CHAPTER 12 ACTIVITY PACKET *solutions*

Instructions. This packet is due on Quercus no later than **11:59pm on Tuesday, April 7th**. Please complete your work directly on this packet. We will spend time together during lecture working on most or all of the activities in this packet. You are responsible for completing all portions of this packet, including lecture activities not discussed in class, and completing the definitions included in the packet. Solutions will be posted to the course website after the assignment due date.

Definition 12.1. An $n \times n$ matrix A is called ORTHOGONALLY DIAGONALIZABLE if . . .

there exists an orthogonal matrix Q and a diagonal matrix D so that $Q^\top AQ = D$.

Lecture Activity 12.1. Consider the matrix $A = \begin{pmatrix} 3 & 2 \\ 2 & 6 \end{pmatrix}$.

P1. Find an orthonormal basis for $\mathbb{R}^2$ of eigenvectors for A, given that

$$\chi_A(x) = (x-2)(x-7).$$

Solution. We see that A has eigenvalues 2 and 7 and we calculate its eigenspaces to be

$$E_7 = \text{Span}\left(\begin{pmatrix} 1 \\ 2 \end{pmatrix}\right), E_2 = \text{Span}\left(\begin{pmatrix} -2 \\ 1 \end{pmatrix}\right).$$

Observe that $\left\{\begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} -2 \\ 1 \end{pmatrix}\right\}$ is an orthogonal basis for $\mathbb{R}^2$ of eigenvectors for A. Dividing each vector by its norm, we obtain an orthonormal basis of eigenvectors

$$\mathcal{B} = \left\{\begin{pmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{pmatrix}, \begin{pmatrix} -2/\sqrt{5} \\ 1/\sqrt{5} \end{pmatrix}\right\}$$

P2. Find an orthogonal matrix Q and diagonal matrix D so that

$$D = Q^\top AQ.$$

Solution. Since $\mathcal{B}$ is a basis of eigenvectors, then the Diagonalization Theorem we have that $D = Q^{-1}AQ$ where

$$D = \begin{pmatrix} 7 & 0 \\ 0 & 2 \end{pmatrix} \text{ and } Q = \begin{pmatrix} 1/\sqrt{5} & -2/\sqrt{5} \\ 2/\sqrt{5} & 1/\sqrt{5} \end{pmatrix}.$$

But since $\mathcal{B}$ is orthonormal the matrix Q is orthogonal, and so $Q^{-1} = Q^T$.

P3. Conclude that A is orthogonally diagonalizable.

This follows by P2.

Proposition 12.2. An $n \times n$ matrix A is orthogonally diagonalizable if and only if there is an orthonormal basis for $\mathbb{R}^n$ consisting of eigenvectors for A.

Proof. Suppose first that A is orthogonally diagonalizable.

Complete the proof: show that there is an orthonormal basis for $\mathbb{R}^n$ consisting of eigenvectors for A.

Since A is orthogonally diagonalizable, we can write $D = Q^\top AQ$ for an orthogonal matrix Q and diagonal matrix D. Letting Q has column vectors $Q = \begin{pmatrix} \vec{v}_1 & \cdots & \vec{v}_n \end{pmatrix}$, then by definition of orthogonal matrices we have that $\mathcal{B} = \{\vec{v}_1, \dots, \vec{v}_n\}$ is an orthonormal basis for $\mathbb{R}^n$. Furthermore, since $Q^\top = Q^{-1}$ then by the Diagonalization Theorem we know that $\mathcal{B}$ is a set of eigenvector for A.

Conversely, suppose that $\mathcal{B} = \{\vec{v}_1, \dots, \vec{v}_n\}$ is an orthonormal basis for $\mathbb{R}^n$ consisting of eigenvectors for A.

Complete the proof: show that A is orthogonally diagonalizable.

By the Diagonalization Theorem we know that A is diagonalizable with $D = Q^{-1}AQ$ where $Q = \begin{pmatrix} \vec{v}_1 & \cdots & \vec{v}_n \end{pmatrix}$. But since $\mathcal{B}$ is an *orthonormal* basis for $\mathbb{R}^n$, we know that Q and orthogonal, and so $Q^{-1} = Q^\top$. So, $D = Q^\top AQ$, as needed.

□

Theorem 12.3 (The Spectral Theorem)**.** An $n \times n$ matrix A is orthogonally diagonalizable if and only if ...

it is symmetric; that is, $A = A^\top$.

Lemma 12.4. Let A be symmetric. Then,

(1) A has at least one eigenvalue and all eigenvalues of A are real, and

(2) if λ, μ are distinct eigenvalues of A, then for any $\vec{x} \in E_\lambda$ and $\vec{y} \in E_\mu$ we have that $\vec{x}$ and $\vec{y}$ are orthogonal.

