\begin{pmatrix} [\vec{b}_1]_{\mathcal{C}} & \cdots & [\vec{b}_n]_{\mathcal{C}} \end{pmatrix}$ is invertible.

Complete the proof: show that the set $\mathcal{B} = \{\vec{b}_1, \ldots, \vec{b}_n\}$ is linearly independent.

□

Lecture Activity 9.5. Let V be the plane in $\mathbb{R}^3$ spanned by

$$\mathcal{C} = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

P1. Show that $\mathcal{B}$ is also a basis for V, where

$$\mathcal{B} = \left\{ \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 4 \\ 2 \\ 3 \end{pmatrix} \right\}.$$

P2. Find the change of basis matrices $M_{\mathcal{C}\leftarrow\mathcal{B}}$ and $M_{\mathcal{B}\leftarrow\mathcal{C}}$.

Theorem 10.1. Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation and $\mathcal{B}$ be any basis for $\mathbb{R}^n$. Then, there exists a unique $n \times n$ matrix M so that $[F(\vec{x})]_\mathcal{B} = M[\vec{x}]_\mathcal{B}$. Furthermore, we have

$$M = \left([F(\vec{b}_1)]_\mathcal{B} \quad \cdots \quad [F(\vec{b}_n)]_\mathcal{B} \right).$$

Prove Theorem 10.1

Definition 10.5. Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation, and $\mathcal{B}$ be any basis for $\mathbb{R}^n$. Then, the DEFINING MATRIX OF F WITH RESPECT TO THE BASIS $\mathcal{B}$ is the matrix M so that ...

We use the notation $M = M_{F,\mathcal{B}}$.

Lecture Activity 10.1. Let $\mathcal{B} = \{\vec{b}_1, \vec{b}_2\}$ be the basis with

$$\vec{b}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \vec{b}_2 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}.$$

P1. Let $F : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation which stretches vectors in the $\vec{b}_1$ direction by 2 and leaves vectors in the $\vec{b}_2$ direction fixed. That is,

$$F(x_1\vec{b}_1 + x_2\vec{b}_2) = 2x_1\vec{b}_1 + x_2\vec{b}_2.$$

Find the defining matrix $M_{F,\mathcal{B}}$.

P2. Let $G : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation with linearly independent eigenvectors $\vec{v}$ and $\vec{w}$ with corresponding eigenvalues λ_1, λ_2. Letting $\mathcal{B} = \{\vec{v}, \vec{w}\}$, find the defining matrix $M_{G,\mathcal{B}}$.