CHAPTER 5 ACTIVITY PACKET *solutions*

Instructions. This packet is due on Quercus no later than **11:59pm on Monday, February 9th**. Please complete your work directly on this packet. We will spend time together during lecture working on most or all of the activities in this packet. You are responsible for completing all portions of this packet, including lecture activities not discussed in class, and completing the definitions included in the packet. Solutions will be posted to the course website after the assignment due date.

Definition 4.18. A function $f : X \to Y$ is called BIJECTIVE if ...

> f is both injective and surjective.

Lecture Activity 4.9. Show that a linear tranformation $F : \mathbb{R}^n \to \mathbb{R}^m$ can be bijective if and only if $n = m$.

Proof. Let $M = M_F$ be the defining matrix for F, and note that M is an $m \times n$ matrix. By Theorem 4.14 and Theorem 4.17, $\mathrm{rref}(M)$ must have a pivot in every row and every column. Since a matrix can have at most one pivot in every row, and a matrix in reduced row echelon form can have at most one pivot in every column, we see that $m = n$. □

Definition 4.19. Let V be a subspace of $\mathbb{R}^n$ and W a subspace of $\mathbb{R}^m$. An ISOMORPHISM between V and W is ...

> any linear bijective map $F : V \to W$.

If an isomorphism exists between two vector spaces, we say these spaces are ISOMORPHIC, and we write $V \cong W$.

Definition 5.2. Let $F : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation.

1. The KERNEL of F is the subset $\ker(F) \subseteq \mathbb{R}^n$ defined by

$$\ker(F) := \{\vec{x} \in \mathbb{R}^n \mid F(\vec{x}) = \vec{0}\}.$$

2. The IMAGE of F is the subset $\operatorname{im}(F) \subseteq \mathbb{R}^m$ defined by

$$\operatorname{im}(F) := \{F(\vec{x}) \mid \vec{x} \in \mathbb{R}^n\}.$$

Lecture Activity 5.1. Let $F = T_C$ where C is our matrix from Lecture Activity 4.3

$$C = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

P1. Find a vector $\vec{x} \in \ker(F)$.

Solution. We need to find a solution to the matrix-vector equation

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

which has solution $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$, for example, and so $\begin{pmatrix} 1 \\ -1 \end{pmatrix} \in \ker(F)$.

P2. Find a vector $\vec{y} \in \operatorname{im}(F)$.

Solution. We have

$$F\left(\begin{pmatrix} 1 \\ 2 \end{pmatrix}\right) = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \end{pmatrix},$$

and so $\begin{pmatrix} 3 \\ 3 \end{pmatrix} \in \operatorname{im}(F)$.

P3. Find a vector $\vec{v}$ so that $\ker(F) = \text{Span}(\vec{v})$. Conclude that $\ker(F)$ is a vector space.

Note that $\ker(F)$ consists of all solutions to the matrix-vector equation

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

which has the same solution set as the system of linear equations with augmented matrix

$$\left(\begin{array}{cc|c} 1 & 1 & 0 \\ 1 & 1 & 0 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right).$$

Hence, solving for our basic variable x in terms of our free variable y, we get

$$\begin{aligned} \ker(F) &= \left\{ \begin{pmatrix} -y \\ y \end{pmatrix} : y \in \mathbb{R} \right\} \\ &= \left\{ y \begin{pmatrix} -1 \\ 1 \end{pmatrix} : y \in \mathbb{R} \right\} \\ &= \text{Span}\left(\begin{pmatrix} -1 \\ 1 \end{pmatrix} \right). \end{aligned}$$

Since we've written $\ker(F)$ as a span, $\ker(F)$ is a vector space by Proposition 3.2.

P4. Find a vector $\vec{w}$ so that $\text{im}(F) = \text{Span}(\vec{w})$. Conclude that $\text{im}(F)$ is a vector space.

