Instructions. This packet is due on Quercus no later than **11:59pm on Monday, February 2nd**. Please complete your work directly on this packet. We will spend time together during lecture working on most or all of the activities in this packet. You are responsible for completing all portions of this packet, including lecture activities not discussed in class, and completing the definitions included in the packet. Solutions will be posted to the course website after the assignment due date.

Definition 4.1. Let A be an $m \times n$ matrix with column vectors $A = \begin{pmatrix} \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \end{pmatrix}$. Then, for a vector $\vec{x}$ in $\mathbb{R}^n$ the MATRIX-VECTOR PRODUCT of A and $\vec{x}$ is the vector in $\mathbb{R}^m$ defined by ...

$$A\vec{x} := x_1\vec{v}_1 + x_2\vec{v}_2 + \cdots + x_n\vec{v}_n.$$

Lecture Activity 4.1. Consider the matrices

$$A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \\ -1 & 1 \end{pmatrix} \text{ and } B = \begin{pmatrix} 3 & -1 & 1 \\ 2 & 0 & 1 \end{pmatrix}.$$

P1. Calculate the matrix-vector product $A\vec{x}$ where $\vec{x} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$.

Solution. We have

$$A\vec{x} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} = 2\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + 1\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} = \boxed{\begin{pmatrix} 4 \\ 1 \\ -1 \end{pmatrix}}$$

P2. Calculate the matrix-vector product $B\vec{y}$ where $\vec{y} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$.

Solution. We have

$$B\vec{y} = \begin{pmatrix} 3 & -1 & 1 \\ 2 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = 1\begin{pmatrix} 3 \\ 2 \end{pmatrix} + 2\begin{pmatrix} -1 \\ 0 \end{pmatrix} + 3\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \boxed{\begin{pmatrix} 4 \\ 5 \end{pmatrix}}$$

P3. Let $\vec{x}$ and $\vec{y}$ be as in the previous problems. Explain why the matrix-vector products $A\vec{y}$ and $B\vec{x}$ are not defined.

Solution. Note that Definition 4.1 requires we multiply each component of $\vec{y}$ by one of the columns of A. Since $\vec{y}$ has three components but A only has two columns, this is not possible. Similarly, observe that Definition 4.1 requires we multiply each column of B by one of the components of $\vec{x}$. Since B has three columns but $\vec{x}$ has only two components, this is not possible.

P4. Let $\vec{z}$ be a vector in $\mathbb{R}^2$. How many components does the vector $A\vec{z}$ have?

Solution. Let $\vec{z} = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$. Then we have

$$A\vec{z} = z_1 \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + z_2 \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix},$$

which is a linear combination of vectors in $\mathbb{R}^3$. Hence, $A\vec{z}$ is a vector in $\mathbb{R}^3$.

P5. Let $\vec{w}$ be a vector in $\mathbb{R}^3$. How many components does the vector $B\vec{w}$ have?

Solution. Let $\vec{w} = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}$. Then we have

$$B\vec{w} = w_1 \begin{pmatrix} 3 \\ 2 \end{pmatrix} + w_2 \begin{pmatrix} -1 \\ 0 \end{pmatrix} + w_3 \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

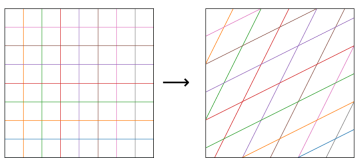
which is a linear combination of vectors in $\mathbb{R}^2$. Hence, $B\vec{w}$ is a vector in $\mathbb{R}^2$.

Definition 4.2. Let A be an $m \times n$ matrix. Then, the MATRIX TRANSFORMATION associated to A is the function ...

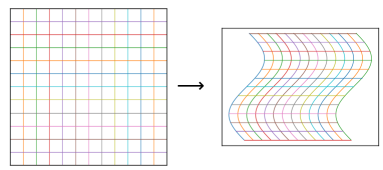
$T_A : \mathbb{R}^n \to \mathbb{R}^m$ defined by $T_A(\vec{x}) := A\vec{x}$.

Lecture Activity 4.2. In the images below, we've plotted where the indicated function sends the standard coordinate grid for $\mathbb{R}^2$. What do you notice?

