Instructions. This packet is due on Quercus no later than **11:59pm on Monday, March 2nd**. Please complete your work directly on this packet. We will spend time together during lecture working on most or all of the activities in this packet. You are responsible for completing all portions of this packet, including lecture activities not discussed in class, and completing the definitions included in the packet. Solutions will be posted to the course website after the assignment due date.

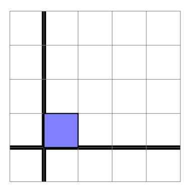
Definition 7.1. The UNIT SQUARE is the subset of $\mathbb{R}^2$ given by ...

$$S := \{\alpha_1\vec{e}_1 + \alpha_2\vec{e}_2 : 0 \leq \alpha_1, \alpha_2 \leq 1\}.$$

Lecture Activity 7.1. Let $F : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation defined by

$$F(\vec{e}_1) = 3\vec{e}_1 \text{ and } F(\vec{e}_2) = 2\vec{e}_2.$$

P1. Sketch a picture of the unit square S.



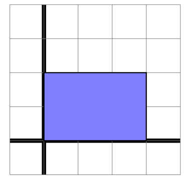
P2. Consider the set $F(S) := \{F(\vec{v}) : \vec{v} \in S\}$. Use the fact that F is linear to show that

$$F(S) = \{\alpha_1(3\vec{e}_1) + \alpha_2(2\vec{e}_2) : 0 \leq \alpha_1, \alpha_2 \leq 1\}.$$

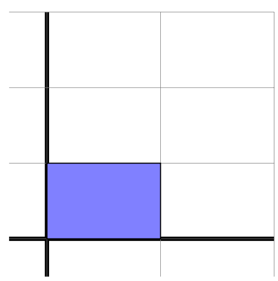
Solution. We have

$$\begin{aligned} F(V) &= \{F(\vec{v}) : \vec{v} \in V\} \\ &= \{F(\alpha_1\vec{e}_1 + \alpha_2\vec{e}_2) : 0 \leq \alpha_1, \alpha_2 \leq 1\} \\ &= \{\alpha_1 F(\vec{e}_1) + \alpha_2 F(\vec{e}_2) : 0 \leq \alpha_1, \alpha_2 \leq 1\}, \text{ since } F \text{ is linear} \\ &= \{\alpha_1(3\vec{e}_1) + \alpha_2(2\vec{e}_2) : 0 \leq \alpha_1, \alpha_2 \leq 1\}. \end{aligned}$$

P3. Use your work in P2 to sketch a picture of $F(S)$ as a subset of $\mathbb{R}^2$.



P4. Sketch the image of the “standard coordinate grid” for $\mathbb{R}^2$ under F.



Lecture Activity 7.2. Let $G : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation defined by

$$G(\vec{e}_1) = \vec{e}_1 + \vec{e}_2 \text{ and } G(\vec{e}_2) = 2\vec{e}_2.$$

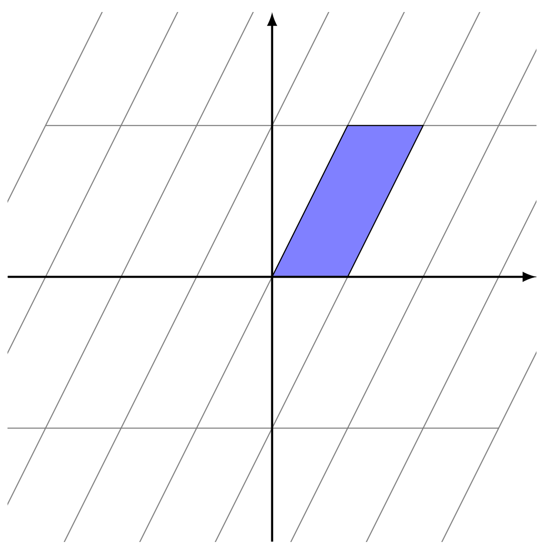
P1. Consider the set $G(S) := \{G(\vec{v}) : \vec{v} \in S\}$. Use the fact that G is linear to show that

$$G(S) = \{\alpha_1(\vec{e}_1 + \vec{e}_2) + \alpha_2(2\vec{e}_2) : 0 \leq \alpha_1, \alpha_2 \leq 1\}.$$

