Instructions. This packet is due on Quercus no later than **11:59pm on Monday, February 2nd**. Please complete your work directly on this packet. We will spend time together during lecture working on most or all of the activities in this packet. You are responsible for completing all portions of this packet, including lecture activities not discussed in class, and completing the definitions included in the packet. Solutions will be posted to the course website after the assignment due date.

Definition 4.1. Let A be an $m \times n$ matrix with column vectors $A = \begin{pmatrix} \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \end{pmatrix}$. Then, for a vector $\vec{x}$ in $\mathbb{R}^n$ the MATRIX-VECTOR PRODUCT of A and $\vec{x}$ is the vector in $\mathbb{R}^m$ defined by ...

Lecture Activity 4.1. Consider the matrices

$$A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \\ -1 & 1 \end{pmatrix} \text{ and } B = \begin{pmatrix} 3 & -1 & 1 \\ 2 & 0 & 1 \end{pmatrix}.$$

P1. Calculate the matrix-vector product $A\vec{x}$ where $\vec{x} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$.

P2. Calculate the matrix-vector product $B\vec{y}$ where $\vec{y} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$.

P3. Let $\vec{x}$ and $\vec{y}$ be as in the previous problems. Explain why the matrix-vector products $A\vec{y}$ and $B\vec{x}$ are not defined.

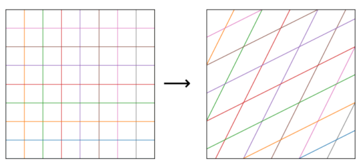
P4. Let $\vec{z}$ be a vector in $\mathbb{R}^2$. How many components does the vector $A\vec{z}$ have?

P5. Let $\vec{w}$ be a vector in $\mathbb{R}^3$. How many components does the vector $B\vec{w}$ have?

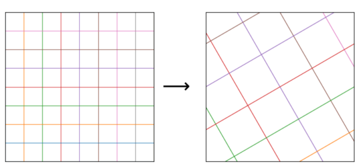
Definition 4.2. Let A be an $m \times n$ matrix. Then, the MATRIX TRANSFORMATION associated to A is the function ...

Lecture Activity 4.2. In the images below, we've plotted where the indicated function sends the standard coordinate grid for $\mathbb{R}^2$. What do you notice?

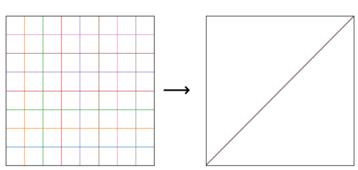
T_A where $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$



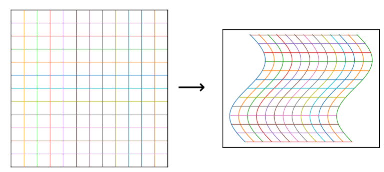
T_B where $B = \begin{pmatrix} \sqrt{2} & -\sqrt{2} \\ \sqrt{2} & \sqrt{2} \end{pmatrix}$



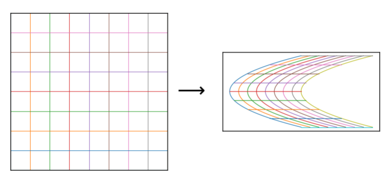
T_C where $C = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$



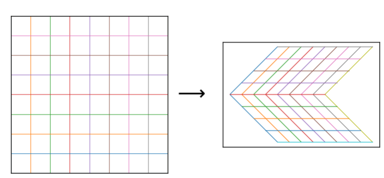
$F\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} x + \sin(y) \\ y \end{pmatrix}$



$G\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} x + y^2 \\ y \end{pmatrix}$



$H\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} x + |y| \\ y \end{pmatrix}$



Definition 4.4. A function $F : \mathbb{R}^n \to \mathbb{R}^m$ is called LINEAR if it satisfies the following two properties for all vectors $\vec{v}, \vec{w} \in \mathbb{R}^n$ and scalars $c \in \mathbb{R}$ ...

Lecture Activity 4.3. Determine which of the following are linear transformations. Give a formal justification for your answer by showing that the function does or does not satisfy the conditions of Definition 4.4.