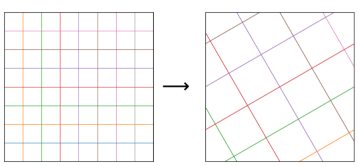
T_A where $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$



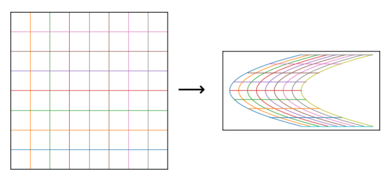
$F\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} x + \sin(y) \\ y \end{pmatrix}$



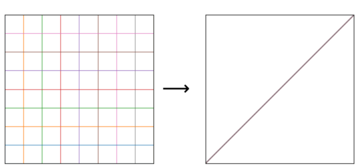
T_B where $B = \begin{pmatrix} \sqrt{2} & -\sqrt{2} \\ \sqrt{2} & \sqrt{2} \end{pmatrix}$



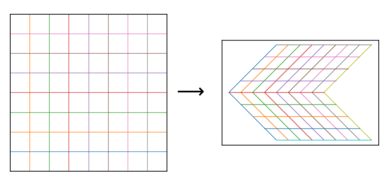
$G\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} x + y^2 \\ y \end{pmatrix}$



T_C where $C = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$



$H\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} x + |y| \\ y \end{pmatrix}$



Solution. Observe that the functions on the left are all matrix transformations, and the lines from the standard coordinate grids remain as lines under our transformation. The functions on the right transforms the lines of the standard coordinate grid into curved lines or kinked lines.

Definition 4.4. A function $F : \mathbb{R}^n \to \mathbb{R}^m$ is called LINEAR if it satisfies the following two properties for all vectors $\vec{v}, \vec{w} \in \mathbb{R}^n$ and scalars $c \in \mathbb{R}$ ...

1. $F(\vec{v} + \vec{w}) = F(\vec{v}) + F(\vec{w})$, and
2. $F(c\vec{v}) = cF(\vec{v})$

Lecture Activity 4.3. Determine which of the following are linear transformations. Give a formal justification for your answer by showing that the function does or does not satisfy the conditions of Definition 4.4.

P1. $F : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$F\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} x^2 \\ y^2 \end{pmatrix}$$

Solution. This function is not linear. For example, we have

$$F\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) = F\left(\begin{pmatrix} 2 \\ 0 \end{pmatrix}\right) = \begin{pmatrix} 4 \\ 0 \end{pmatrix},$$

but

$$F\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) + F\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix},$$

and so

$$F\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) \neq F\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) + F\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right).$$

Thus, F is not linear.

P2. $G : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$G\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} x+y \\ x \end{pmatrix}.$$

Solution. We claim that G is linear. To see this, take any vectors $\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \in \mathbb{R}^2$.

Then,

$$\begin{aligned} G\left(\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}\right) &= G\left(\begin{pmatrix} x_1+x_2 \\ y_1+y_2 \end{pmatrix}\right) \\ &= \begin{pmatrix} (x_1+x_2)+(y_1+y_2) \\ x_1+x_2 \end{pmatrix} \\ &= \begin{pmatrix} x_1+y_1 \\ x_1 \end{pmatrix} + \begin{pmatrix} x_2+y_2 \\ x_2 \end{pmatrix} \\ &= G\left(\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}\right) + G\left(\begin{pmatrix} x_2 \\ y_2 \end{pmatrix}\right). \end{aligned}$$

Now, for any scalar $c \in \mathbb{R}$ we have

$$\begin{aligned} G\left(c\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}\right) &= G\left(\begin{pmatrix} cx_1 \\ cy_1 \end{pmatrix}\right) \\ &= \begin{pmatrix} cx_1+cy_1 \\ cx_1 \end{pmatrix} \\ &= c\begin{pmatrix} x_1+y_1 \\ x_1 \end{pmatrix} \\ &= c\,G\left(\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}\right) \end{aligned}$$

as needed.

P3. $T_A : \mathbb{R}^2 \to \mathbb{R}^2$ where $A = \begin{pmatrix} \vec{v}_1 & \vec{v}_2 \end{pmatrix}$ is any 2×2 matrix.