Solution. We have

$$\begin{aligned} G(V) &= \{G(\vec{v}) : \vec{v} \in V\} \\ &= \{G(\alpha_1\vec{e}_1 + \alpha_2\vec{e}_2) : 0 \leq \alpha_1, \alpha_2 \leq 1\} \\ &= \{\alpha_1 G(\vec{e}_1) + \alpha_2 G(\vec{e}_2) : 0 \leq \alpha_1, \alpha_2 \leq 1\}, \text{ since } G \text{ is linear} \\ &= \{\alpha_1(\vec{e}_1 + \vec{e}_2) + \alpha_2(2\vec{e}_2) : 0 \leq \alpha_1, \alpha_2 \leq 1\}. \end{aligned}$$

P2. Use your work in P1 to sketch a picture of $G(S)$ as a subset of $\mathbb{R}^2$. Then, sketch the image of the "standard coordinate grid" for $\mathbb{R}^2$ under G.



Definition 7.3. An ordered basis $\{\vec{b}_1, \vec{b}_2\}$ for $\mathbb{R}^2$ is called POSITIVELY ORIENTED if ...

we can rotate $\vec{b}_1$ counterclockwise to reach $\vec{b}_2$ without crossing the line spanned by $\vec{b}_2$.

Otherwise, the basis is called NEGATIVELY ORIENTED.

Lecture Activity 7.3. Find the orientation for the following ordered bases for $\mathbb{R}^2$.

P1. $\mathcal{B} = \{\vec{b}_1, \vec{b}_2\}$ where

$$\vec{b}_1 = \begin{pmatrix} 2 \\ 0 \end{pmatrix} \text{ and } \vec{b}_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

Solution. The set $\mathcal{B}$ is positively oriented.

P2. $\mathcal{C} = \{\vec{c}_1, \vec{c}_2\}$ where

$$\vec{c}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \text{ and } \vec{c}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Solution. The set $\mathcal{C}$ is negatively oriented.

Definition 7.4. Let $F : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation. Then, the DETERMINANT of F, denoted by $\det(F)$, is ...

the *oriented area* of $F(S)$. That is,

$$\det(F) := \begin{cases} \text{area}(F(S)) & \text{if } \{F(\vec{e}_1), F(\vec{e}_2)\} \text{ is positively oriented} \\ -\text{area}(F(S)) & \text{if } \{F(\vec{e}_1), F(\vec{e}_2)\} \text{ is negatively oriented} \\ 0 & \text{if area}(F(S)) = 0. \end{cases}$$

If A is a 2×2 matrix, the DETERMINANT OF A, denoted by $\det(A)$, is ...

the determinant of the matrix transformation T_A. That is, $\det(A) := \det(T_A)$.

Lecture Activity 7.4. Find the determinant of matrices

$$A = \begin{pmatrix} 2 & 2 \\ 0 & 1 \end{pmatrix} \text{ and } B = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

Solution. We have

$$T_A(\vec{e}_1) = \begin{pmatrix} 2 \\ 0 \end{pmatrix}, \text{ and } T_A(\vec{e}_2) = \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

Furthermore, $\{T_A(\vec{e}_1), T_A(\vec{e}_2)\}$ is positively oriented by Lecture Activity 7.3. Now the determinant of A is the area of the parallelogram with sides $T_A(\vec{e}_1)$ and $T_A(\vec{e}_2)$. Recall that the area of a parallelogram is equal to the langth of its base multiplied by the length of its height. This parallelogram has a base of length 2, and its height is 1, and so its area is 2×1. Therefore, $\det(A) = 2$.

Next, we have

$$T_B(\vec{e}_1) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \text{ and } T_B(\vec{e}_2) = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Furthermore, $\{T_A(\vec{e}_1), T_A(\vec{e}_2)\}$ is negatively oriented by Lecture Activity 7.3. Now the determinant of B is the area of the parallelogram with sides $T_B(\vec{e}_1)$ and $T_B(\vec{e}_2)$ multiplied by -1. Observe that this parallelogram is the unit square rotated 45° clockwise, and so its area remains 1. Therefore, $\det(B) = -1$.