We have

$$\begin{aligned} \text{im}(F) &= \left\{ F\left(\begin{pmatrix} x \\ y \end{pmatrix} \right) : \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \right\}, \text{ by definition of im}(F) \\ &= \left\{ \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} : x, y \in \mathbb{R} \right\}, \text{ since } F = T_C \\ &= \left\{ x \begin{pmatrix} 1 \\ 1 \end{pmatrix} + y \begin{pmatrix} 1 \\ 1 \end{pmatrix} : x, y \in \mathbb{R} \right\}, \text{ by definition of the matrix-vector product} \\ &= \text{Span}\left(\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right), \text{ by definition of span} \\ &= \text{Span}\left(\begin{pmatrix} 1 \\ 1 \end{pmatrix} \right), \text{ deleting the redundant vector.} \end{aligned}$$

Since we've written $\text{im}(F)$ as a span, $\text{im}(F)$ is a vector space by Proposition 3.2.

Definition 5.4. Let $F : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation.

1. The RANK of F is ...

the dimension of $\operatorname{im}(F)$,

   and is denoted by $\operatorname{rank}(F)$.

2. The NULLITY of F is ...

the dimension of $\ker(F)$,

   and is denoted by $\operatorname{nullity}(F)$.

Lecture Activity 5.2. Let $F : \mathbb{R}^3 \to \mathbb{R}^2$ be given by

$$F\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = \begin{pmatrix} x+y \\ x+z \end{pmatrix}.$$

P1. Calculate $\operatorname{rank}(F)$.

Solution. Observe that F has defining matrix

$$M_F = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix},$$

and so working similarly to Lecture Activity 5.1, we have

$$\begin{aligned}
\operatorname{im}(F) &= \left\{ F\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) : \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \right\}, \text{ by definition of } \operatorname{im}(F) \\
&= \left\{ \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} : x, y, z \in \mathbb{R} \right\}, \text{ since } M_F \text{ is the defining matrix of } F \\
&= \left\{ x \begin{pmatrix} 1 \\ 1 \end{pmatrix} + y \begin{pmatrix} 1 \\ 0 \end{pmatrix} + z \begin{pmatrix} 0 \\ 1 \end{pmatrix} : x, y, z \in \mathbb{R} \right\}, \text{ by definition of the matrix-vector product} \\
&= \operatorname{Span}\left(\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right). \text{ by definition of span.}
\end{aligned}$$

This gives a generating set for $\operatorname{im}(F)$, and so now we can use our strategy from Section 3.3 to find a basis. Since

$$\operatorname{rref}(M_F) = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \end{pmatrix}$$

has two pivots, then by Theorem 3.11 we know that $\operatorname{rank}(F) = \dim \operatorname{im}(F) = 2$.

P2. Calculate nullity(F).

Solution. Since $M_F = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$, then working similarly to Lecture Activity 5.1, we need to find all solutions to the matrix-vector equation

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

which has the same solution set as the system of linear equations with augmented matrix

$$\left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{array}\right)$$

which has reduced row echelon form

$$\left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \end{array}\right).$$

So, our solutions must satisfy $x + z = 0$ and $y - z = 0$. Parameterizing our basic variables x, y in terms of our free variable z gives

$$\begin{aligned} \ker(F) &= \left\{ \begin{pmatrix} -z \\ z \\ z \end{pmatrix} : z \in \mathbb{R} \right\} \\ & \left\{ z \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} : z \in \mathbb{R} \right\} \\ &= \operatorname{Span}\left(\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} \right). \end{aligned}$$

Hence, we found a generating set for $\ker(F)$ with a single element, and so this set must be a basis (noting that single element sets are always linearly independent). Thus, nullity$(F) = \dim \ker(F) = 1$.