Solution. We claim that T_A is linear. To see this, take any vectors $\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \in \mathbb{R}^2$. Then,

$$\begin{aligned}
T_A\left(\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}\right) &= T_A\left(\begin{pmatrix} x_1 + x_2 \\ y_1 + y_2 \end{pmatrix}\right) \\
&= A\begin{pmatrix} x_1 + x_2 \\ y_1 + y_2 \end{pmatrix} \\
&= (x_1 + x_2)\vec{v}_1 + (y_1 + y_2)\vec{v}_2 \\
&= (x_1\vec{v}_1 + y_1\vec{v}_2) + (x_2\vec{v}_1 + y_2\vec{v}_2) \\
&= A\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + A\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \\
&= T_A\left(\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}\right) + T_A\left(\begin{pmatrix} x_2 \\ y_2 \end{pmatrix}\right).
\end{aligned}$$

Now, for any scalar $c \in \mathbb{R}$ we have

$$\begin{aligned}
T_A\left(c\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}\right) &= T_A\left(\begin{pmatrix} cx_1 \\ cy_1 \end{pmatrix}\right) \\
&= A\begin{pmatrix} cx_1 \\ cy_1 \end{pmatrix} \\
&= cx_1\vec{v}_1 + cy_1\vec{v}_2 \\
&= c(x_1\vec{v}_1 + y_1\vec{v}_2) \\
&= cA\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \\
&= cT_A\left(\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}\right),
\end{aligned}$$

as needed

Lecture Activity 4.4. Suppose that $F : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation satisfying

$$F\left(\begin{pmatrix}1\\0\end{pmatrix}\right) = \begin{pmatrix}1\\-1\end{pmatrix} \text{ and } F\left(\begin{pmatrix}0\\1\end{pmatrix}\right) = \begin{pmatrix}1\\2\end{pmatrix}.$$

P1. Find $F\left(\begin{pmatrix}1\\1\end{pmatrix}\right)$ and $F\left(\begin{pmatrix}2\\3\end{pmatrix}\right)$.

Solution. Observe that

$$\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix} + \begin{pmatrix}0\\1\end{pmatrix}, \text{ and } \begin{pmatrix}2\\3\end{pmatrix} = 2\begin{pmatrix}1\\0\end{pmatrix} + 3\begin{pmatrix}0\\1\end{pmatrix}.$$

Since F is a linear transformation, we have

$$\begin{aligned} F\left(\begin{pmatrix}1\\1\end{pmatrix}\right) &= F\left(\begin{pmatrix}1\\0\end{pmatrix} + \begin{pmatrix}0\\1\end{pmatrix}\right) \\ &= F\left(\begin{pmatrix}1\\0\end{pmatrix}\right) + F\left(\begin{pmatrix}0\\1\end{pmatrix}\right) \\ &= \begin{pmatrix}1\\-1\end{pmatrix} + \begin{pmatrix}1\\2\end{pmatrix} = \boxed{\begin{pmatrix}2\\1\end{pmatrix}}. \end{aligned}$$

Similarly,

$$\begin{aligned} F\left(\begin{pmatrix}2\\3\end{pmatrix}\right) &= F\left(2\begin{pmatrix}1\\0\end{pmatrix} + 3\begin{pmatrix}0\\1\end{pmatrix}\right) \\ &= 2F\left(\begin{pmatrix}1\\0\end{pmatrix}\right) + 3F\left(\begin{pmatrix}0\\1\end{pmatrix}\right) \\ &= 2\begin{pmatrix}1\\-1\end{pmatrix} + 3\begin{pmatrix}1\\2\end{pmatrix} = \boxed{\begin{pmatrix}5\\4\end{pmatrix}}. \end{aligned}$$

P2. Find a formula for $F\left(\begin{pmatrix}x\\y\end{pmatrix}\right)$.

Solution. Since $\begin{pmatrix}x\\y\end{pmatrix} = x\begin{pmatrix}1\\0\end{pmatrix} + y\begin{pmatrix}0\\1\end{pmatrix}$, and F is a linear transformation, we have

$$\begin{aligned} F\left(\begin{pmatrix}x\\y\end{pmatrix}\right) &= F\left(x\begin{pmatrix}1\\0\end{pmatrix} + y\begin{pmatrix}0\\1\end{pmatrix}\right) \\ &= xF\left(\begin{pmatrix}1\\0\end{pmatrix}\right) + yF\left(\begin{pmatrix}0\\1\end{pmatrix}\right) \\ &= x\begin{pmatrix}1\\-1\end{pmatrix} + y\begin{pmatrix}1\\2\end{pmatrix} \\ &= \boxed{\begin{pmatrix}x+y\\-x+2y\end{pmatrix}}. \end{aligned}$$

P3. Find a 2×2 matrix M so that $F(\vec{x}) = M\vec{x}$ for all vectors $\vec{x} \in \mathbb{R}^2$.