Lecture Activity 7.5. In this activity, we develop a method to calculate the determinant of a 2×2 matrix completely algebraically. Let

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

For simplicity we'll assume that the set $\left\{ \begin{pmatrix} a \\ c \end{pmatrix}, \begin{pmatrix} b \\ d \end{pmatrix} \right\}$ is a positively oriented ordered basis with $d \neq 0$ and both vectors are in the first quadrant.

P1. Recall that the area of a parallelogram can be computed as the product of its base times its height. Use this observation to calculate the determinant of

$$\begin{pmatrix} e & b \\ 0 & d \end{pmatrix}.$$

Solution. We have

$$T_A(\vec{e}_1) = \begin{pmatrix} e \\ 0 \end{pmatrix}, \text{ and } T_A(\vec{e}_2) = \begin{pmatrix} b \\ d \end{pmatrix}.$$

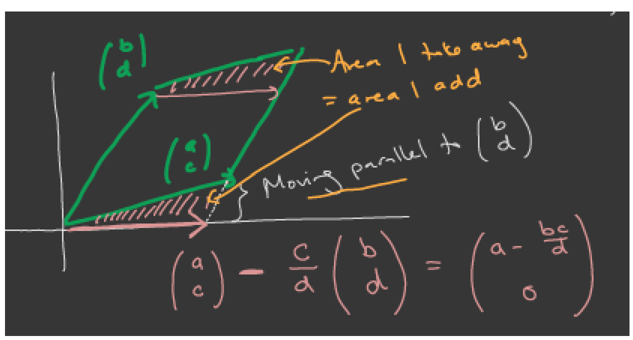
and by assumption the set $\{T_A(\vec{e}_1), T_A(\vec{e}_2)\}$ is a positively oriented basis. The two vectors $T_A(\vec{e}_1), T_A(\vec{e}_2)$ are the sides of a parallelogram with base length equal to e, and height equal to d. Therefore, $\det(A) = ed$.

P2. Show that $\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \det \begin{pmatrix} a - \frac{bc}{d} & b \\ 0 & d \end{pmatrix}$.

Solution. We have

$$T_A(\vec{e}_1) = \begin{pmatrix} a \\ c \end{pmatrix}, \text{ and } T_A(\vec{e}_2) = \begin{pmatrix} b \\ d \end{pmatrix}.$$

The result can be seen in the image below, where we demonstrates that the area of the parallelogram remains the same when we replace the first column vector by the first column vector minus a scalar multiple of the second column vector



P3. Use the previous two parts to conclude that

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc.$$

Solution. Using our work in the previous parts, we have

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \det \begin{pmatrix} a - \frac{bc}{d} & b \\ 0 & d \end{pmatrix} = \left(a - \frac{bc}{d}\right) d = ad - bc.$$

Lecture Activity 7.6. Calculate the determinant of the following functions. Given your calculation, what can be said about how the function defined by the given matrix transforms the domain space $\mathbb{R}^2$?

P1. $A = \begin{pmatrix} 2 & 4 \\ 1 & 5/2 \end{pmatrix}$

Solution. We have $\det(A) = 2 \cdot \frac{5}{2} - 4 = 1$. So, the transformation T_A preserves the orientation of $\mathbb{R}^2$ and the image of the unit square remains as 1. Observe that this tells us T_A is bijective.

P2. $B = \begin{pmatrix} 1/2 & 5 \\ 4 & 2 \end{pmatrix}$

Solution. We have $\det(B) = 2 \cdot \frac{1}{2} - 20 = -19$. So, the transformation T_A reverses the orientation and "stretches out" the domain space. Observe in this case that T_B is bijective.

P3. $C = \begin{pmatrix} 4 & 1/2 \\ 8 & 1 \end{pmatrix}$

Solution. We have $\det(C) = 4 \cdot 1 - 8 \cdot \frac{1}{2} = 0$. Since the image of the unit square is mapped to something with 0 area, the function T_C must "collapse" the domain space. Observe that this tells us T_C is not injective (and hence not surjective).