Proof. See the course lecture notes for the proof of (1). For (2), suppose that A has eigenvalues $\lambda \neq \mu$ and let $\vec{x} \in E_\lambda$ and $\vec{y} \in E_\mu$. Then we have ...

Complete the proof: show that $\vec{x} \cdot \vec{y} = 0$

$$\begin{aligned} \lambda\vec{x} \cdot \vec{y} &= A\vec{x} \cdot \vec{y} \\ &= (A\vec{x})^\top \vec{y} \\ &= \vec{x}^\top A^\top \vec{y} \\ &= \vec{x}^\top A\vec{y}, \text{ since } A = A^\top \\ &= \vec{x} \cdot \mu\vec{y} \\ &= \mu\vec{x} \cdot \vec{y}, \end{aligned}$$

and so $(\lambda - \mu)\vec{x} \cdot \vec{y} = 0 \Rightarrow \vec{x} \cdot \vec{y} = 0$ since we've assumed that $\lambda \neq \mu$.

$\square$

Definition 12.5. Suppose that A is an $n \times n$ symmetric matrix with eigenvalues $\lambda_1, \dots, \lambda_n$ and orthonormal basis of eigenvectors $\{\vec{v}_1, \dots, \vec{v}_n\}$. We call the equality

$$\boxed{A = QDQ^\top}$$

a SPECTRAL DECOMPOSITION of A, where

$$\boxed{D = \operatorname{diag}(\lambda_1, \dots, \lambda_n), \text{ and } Q = \begin{pmatrix} \vec{v}_1 & \cdots & \vec{v}_n \end{pmatrix}.}$$

Lecture Activity 12.2. Consider the matrix $A = \begin{pmatrix} 3 & 2 \\ 2 & 6 \end{pmatrix}$ from Lecture Activity 12.1.

P1. Use your work from P2 of Lecture Activity 12.2 to find a spectral decomposition for A. That is, find an orthogonal matrix Q and diagonal matrix D so that $A = QDQ^\top$.

Solution. By P2 of Lecture Activity 12.2 we know that $\mathbb{R}^2$ has an orthonormal basis of eigenvectors for A given by

$$\left\{ \begin{pmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{pmatrix}, \begin{pmatrix} -2/\sqrt{5} \\ 1/\sqrt{5} \end{pmatrix} \right\}.$$

So, A has spectral decomposition

$$A = \underbrace{\begin{pmatrix} 1/\sqrt{5} & -2/\sqrt{5} \\ 2/\sqrt{5} & 1/\sqrt{5} \end{pmatrix}}_{Q} \underbrace{\begin{pmatrix} 7 & 0 \\ 0 & 2 \end{pmatrix}}_{D} \underbrace{\begin{pmatrix} 1/\sqrt{5} & -2/\sqrt{5} \\ 2/\sqrt{5} & 1/\sqrt{5} \end{pmatrix}}_{Q^\top}$$

P2. Use Chapter Exercise P11.8 to show that Q is a rotation matrix. Recalling that we can write rotation matrices in the form

$$Q = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix},$$

find the angle θ.

Solution. Observe that $\det(Q) = 1$, and so by Chapter Exercise P11.8, T_Q rotates vectors counterclockwise by an angle of θ. So we must have $\cos\theta = 1/\sqrt{5} \Rightarrow \theta \approx 63^\circ$.

P3. Using your work in the previous parts, give a geometric description of how T_A transforms the standard coordinate grid for $\mathbb{R}^2$.

Since $T_Q = T_Q \circ T_D \circ T_{Q^\top}$ then from our work above we see that T_Q first rotates vectors $\approx 63^\circ$ counterclockwise about the origin, then dilates by stretching horizontally by 7 and vertically by 2, then rotates vectors *clockwise* about the origin by $\approx 63^\circ$.

Proposition 12.7. Let A be an $m \times n$ matrix. Then, there exists an orthonormal basis for $\mathbb{R}^n$ of eigenvectors of $A^\top A$ so that $\{A\vec{v}_1, \dots, A\vec{v}_n\}$ is an orthogonal subset of $\mathbb{R}^m$. Furthermore, if we reindex our basis so that $A\vec{v}_1, \dots, A\vec{v}_r$ are nonzero, and $A\vec{v}_{r+1} = \cdots = A\vec{v}_n = \vec{0}$, then $\{A\vec{v}_1, \dots, A\vec{v}_r\}$ forms an orthogonal basis for $\mathrm{Col}(A)$.

Complete the proof: show that $A^\top A$ is symmetric.

We have $(A^\top A)^\top = A^\top (A^\top)^\top = A^\top A$, and so A is symmetric

So, by the Spectral Theorem, there there exists an orthonormal basis $\{\vec{v}_1, \dots, \vec{v}_n\}$ for $\mathbb{R}^n$ of eigenvectors of $A^\top A$. Suppose that each $\vec{v}_i$ has eigenvalue λ_i.