Solution. From the previous part, we have

$$F\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = x\begin{pmatrix} 1 \\ -1 \end{pmatrix} + y\begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

Rewriting the expression on the right in matrix-vector form yields

$$F\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} 1 & 1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix},$$

and so $M = \begin{pmatrix} 1 & 1 \\ -1 & 2 \end{pmatrix}$. Observe that

$$M = \begin{pmatrix} F\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) & F\left(\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right) \end{pmatrix}.$$

That is, the columns of M are determined by where F sends the standard basis vectors.

Definition 4.9. Let $F : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Then, the DEFINING MATRIX of F is the $m \times n$ matrix M satisfying

$$F(\vec{x}) = M\vec{x}$$

for all vectors $\vec{x}$ in $\mathbb{R}^n$, and is denoted by $M = M_F$.

Lecture Activity 4.5. Let $F : \mathbb{R}^2 \to \mathbb{R}^2$ be the transformation which rotates every vector $\theta°$ counterclockwise about the origin.

P1. Use geometric reasoning to argue that F is a linear transformation.

Solution. Since F only rotates the plane, we can see that all lines will remain as lines and the origin remains fixed.

P2. Find the defining matrix M_F when $\theta = 90°$.

Solution. We see that $F : \vec{e}_1 \mapsto \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and $F : \vec{e}_2 \mapsto \begin{pmatrix} -1 \\ 0 \end{pmatrix}$ and so

$$\boxed{M_F = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}}.$$

P3. Find the defining matrix M_F for any value of θ. Note that your matrix will depend on the unknown angle θ.

Solution. We see that $F : \vec{e}_1 \mapsto \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$ and $F : \vec{e}_2 \mapsto \begin{pmatrix} \cos(\theta + 90°) \\ \sin(\theta + 90°) \end{pmatrix} = \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}$. So,

$$\boxed{M_F = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}}.$$

Definition 4.13. A function $f : X \to Y$ is called ONE-TO-ONE (or INJECTIVE) if the following property holds ...

for every $y \in Y$, there is *at most* one input $x \in X$ so that $f(x) = y$.

Lecture Activity 4.6. Determine which of the following functions are injective. Give a formal justification for your answer by showing that the function does or does not satisfy the conditions of Definition 4.13.

P1. $T_A : \mathbb{R}^2 \to \mathbb{R}^2$ where

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Solution. Take any vector $\vec{y} = \begin{pmatrix} a \\ b \end{pmatrix} \in \mathbb{R}^2$. Note that the matrix-vector equation $A\vec{x} = \vec{y}$ has the same solution set as the system of linear equations with augmented matrix

$$\left(\begin{array}{cc|c} 1 & 0 & a \\ 0 & 1 & b \end{array}\right).$$

By Rouché-Capelli, this system has *exactly one* solution. Hence, there is *at most one* value $\vec{x} \in \mathbb{R}^2$ so that $T_A(\vec{x}) = \vec{y}$, and so T_A $\boxed{\text{is injective}}$.

P2. $T_B : \mathbb{R}^3 \to \mathbb{R}^2$ where

$$B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}.$$

Solution. We have

$$T_B\left(\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}\right) = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \text{ and } T_B\left(\begin{pmatrix} 0 \\ 2 \\ -2 \end{pmatrix}\right) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Hence, there is more than one solution to the equation $T_B(\vec{x}) = \vec{0}$, and so T_B is $\boxed{\text{not injective}}$.

P3. $T_D : \mathbb{R}^2 \to \mathbb{R}^3$ where

$$D = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Solution. Take any vector $\vec{y} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \in \mathbb{R}^3$. Note that the matrix-vector equation $D\vec{x} = \vec{y}$ has the same solution set as the system of linear equations with augmented matrix

$$\left(\begin{array}{cc|c} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & c \end{array}\right).$$

By Rouché-Capelli, this system either has no solutions (if $c \neq 0$) or has exactly one solution (when $c = 0$). Hence, there is at most one value $\vec{x} \in \mathbb{R}^2$ so that $T_D(\vec{x}) = \vec{y}$, and so T_D $\boxed{\text{is injective}}$.

Theorem 4.14. A linear transformation F is injective if and only if every column of $\text{rref}(M_F)$ has a pivot.

Proof. Let $M = M_F$ be the defining matrix of a linear transformation $F : \mathbb{R}^n \to \mathbb{R}^m$. By definition of the defining matrix, for any $\vec{y} \in \mathbb{R}^m$, the set of vectors $\vec{x} \in \mathbb{R}^n$ satisfying

$$F(\vec{x}) = \vec{y}$$

is precisely the set of vectors satisfying the matrix-vector equation $M\vec{x} = \vec{y}$.