Definition 5.5. Let A be an $m \times n$ matrix with column vectors $A = \begin{pmatrix} \vec{v}_1 & \cdots & \vec{v}_n \end{pmatrix}$. Then, the COLUMN SPACE of A is the subspace of $\mathbb{R}^m$ given by ...

$$\operatorname{Col}(A) := \operatorname{Span}(\vec{v}_1, \ldots, \vec{v}_n).$$

The NULL SPACE of A is the subspace of $\mathbb{R}^n$ given by

$$\operatorname{Nul}(A) := \{\vec{x} \in \mathbb{R}^m \mid A\vec{x} = \vec{0}\}.$$

Proposition 5.6. Let $F : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation with defining matrix M_F. Then, $\ker(F) = \operatorname{Nul}(M_F)$ and $\operatorname{im}(F) = \operatorname{Col}(M_F)$.

Proof. Let F have defining matrix $M = M_F$.

Complete the proof: show that $\ker(F) = \operatorname{Nul}(M)$

We have

$$\begin{aligned}
\ker(F) &= \left\{\vec{x} \in \mathbb{R}^n : F(\vec{x}) = \vec{0}\right\}, \text{ by definition of } \ker(F) \\
&= \left\{\vec{x} \in \mathbb{R}^n : M\vec{x} = \vec{0}\right\}, \text{ since } M \text{ is the defining matrix of } F \\
&= \operatorname{Nul}(M), \text{ by definition of } \operatorname{Nul}(M).
\end{aligned}$$

Next, suppose that M has column vectors $M = \begin{pmatrix} \vec{v}_1 & \cdots & \vec{v}_n \end{pmatrix}$.

Complete the proof: show that $\operatorname{im}(F) = \operatorname{Col}(M)$

$$
\begin{aligned}
\operatorname{im}(F) &= \left\{ F\left(\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}\right) : \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n \right\}, \text{ by definition of } \operatorname{im}(F) \\
&= \left\{ M \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} : x_1, \ldots, x_n \in \mathbb{R} \right\}, \text{ since } M \text{ is the defining matrix of } F \\
&= \left\{ \begin{pmatrix} \vec{v}_1 & \cdots & \vec{v}_n \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} : x_1, \ldots, x_n \in \mathbb{R} \right\}, \text{ since } M = \begin{pmatrix} \vec{v}_1 & \cdots & \vec{v}_n \end{pmatrix} \\
&= \{ x_1 \vec{v}_1 + \cdots + x_n \vec{v}_n : x_1, \ldots, x_n \in \mathbb{R} \}, \text{ by definition of the matrix-vector product} \\
&= \operatorname{Span}(\vec{v}_1, \ldots, \vec{v}_n), \text{ by definition of span} \\
&= \operatorname{Col}(M), \text{ by definition of the column space.}
\end{aligned}
$$

□

Definition 5.7. Let A be a matrix.

1. The NULLITY of A is ...

   the dimension of $\operatorname{Nul}(A)$,

   and is denoted by $\operatorname{nullity}(A)$.

2. The RANK of A is ...

   the dimension of $\operatorname{Col}(A)$,

   and is denoted by $\operatorname{rank}(A)$.

Lecture Activity 5.3. Calculate the rank and nullity of

$$A = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 2 & 1 & 1 \end{pmatrix}$$

Solution. We have that $\operatorname{rref}(A) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, and so the column vectors of A are linearly independent, and hence form a basis for $\operatorname{Col}(A)$. Hence, $\operatorname{rank}(A) = \dim \operatorname{Col}(A) = 3$. To calculate the nullity, note that the matrix-vector equation $A\vec{x} = \vec{0}$ has the same solution set as the system of linear equations with augmented matrix

$$\left(\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 2 & 1 & 1 & 0 \end{array}\right)$$

which has reduced row echelon form

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right).$$

Hence, the only solution to $A\vec{x} = \vec{0}$ is $\vec{x} = \vec{0}$ and so $\operatorname{Nul}(A) = \{\vec{0}\}$ is the trivial space. Hence, $\operatorname{nullity}(A) = \dim(\operatorname{Nul}(A)) = 0$.

Lecture Activity 5.4. Let A and B be $3 \times n$ matrices.

P1. Suppose that A has exactly two pivot columns, which are located in columns 1 and 3. Show that $\text{rank}(A) = 2$ and $\text{nullity}(A) = 1$.