P1. $F : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$F\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} x^2 \\ y^2 \end{pmatrix}$$

P2. $G : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$G\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} x + y \\ x \end{pmatrix}.$$

P3. $T_A : \mathbb{R}^2 \to \mathbb{R}^2$ where $A = \begin{pmatrix} \vec{v}_1 & \vec{v}_2 \end{pmatrix}$ is any 2×2 matrix.

Lecture Activity 4.4. Suppose that $F : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation satisfying

$$F\left(\begin{pmatrix}1\\0\end{pmatrix}\right) = \begin{pmatrix}1\\-1\end{pmatrix} \text{ and } F\left(\begin{pmatrix}0\\1\end{pmatrix}\right) = \begin{pmatrix}1\\2\end{pmatrix}.$$

P1. Find $F\left(\begin{pmatrix}1\\1\end{pmatrix}\right)$ and $F\left(\begin{pmatrix}2\\3\end{pmatrix}\right)$.

P2. Find a formula for $F\left(\begin{pmatrix}x\\y\end{pmatrix}\right)$.

P3. Find a 2×2 matrix M so that $F(\vec{x}) = M\vec{x}$ for all vectors $\vec{x} \in \mathbb{R}^2$.

Definition 4.9. Let $F : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Then, the DEFINING MATRIX of F is the $m \times n$ matrix M satisfying

for all vectors $\vec{x}$ in $\mathbb{R}^n$, and is denoted by $M = M_F$.

Lecture Activity 4.5. Let $F : \mathbb{R}^2 \to \mathbb{R}^2$ be the transformation which rotates every vector θ° counterclockwise about the origin.

P1. Use geometric reasoning to argue that F is a linear transformation.

P2. Find the defining matrix M_F when $\theta = 90^\circ$.

P3. Find the defining matrix M_F for any value of θ. Note that your matrix will depend on the unknown angle θ.

Definition 4.13. A function $f : X \to Y$ is called ONE-TO-ONE (or INJECTIVE) if the following property holds ...

Lecture Activity 4.6. Determine which of the following functions are injective. Give a formal justification for your answer by showing that the function does or does not satisfy the conditions of Definition 4.13.

P1. $T_A : \mathbb{R}^2 \to \mathbb{R}^2$ where

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

P2. $T_B : \mathbb{R}^3 \to \mathbb{R}^2$ where

$$B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}.$$

P3. $T_D : \mathbb{R}^2 \to \mathbb{R}^3$ where

$$D = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Theorem 4.14. A linear transformation F is injective if and only if every column of $\text{rref}(M_F)$ has a pivot.

Proof. Let $M = M_F$ be the defining matrix of a linear transformation $F : \mathbb{R}^n \to \mathbb{R}^m$. By definition of the defining matrix, for any $\vec{y} \in \mathbb{R}^m$, the set of vectors $\vec{x} \in \mathbb{R}^n$ satisfying

$$F(\vec{x}) = \vec{y}$$

is precisely the set of vectors satisfying the matrix-vector equation $M\vec{x} = \vec{y}$.

Use Rouché-Capelli to complete the proof.

□

Definition 4.16. A function $f : X \to Y$ is called ONTO (or SURJECTIVE) if the following property holds ...

Lecture Activity 4.7. Determine which of the following functions are surjective. Give a formal justification for your answer by showing that the function does or does not satisfy the conditions of Definition 4.16.

P1. $T_A : \mathbb{R}^2 \to \mathbb{R}^2$ where

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

P2. $T_B : \mathbb{R}^3 \to \mathbb{R}^2$ where

$$B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}.$$

P3. $T_D : \mathbb{R}^2 \to \mathbb{R}^3$ where

$$D = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Lecture Activity 4.8. Use Theorems 4.14 and 4.17 to determine which of the following functions are injective, surjective, or neither.

P1. $F : \mathbb{R}^2 \to \mathbb{R}^3, \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x \\ y \\ 0 \end{pmatrix}$

P2. $G : \mathbb{R}^3 \to \mathbb{R}^3, \begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} x - y \\ y + z \\ x + z \end{pmatrix}$

P3. $H : \mathbb{R}^3 \to \mathbb{R}^3, \begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} x - y \\ y + z \\ z \end{pmatrix}$