Use Rouché-Capelli to complete the proof.

Observe that the matrix-vector equation $M\vec{x} = \vec{y}$ has the same solution set as the system of linear equations with augmented matrix

$$\left(M \mid \vec{y}\right).$$

By Rouché-Capelli, $\text{rref}\left(M \mid \vec{y}\right)$ has a pivot in the last column if and only if the system has no solutions. Otherwise, the system has exactly one solution if and only if $\text{rref}(M)$ has a pivot in every column.

Therefore, the system has *at most* one solution if and only if $\text{rref}(M)$ has a pivot in every column, as needed. □

Definition 4.16. A function $f : X \to Y$ is called ONTO (or SURJECTIVE) if the following property holds ...

for every $y \in Y$, there is *at least* one input $x \in X$ so that $f(x) = y$.

Lecture Activity 4.7. Determine which of the following functions are surjective. Give a formal justification for your answer by showing that the function does or does not satisfy the conditions of Definition 4.16.

P1. $T_A : \mathbb{R}^2 \to \mathbb{R}^2$ where

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Solution. Take any vector $\vec{y} = \begin{pmatrix} a \\ b \end{pmatrix} \in \mathbb{R}^2$. Note that the matrix-vector equation $A\vec{x} = \vec{y}$ has the same solution set as the system of linear equations with augmented matrix

$$\left(\begin{array}{cc|c} 1 & 0 & a \\ 0 & 1 & b \end{array}\right).$$

By Rouché-Capelli, this system has *exactly one* solution. Hence, there is *at least one* value $\vec{x} \in \mathbb{R}^2$ so that $T_A(\vec{x}) = \vec{y}$, and so T_A $\boxed{\text{is surjective}}$.

P2. $T_B : \mathbb{R}^3 \to \mathbb{R}^2$ where

$$B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}.$$

Solution. Take any vector $\vec{y} = \begin{pmatrix} a \\ b \end{pmatrix} \in \mathbb{R}^2$. Note that the matrix-vector equation $B\vec{x} = \vec{y}$ has the same solution set as the system of linear equations with augmented matrix

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & a \\ 0 & 1 & 1 & b \end{array}\right).$$

By Rouché-Capelli, this system has *infinitely many* solutions. Hence, there is *at least one* value $\vec{x} \in \mathbb{R}^3$ so that $T_B(\vec{x}) = \vec{y}$, and so T_B $\boxed{\text{is surjective}}$.

P3. $T_D : \mathbb{R}^2 \to \mathbb{R}^3$ where

$$D = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Solution. Consider the vector $\vec{y} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$. Note that the matrix-vector equation $D\vec{x} = \vec{y}$ has the same solution set as the augmented matrix

$$\left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right).$$

By Rouché-Capelli, this system *does not have a solution.* Hence, there does not exist an input $\vec{x} \in \mathbb{R}^3$ so that $T_D(\vec{x}) = \vec{y}$), and so T_D $\boxed{\text{is not surjective}}$.

Lecture Activity 4.8. Use Theorems 4.14 and 4.17 to determine which of the following functions are injective, surjective, or neither.

P1. $F : \mathbb{R}^2 \to \mathbb{R}^3$, $\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x \\ y \\ 0 \end{pmatrix}$

Solution. Observe that F has defining matrix

$$M_F = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

This matrix is in reduced row echelon form, and has a pivot in every column but not every row. Hence, F is $\boxed{\text{injective but not surjective}}$.

P2. $G : \mathbb{R}^3 \to \mathbb{R}^3$, $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} x - y \\ y + z \\ x + z \end{pmatrix}$

Solution. Observe that G has defining matrix

$$M_G = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$$

and we have

$$\mathrm{rref}(M_G) = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

This matrix does not have a pivot in the third column, and does not have a pivot in the third row. Hence, G is $\boxed{\text{neither injective nor surjective}}$.

P3. $H : \mathbb{R}^3 \to \mathbb{R}^3$, $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} x - y \\ y + z \\ z \end{pmatrix}$

Solution. Observe that H has defining matrix

$$M_H = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

and we have

$$\text{rref}(M_H) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

This matrix has a pivot in every column and every row. Hence, H is $\boxed{\text{both injective and surjective}}$.