Solution. Let $A = \begin{pmatrix} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \end{pmatrix}$. Since $\text{rref}(A)$ has a pivot in the first and third column, then by Theorem 3.11 we know that $\{\vec{v}_1, \vec{v}_3\}$ forms a basis for $\text{Col}(A)$. Hence, $\text{rank}(A) = \dim(\text{Col}(A)) = 2$. Next, note that

$$\begin{aligned} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \text{Nul}(A) &\Leftrightarrow \begin{pmatrix} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \\ &\Leftrightarrow x\vec{v}_1 + y\vec{v}_2 + z\vec{v}_3 = \vec{0} \end{aligned}$$

Since the second column of A does not have a pivot, we know that the variable y is free in the vector equation above. So, we can parameterize our basic variables x and z in terms of our free variable y, say that $x = ay$ and $z = by$ for some $a, b \in R$. Then

$$\begin{aligned} \text{Nul}(A) &= \left\{ \begin{pmatrix} ay \\ y \\ by \end{pmatrix} : y \in \mathbb{R} \right\} \\ &= \left\{ y \begin{pmatrix} a \\ 1 \\ b \end{pmatrix} : y \in \mathbb{R} \right\} \\ &= \text{Span}\left(\begin{pmatrix} a \\ 1 \\ b \end{pmatrix} \right). \end{aligned}$$

Since we found a generating set for $\text{Nul}(A)$ with a single element, this set must be a basis (noting that any set containing a single element is linearly independent). So, $\text{nullity}(A) = \dim(\text{Nul}(A)) = 1$.

P2. Suppose that B has exactly one pivot column, which is located in column 1. Show that $\text{rank}(B) = 1$ and $\text{nullity}(B) = 2$.

Solution. Let $B = \begin{pmatrix} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \end{pmatrix}$. Since $\text{rref}(B)$ has a pivot in the first column, then by Theorem 3.11 we know that $\{\vec{v}_1\}$ forms a basis for $\text{Col}(B)$. Hence, $\text{rank}(B) = \dim(\text{Col}(B)) = 1$. Next, note that

$$\begin{aligned} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \text{Nul}(A) &\Leftrightarrow \begin{pmatrix} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \\ &\Leftrightarrow x\vec{v}_1 + y\vec{v}_2 + z\vec{v}_3 = \vec{0} \end{aligned}$$

Since the second and third columns of A do not have pivots, we know that the variables y and z are free in the vector equation above. So, we can parameterize our basic variable x in terms of our free variables y and z, say that $x = ay + bz$ for some $a, b \in R$. Then

$$\begin{aligned} \text{Nul}(B) &= \left\{ \begin{pmatrix} ay + bz \\ y \\ z \end{pmatrix} : y, z \in \mathbb{R} \right\} \\ &= \left\{ \begin{pmatrix} ay \\ y \\ 0 \end{pmatrix} + \begin{pmatrix} bz \\ 0 \\ z \end{pmatrix} : y, z \in \mathbb{R} \right\} \\ &= \left\{ y \begin{pmatrix} a \\ 1 \\ 0 \end{pmatrix} + z \begin{pmatrix} b \\ 0 \\ 1 \end{pmatrix} : y, z \in \mathbb{R} \right\} \\ &= \text{Span}\left(\begin{pmatrix} a \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} b \\ 0 \\ 1 \end{pmatrix} \right). \end{aligned}$$

Note that

$$\begin{pmatrix} a & b \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$$

and so the set

$$\left\{ \begin{pmatrix} a \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} b \\ 0 \\ 1 \end{pmatrix} \right\}$$

is linearly independent by Proposition 2.12. Hence, this set forms a basis, and so $\text{nullity}(B) = \dim(\text{Nul}(B)) = 2$.

Definition 5.10. A system of linear equations is called HOMOGENEOUS if ...

the constant coefficients are all equal to zero.