P4. $D = \begin{pmatrix} 3 & 1/2 \\ 2 & 1/2 \end{pmatrix}$

Solution. We have $\det(D) = 3 \cdot \frac{1}{2} - 2 \cdot \frac{1}{2} = \frac{1}{2}$, so T_D shrinks the domain space. Observe in this case that T_D is bijective.

Definition 7.6. The UNIT CUBE is the subset of $\mathbb{R}^3$ given by ...

$$C := \{\alpha_1\vec{e}_1 + \alpha_2\vec{e}_2 + \alpha_3\vec{e}_3 : 0 \le \alpha_1, \alpha_2, \alpha_3 \le 1\}.$$

Lecture Activity 7.7. Find the orientation of the following ordered bases for $\mathbb{R}^3$.

P1. $\mathcal{B} = \{\vec{b}_1, \vec{b}_2, \vec{b}_3\}$ where

$$\vec{b}_1 = \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix}, \vec{b}_2 = \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}, \text{ and } \vec{b}_3 = \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}.$$

Solution. The basis $\mathcal{B}$ is positively oriented because it satisfies the right-hand rule.

P2. $\mathcal{C} = \{\vec{c}_1, \vec{c}_2, \vec{c}_3\}$ where

$$\vec{c}_1 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \vec{c}_2 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \vec{c}_3 = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}.$$

Solution. The basis $\mathcal{C}$ is negatively oriented because it does not satisfy the right-hand rule.

Lecture Activity 7.8. Calculate the determinant of the matrices

$$A = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 1 \end{pmatrix}, \text{ and } B = \begin{pmatrix} 0 & 0 & 2 \\ 1 & 1 & 0 \\ 1 & -1 & 0 \end{pmatrix}$$

Solution. We have

$$T_A(\vec{e}_1) = \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix}, T_A(\vec{e}_2) = \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}, T_A(\vec{e}_3) = \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}.$$

Observe that $T_A(C)$ is a parallelepiped with height 3 and base given by the parallelogram in the yz-plane with sides $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$. Hence, the volume of $T_A(C)$ is

$$3 \left| \det \begin{pmatrix} 2 & 2 \\ 0 & 1 \end{pmatrix} \right| = 6.$$

Furthermore, the set

$$\left\{ \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} \right\}$$

is positively oriented (as we saw in Lecture Activity 7.7) and so $\det(A) = 6$.

Now, to calculate $\det(B)$, we have

$$T_B(\vec{e}_1) = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, T_B(\vec{e}_2) = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, T_B(\vec{e}_3) = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}.$$

Observe that $T_B(C)$ is a parallelepiped with height 2 and base given by the parallelogram in the yz-plane with sides $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$. Hence, the volume of $T_B(C)$ is

$$2 \left| \det \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \right| = 4.$$

Furthermore, the set

$$\left\{ \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} \right\}$$

is negatively oriented (as we saw in Lecture Activity 7.7) and so $\det(A) = -4$.

Lecture Activity 7.9. Use Proposition 7.10 to calculate $\det(A)$, where

$$A = \begin{pmatrix} 1 & 2 & -1 \\ 1 & 3 & 1 \\ 1 & -1 & -1 \end{pmatrix}.$$

Given this calculation, what can you say about the matrix transformation $T_A : \mathbb{R}^3 \to \mathbb{R}^3$?

Solution. First, we the Row Linearity Property to get

$$\det(A) = \det \begin{pmatrix} 1 & 0 & 0 \\ 1 & 3 & 1 \\ 1 & -1 & -1 \end{pmatrix} + \begin{pmatrix} 0 & 2 & 0 \\ 1 & 3 & 1 \\ 1 & -1 & -1 \end{pmatrix} + \begin{pmatrix} 0 & 0 & -1 \\ 1 & 3 & 1 \\ 1 & -1 & -1 \end{pmatrix}.$$

Next, we use the Column Swapping Property, which gives

$$\det(A) = \det \begin{pmatrix} 1 & 0 & 0 \\ 1 & 3 & 1 \\ 1 & -1 & -1 \end{pmatrix} - \begin{pmatrix} 2 & 0 & 0 \\ 3 & 1 & 1 \\ -1 & 1 & -1 \end{pmatrix} + \begin{pmatrix} -1 & 0 & 0 \\ 1 & 1 & 3 \\ -1 & 1 & -1 \end{pmatrix}.$$

