Instructions. This packet is due on Quercus no later than **11:59pm on Monday, February 16th**. Please complete your work directly on this packet. We will spend time together during lecture working on most or all of the activities in this packet. You are responsible for completing all portions of this packet, including lecture activities not discussed in class, and completing the definitions included in the packet. Solutions will be posted to the course website after the assignment due date.

Lecture Activity 6.1. Let $F : \mathbb{R}^n \to \mathbb{R}^m$ and $G : \mathbb{R}^n \to \mathbb{R}^m$ be linear transformations.

P1. Show that $F + G$ is a linear transformations.

Solution. Take any $\vec{x}, \vec{y} \in \mathbb{R}^n$. Then,

$$\begin{aligned}(F+G)(\vec{x}+\vec{y}) &= F(\vec{x}+\vec{y}) + G(\vec{x}+\vec{y}), \text{ by definition of function addition}\\ &= F(\vec{x}) + G(\vec{x}) + F(\vec{y}) + G(\vec{y}), \text{ since } F, G \text{ are linear}\\ &= (F+G)(\vec{x}) + (F+G)(\vec{y}), \text{ by definition of function addition.}\end{aligned}$$

Now, for any scalar $c \in \mathbb{R}$ we have

$$\begin{aligned}(F+G)(c\vec{x}) &= F(c\vec{x}) + G(c\vec{x}), \text{ by definition of function addition}\\ &= cF(\vec{x}) + cG(\vec{x}), \text{ since } F, G \text{ are linear}\\ &= c\left(F(\vec{x}) + G(\vec{x})\right)\\ &= c(F+G)(\vec{x}), \text{ by definition of function addition.}\end{aligned}$$

Hence, $F + G$ is a linear transformation.

P2. Let $F = T_A$ and $G = T_B$ where

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \text{ and } B = \begin{pmatrix} -1 & 3 \\ 1 & 2 \end{pmatrix}.$$

By P1, we know that $F + G$ is linear. Find the defining matrix M_{F+G}.

Solution. We have

$$\begin{aligned}(F+G)\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) &= F\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) + G\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right)\\ &= \begin{pmatrix} 1 \\ 3 \end{pmatrix} + \begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 4 \end{pmatrix}\end{aligned}$$

and

$$\begin{aligned}(F+G)\left(\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right) &= F\left(\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right) + G\left(\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right)\\ &= \begin{pmatrix} 2 \\ 4 \end{pmatrix} + \begin{pmatrix} 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 5 \\ 6 \end{pmatrix}.\end{aligned}$$

Hence, by Theorem 4.8, we have that $M_{F+G} = \begin{pmatrix} 0 & 5 \\ 4 & 6 \end{pmatrix}$.

Definition 6.1. Let $A = \begin{pmatrix} \vec{v}_1 & \cdots & \vec{v}_n \end{pmatrix}$ and $B = \begin{pmatrix} \vec{w}_1 & \cdots & \vec{w}_n \end{pmatrix}$ be $m \times n$ matrices and $c \in \mathbb{R}$ be a scalar.

1. The SUM of A and B is the $m \times n$ matrix given by ...

$$A + B := \begin{pmatrix} \vec{v}_1 + \vec{w}_1 & \cdots & \vec{v}_n + \vec{w}_n \end{pmatrix}.$$

2. The SCALAR PRODUCT of A with c is the $m \times n$ matrix given by ...

$$cA := \begin{pmatrix} c\vec{v}_1 & \cdots & c\vec{v}_n \end{pmatrix}.$$

Lecture Activity 6.2. Let $B = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$. Find the matrix A given that

$$2(A + (B + 3A)) = 7A - (B + A).$$

Solution. Using the properties in Proposition 6.2, we have

$$2B + 8A = 6A - B$$

$$\Rightarrow 2A = -\frac{3}{2}B = \begin{pmatrix} -3/2 & -3 \\ -9/2 & -6 \end{pmatrix}.$$