Lecture Activity 5.5. Consider the system of linear equations

$$\begin{cases} x + 2y + 4z = 0 \\ x + y - z = 0 \\ y + 5z = 0 \end{cases}$$

P1. Find a matrix C so that the vector form of the solution set to this system is equal to $\mathrm{Nul}(C)$.

Solution. This system of linear equations has the same solution set as the matrix-vector equation

$$\begin{pmatrix} 1 & 2 & 4 \\ 1 & 1 & -1 \\ 0 & 1 & 5 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

So, by definition, this system has solution set equal to $\mathrm{Nul}(C)$ where

$$C = \begin{pmatrix} 1 & 2 & 4 \\ 1 & 1 & -1 \\ 0 & 1 & 5 \end{pmatrix}$$

P2. Calculate $\mathrm{nullity}(C)$.

Solution. We have

$$\mathrm{rref}(C) = \begin{pmatrix} 1 & 0 & -6 \\ 0 & 1 & 5 \\ 0 & 0 & 0 \end{pmatrix},$$

and so by Theorem 5.8, $\mathrm{nullity}(C) = 1$.

P3. Recall from Section 1.7 that the solution set to this system is equal to the set of intersection points of planes in $\mathbb{R}^3$. Given your work in the previous parts, do these planes intersect at a point, a line, or a plane in $\mathbb{R}^3$?

Solution. Since the solution set is equal to $\mathrm{Nul}(C)$ by P1, which is a one-dimensional space by P2, these planes intersect at a line.

Lecture Activity 5.6. Determine whether the solution set for each of the following systems is empty, a point, a line, or a plane in $\mathbb{R}^3$.

P1. $\begin{cases} x + 2y + 4z = 1 \\ x + y - z = 2 \\ y + 5z = -1 \end{cases}$

Solution. We have

$$\operatorname{rref}\left(\begin{array}{ccc|c} 1 & 2 & 4 & 1 \\ 1 & 1 & -1 & 2 \\ 0 & 1 & 5 & -1 \end{array}\right) = \left(\begin{array}{ccc|c} 1 & 0 & -6 & 3 \\ 0 & 1 & 5 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

and so the system is consistent. Furthermore, if we let C denote the coefficient matrix, we see that nullity(C) is 1-dimensional. By Theorem 5.12 we know that the solution set is equal to $\vec{p} + \text{Nul}(C)$ for some fixed vector $\vec{p}$, and so the solution set describes a **line** in $\mathbb{R}^3$.

P2. $\begin{cases} x + 2y + 2z = 5 \\ x + y + z = 0 \\ 3x + 3z = 1 \end{cases}$

Solution. We have

$$\operatorname{rref}\left(\begin{array}{ccc|c} 1 & 2 & 2 & 5 \\ 1 & 1 & 1 & 0 \\ 3 & 0 & 3 & 1 \end{array}\right) = \left(\begin{array}{ccc|c} 1 & 0 & 0 & -5 \\ 0 & 1 & 0 & -\frac{1}{3} \\ 0 & 0 & 1 & \frac{16}{3} \end{array}\right).$$

and so the system is consistent. Furthermore, if we let C denote the coefficient matrix, we see that nullity(C) is 0-dimensional. By Theorem 5.12 we know that the solution set is equal to $\vec{p} + \text{Nul}(C)$ for some fixed vector $\vec{p}$, and so the solution set describes a **point** in $\mathbb{R}^3$.

P3. $\begin{cases} x + 2y + 4z = 1 \\ x + y - z = 2 \\ y + 5z = 1 \end{cases}$

Solution. We have

$$\operatorname{rref}\left(\begin{array}{ccc|c} 1 & 2 & 4 & 1 \\ 1 & 1 & -1 & 2 \\ 0 & 1 & 5 & 1 \end{array}\right) = \left(\begin{array}{ccc|c} 1 & 0 & -6 & 0 \\ 0 & 1 & 5 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right),$$

which has a pivot in the last column, and so this system is inconsistent. Hence, the solution set is **empty**.