Finally, we can use the Reduction Property to get

$$\begin{aligned} \det(A) &= 1 \det \begin{pmatrix} 3 & 1 \\ -1 & -1 \end{pmatrix} - 2 \det \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} + (-1) \begin{pmatrix} 1 & 3 \\ 1 & -1 \end{pmatrix} \\ &= 1 \cdot (-2) - 2 \cdot (-2) - 1 \cdot -4 \\ &= 6. \end{aligned}$$

Observe that T_A stretches the domain space, and since C is sent to a parallelepiped with nonzero volume we see that T_A is bijective.

Definition 7.12. For an $n \times n$ matrix $A = \big(a_{ij}\big)$, the ij-MINOR of A is ...

the $(n-1) \times (n-1)$ matrix A_{ij} with the ith row and jth column deleted.

Definition 7.13. Let A be the $n \times n$ matrix with ij-entry equal to a_{ij}. We define the determinant of A by the following *cofactor expansion* formula:

$$\det(A) := a_{11}\det(A_{11}) - a_{12}\det(A_{12}) + \cdots + (-1)^{n+1} a_{1n}\det(A_{1n}).$$

Lecture Activity 7.10. Find the determinant of the following matrix

$$A = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 2 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 2 \end{pmatrix}.$$

Solution. We have

$$\begin{aligned}
\det(A) &= 1 \cdot \det\begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 1 & 1 & 2 \end{pmatrix} - 0 \cdot \det\begin{pmatrix} 0 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & 1 & 2 \end{pmatrix} + 1 \cdot \det\begin{pmatrix} 0 & 2 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 2 \end{pmatrix} - 0 \cdot \det\begin{pmatrix} 0 & 2 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{pmatrix} \\
&= \det\begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 1 & 1 & 2 \end{pmatrix} + \det\begin{pmatrix} 0 & 2 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 2 \end{pmatrix} \\
&= \left(2\det\begin{pmatrix} -1 & 0 \\ 1 & 2 \end{pmatrix}\right) + \left(-2\det\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}\right) \\
&= 2 \cdot (-2) - 2 \cdot 2 \\
&= -8.
\end{aligned}$$

Proposition 7.1 (Determinant Properties)**.** Let A be an $n \times n$ matrix.

1. If $B = A^\top$ is the transpose of A, then $\det(B) =$ $\boxed{\det(A)}$

2. If B is obtained by interchanging two rows of A, then $\det(B) =$ $\boxed{-\det(A)}$

3. If B is obtained by multiplying one row of A by a constant c, then $\det(B) =$ $\boxed{c\det(A)}$

4. If B is obtained by replacing a row of A by that row and a scalar multiple of another row of A, then $\det(B) =$ $\boxed{\det(A)}$

5. If B is any $n \times n$ matrix, then $\det(AB) =$ $\boxed{\det(A)\det(B)}$

Lecture Activity 7.11. Use Proposition 7.1 to calculate the determinant of the following matrix

$$A = \begin{pmatrix} 1 & 0 & 3 & 0 \\ 1 & 0 & 2 & 0 \\ 4 & 0 & 0 & 1 \\ 1 & 2 & 3 & 0 \end{pmatrix}$$

Solution. We have

$$\begin{aligned}
\det(A) &= \det \begin{pmatrix} 1 & 1 & 4 & 1 \\ 0 & 0 & 0 & 2 \\ 3 & 2 & 0 & 3 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \text{using property (1)} \\
&= -\det \begin{pmatrix} 0 & 0 & 0 & 2 \\ 1 & 1 & 4 & 1 \\ 3 & 2 & 0 & 3 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \text{using property (2)} \\
&= 2\det \begin{pmatrix} 1 & 1 & 4 \\ 3 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\
&= -2\det \begin{pmatrix} 0 & 0 & 1 \\ 3 & 2 & 0 \\ 1 & 1 & 4 \end{pmatrix}, \text{using property (2)} \\
&= -2\det \begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix} \\
&= -2.
\end{aligned}$$