Lecture Activity 6.3. Let $F : \mathbb{R}^k \to \mathbb{R}^m$ and $G : \mathbb{R}^n \to \mathbb{R}^k$ be linear transformations with defining matrices

$$A = M_F \text{ and } B = M_G.$$

Recall from Chapter Exercise 4.3 that the composition of linear functions is linear. Show that the defining matrix $M = M_{F \circ G}$ of the composition $F \circ G$ is given by

$$\begin{pmatrix} A\vec{b}_1 & A\vec{b}_2 & \cdots & A\vec{b}_m \end{pmatrix}$$

where $B = \begin{pmatrix} \vec{b}_1 & \vec{b}_2 & \cdots & \vec{v}_n \end{pmatrix}$.

Solution. Taken any $\vec{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^m$. Then,

$$\begin{aligned} (F \circ G)(\vec{x}) - A(B\vec{x}) & \\ &= A(x_1\vec{b}_1 + x_2\vec{b}_2 + \cdots + x_n\vec{b}_n) \\ &= x_1A\vec{b}_1 + x_2A\vec{b}_2 + \cdots + x_nA\vec{b}_n \\ &= \begin{pmatrix} A\vec{b}_1 & A\vec{b}_2 & \cdots & A\vec{b}_n \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \\ &= \begin{pmatrix} A\vec{b}_1 & A\vec{b}_2 & \cdots & A\vec{b}_n \end{pmatrix} \vec{x}. \end{aligned}$$

So, $M_{F \circ G} = \begin{pmatrix} A\vec{b}_1 & A\vec{b}_2 & \cdots & A\vec{b}_n \end{pmatrix}$.

Definition 6.3. Let A be an $m \times k$ matrix and $B = \begin{pmatrix} \vec{b}_1 & \cdots & \vec{b}_n \end{pmatrix}$ be a $k \times n$ matrix. Then, the MATRIX PRODUCT of A and B is the $m \times n$ matrix …

$$AB = \begin{pmatrix} A\vec{b}_1 & \cdots & A\vec{b}_n \end{pmatrix}.$$

As we saw in Lecture Activity 6.3, note that $T_{AB} = T_A \circ T_B$.

Lecture Activity 6.4. Let

$$A = \begin{pmatrix} 1 & 0 & 1 & -1 \\ 0 & 2 & 3 & 1 \end{pmatrix}, \ B = \begin{pmatrix} 0 & -1 \\ 3 & 0 \\ 1 & 0 \\ -1 & 4 \end{pmatrix} \text{ and } C = \begin{pmatrix} 1 & 1 & 2 & 0 \\ 0 & 1 & 4 & 1 \\ 0 & 0 & -2 & 1 \end{pmatrix}.$$

Calculate all possible matrix products AB, BA, AC, CA, BC, CB. If a matrix product is not defined, explain why not.

Solution. We have

$$AB = \begin{pmatrix} 2 & -5 \\ 8 & 4 \end{pmatrix}, BA = \begin{pmatrix} 0 & -2 & -3 & -1 \\ 3 & 0 & 3 & -3 \\ 1 & 0 & 1 & -1 \\ -1 & 8 & 11 & 5 \end{pmatrix}, CB = \begin{pmatrix} 5 & -1 \\ 6 & 4 \\ -3 & 4 \end{pmatrix}$$

Note that AC, CA, and BC are undefined, since they have incompatible dimensions.

Lecture Activity 6.5. Let $F : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation, and suppose that the inverse function $F^{-1} : \mathbb{R}^m \to \mathbb{R}^n$ is known to exist.

P1. Use Lecture Activity 4.9 to show that $m = n$.

Solution. Recall that a function is invertible if and only if it's bijective. Furthermore, in Lecture Activity 4.9 we showed that a linear transformation $F : \mathbb{R}^n \to \mathbb{R}^m$ is invertible if and only if $m = n$. Since F^{-1} is supposed to exist, we must have $m = n$.

P2. Show that F^{-1} is a linear transformation.

Solution. Take any $\vec{y}, \vec{z} \in \mathbb{R}^m$. Since F is invertible, it must be surjective. So, there exists vectors $\vec{u}, \vec{v} \in \mathbb{R}^n$ so that $F(\vec{u}) = \vec{y}$ and $F(\vec{v}) = \vec{z}$. So, we have

$$\begin{aligned} F^{-1}(\vec{y} + \vec{z}) &= F^{-1}(F(\vec{u}) + F(\vec{v})) \\ &= F^{-1}(F(\vec{u} + \vec{v})), && \text{since } F \text{ is linear} \\ &= \vec{u} + \vec{v}, && \text{since } F^{-1} \text{ is the inverse of } F \\ &= F^{-1}(\vec{y}) + F^{-1}(\vec{z}), \end{aligned}$$

where the final equality follows because $F(\vec{u}) = \vec{y} \Rightarrow F^{-1}(\vec{y}) = \vec{u}$ and $F(\vec{v}) = \vec{z} \Rightarrow F^{-1}(\vec{z}) = \vec{v}$. Similarly, for any $c \in \mathbb{R}$ we have

$$\begin{aligned} F^{-1}(c\vec{y}) &= F^{-1}(cF(\vec{u})) \\ &= F^{-1}(F(c\vec{u})) \\ &= c\vec{u} \\ &= cF^{-1}(\vec{y}), \end{aligned}$$

as needed.

P3. Suppose that F has defining matrix $M_F = A$ and F^{-1} has defining matrix $M_{F^{-1}} = B$. What matrix does AB need to be equal to? What about BA?

Solution. Let $\vec{e}_1, \ldots, \vec{e}_n$ denote the standard basis for $\mathbb{R}^n$. By definition of the matrix inverse, we have

$$(F \circ F^{-1})(\vec{e}_i) = (F^{-1} \circ F)(\vec{e}_i) = \vec{e}_i,$$

for all $i = 1, \ldots, n$. So, by Theorem 4.8 (and recalling that $M_F = A$ and $M_{F^{-1}} = B$) we must have

$$AB = BA = \begin{pmatrix} \vec{e}_1 & \cdots & \vec{e}_n \end{pmatrix}.$$

Definition 6.6. The IDENTITY MATRIX I_n is ...

> the $n \times n$ matrix $I_n = \begin{pmatrix} \vec{e}_1 & \cdots & \vec{e}_n \end{pmatrix}$. That is,
>
> $$I_n = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}.$$

Definition 6.7. Let A be an $n \times n$ matrix. Then INVERSE OF A, if it exists, is ...

> the matrix B so that $AB = BA = I_n$. In this case, we write $B = A^{-1}$.

Lecture Activity 6.6. Use Theorem 6.11 to find the inverse of $A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & -1 \end{pmatrix}$.

Solution. We have

$$\begin{aligned}
\left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 4 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 \end{array}\right) &\sim \left(\begin{array}{ccc|ccc} 1 & 0 & -5 & 1 & -2 & 0 \\ 0 & 1 & 4 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 \end{array}\right) \text{ via } R_1 - 2R_2 \\
&\sim \left(\begin{array}{ccc|ccc} 1 & 0 & -5 & 1 & -2 & 0 \\ 0 & 1 & 4 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 \end{array}\right) \text{ via } (-1)R_3 \\
&\sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -2 & -5 \\ 0 & 1 & 4 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 \end{array}\right) \text{ via } R_1 + 5R_3 \\
&\sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -2 & -5 \\ 0 & 1 & 0 & 0 & 1 & 4 \\ 0 & 0 & 1 & 0 & 0 & -1 \end{array}\right) \text{ via } R_2 - 4R_3
\end{aligned}$$

Hence, by Theorem 6.10 we have $A^{-1} = \begin{pmatrix} 1 & -2 & -5 \\ 0 & 1 & 4 \\ 0 & 0 & -1 \end{pmatrix}$. Indeed, we can check that

$$AA^{-1} = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & -2 & -5 \\ 0 & 1 & 4 \\ 0 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \checkmark$$

Definition 6.12. An $n \times n$ matrix is called ELEMENTARY if ...

it can be obtained by performing exactly one row operation to the identity matrix.

Lecture Activity 6.7. Consider the matrix

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}.$$

P1. Let E_1 be the elementary matrix obtained by performing the row operation $R_1 \leftrightarrow R_3$ to I_3. Find E_1, and then calculate E_1A. What do you notice?

Solution. We have

$$E_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \text{ and } E_1A = \begin{pmatrix} 7 & 8 & 9 \\ 4 & 5 & 6 \\ 1 & 2 & 3 \end{pmatrix}.$$

Observe that E_1A is equal to the matrix obtained by performing the row operation $R_1 \leftrightarrow R_3$ to A.

P2. Let E_2 be the elementary matrix obtained by performing the row operation $5R_2$ to I_3. Find E_2 and then calculate E_2A. What do you notice?

Solution. We have

$$E_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ and } E_2A = \begin{pmatrix} 1 & 2 & 3 \\ 20 & 25 & 30 \\ 7 & 8 & 9 \end{pmatrix}.$$

Observe that E_2A is equal to the matrix obtained by performing the row operation $5R_2$ to A.

P3. Let E_3 be the elementary matrix obtained by performing the row operation $R_1 + 2R_2$ to I_3. Find E_3 and then calculate E_3A. What do you notice?

Solution. We have

$$E_3 = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ and } E_3A = \begin{pmatrix} 9 & 12 & 15 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}.$$

Observe that E_3A is equal to the matrix obtained by performing the row operation $R_1 + 2R_2$ to A.

Lecture Activity 6.8. Let's see how we can capture Gauss-Jordan via matrix products. Consider the matrix

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & -1 \end{pmatrix}$$

from Lecture Activity 6.6.

P1. Use your work in Lecture Activity 6.6 to find elementary matrices $E_1, \dots, E_k$ so that

$$E_1 \cdots E_k A = I_3.$$

Solution. Using our row operations from Lecture Activity 6.6, we obtain

$$\underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -4 \\ 0 & 0 & 1 \end{pmatrix}}_{E_1} \underbrace{\begin{pmatrix} 1 & 0 & 5 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{E_2} \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}}_{E_3} \underbrace{\begin{pmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{E_4} \underbrace{\begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & -1 \end{pmatrix}}_{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

P2. Reverse your work from Lecture Activity 6.7 to show that $I_3 \sim A$. That is, find a sequence of elementary operations to perform to I_3 to obtain A.

Solution. Working backwards, we have

$$\begin{aligned} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} &\sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{pmatrix}, \text{ via } R_2 + 4R_3 \\ &\sim \begin{pmatrix} 1 & 0 & -5 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{pmatrix}, \text{ via } R_1 - 5R_3 \\ &\sim \begin{pmatrix} 1 & 0 & -5 \\ 0 & 1 & 4 \\ 0 & 0 & -1 \end{pmatrix}, \text{ via } (-1)R_3 \\ &\sim \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & -1 \end{pmatrix}, \text{ via } R_1 + 2R_2 \end{aligned}$$

P3. Find elementary matrices $\tilde{E}_1, \dots, \tilde{E}_k$ so that $\tilde{E}_k \cdots \tilde{E}_1 I_3 = A$. Conclude that A is a product of elementary matrices.

Solution. Using our row operations above, we obtain

$$\underbrace{\begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{\tilde{E}_4} \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}}_{\tilde{E}_3} \underbrace{\begin{pmatrix} 1 & 0 & -5 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{\tilde{E}_2} \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{pmatrix}}_{\tilde{E}_1} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & -1 \end{pmatrix}}_{A}.$$

Hence, $A = \tilde{E}_4 \tilde{E}_3 \tilde{E}_2 \tilde{E}_1$.

Observe that, in the activity above, $\tilde{E}_i = E_i^{-1}$. Another approach would have been to apply inverses on the left-hand side of our equation in P1 and observe that, to obtain the inverse of an elementary matrix, we perform the “reverse” operation to the identity matrix.