Complete the proof: show that $(A\vec{v}_i) \cdot (A\vec{v}_j) = 0$ for all $i \neq j$.

For any $i \neq j$ we have

$$\begin{aligned}(A\vec{v}_i) \cdot (A\vec{v}_j) &= (A\vec{v}_i)^\top (A\vec{v}_j) \\ &= \vec{v}_i^\top A^\top A\vec{v}_j \\ &= \vec{v}_i^\top (\lambda_j \vec{v}_j) \\ &= \vec{v}_i \cdot (\lambda_j \vec{v}_j) \\ &= \lambda_j (\vec{v}_i \cdot \vec{v}_j) \\ &= 0,\end{aligned}$$

where the final equality follows because $\vec{v}_i$ and $\vec{v}_j$ are orthogonal when $i \neq j$. Hence, $\{A\vec{v}_1, \dots, A\vec{v}_m\}$ is an orthogonal subset of $\mathbb{R}^m$.

Next, reindex our basis as in the theorem statement.

Complete the proof: show that $\mathrm{Col}(A) = \mathrm{Span}(A\vec{v}_1, \dots, A\vec{v}_r)$.

Noting that $\{\vec{v}_1, \dots, \vec{v}_n\}$ is a basis for $\mathbb{R}^n$, we have $A\vec{y} \in \mathrm{Col}(A)$ if and only if

$$A\vec{y} = A(x_1\vec{v}_1 + \cdots + x_n\vec{v}_n) = x_1 A\vec{v}_1 + \cdots + x_r A\vec{v}_r + \vec{0},$$

and so $\mathrm{Col}(A) = \mathrm{Span}(A\vec{v}_1, \dots, A\vec{v}_r)$.

Since $\{A\vec{v}_1, \dots, A\vec{v}_r\}$ is orthogonal, then this set is linearly independent, and hence is a basis. ☐

Lecture Activity 12.3. Consider the 3×2 matrix $A = \begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 1 & -1 \end{pmatrix}$.

P1. Verify that $A^\top A = \begin{pmatrix} 3 & 2 \\ 2 & 6 \end{pmatrix}$. Then, use your work from Lecture Activity 12.1 to find an orthonormal basis for $\mathbb{R}^2$ of eigenvectors $\{\vec{v}_1, \vec{v}_2\}$ for $A^\top A$.

Solution. We have

$$A^\top A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & -1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 3 & 2 \\ 2 & 6 \end{pmatrix}$$

and so by Lecture Activity 12.1 we know that $\mathbb{R}^2$ has an orthonormal basis of eigenvectors for $A^\top A$ given by $\{\vec{v}_1, \vec{v}_2\}$ where

$$\vec{v}_1 = \begin{pmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{pmatrix}, \vec{v}_2 = \begin{pmatrix} -2/\sqrt{5} \\ 1/\sqrt{5} \end{pmatrix}.$$

P2. Use Proposition 12.7 to verify that $\mathcal{B} = \{\vec{u}_1, \vec{u}_2\}$ where

$$\vec{u}_1 = \frac{A\vec{v}_1}{\|A\vec{v}_1\|}, \vec{u}_2 = \frac{A\vec{v}_2}{\|A\vec{v}_2\|}$$

is an orthonormal basis for $\operatorname{im}(T_A)$.

Solution. We have

$$A\vec{v}_1 = \begin{pmatrix} 3/\sqrt{5} \\ \sqrt{5} \\ -1/\sqrt{5} \end{pmatrix}, A\vec{v}_2 = \begin{pmatrix} -1/\sqrt{5} \\ 0 \\ -3/\sqrt{5} \end{pmatrix}$$

$$\Rightarrow \vec{u}_1 = \begin{pmatrix} 3/\sqrt{35} \\ \sqrt{35}/7 \\ -1/\sqrt{35} \end{pmatrix}, \vec{u}_2 = \begin{pmatrix} -1/\sqrt{10} \\ 0 \\ -3/\sqrt{10} \end{pmatrix}$$

which is an orthonormal basis for $\operatorname{im}(T_A)$ by Proposition 12.7.

P3. Show that there exists a vector $\vec{u}_3$ so that $\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$ is an orthonormal basis for $\mathbb{R}^3$.

Solution. Observe that if we take any $\vec{v}_3$ not in $\operatorname{Span}(\vec{u}_1, \vec{u}_2)$, then $\{\vec{u}_1, \vec{u}_2, \vec{v}_3\}$ is linearly independent, and hence a basis for $\mathbb{R}^3$, and so we can use Gram-Schmidt to obtain a new vector $\vec{u}_3$ which is orthogonal to $\vec{u}_1, \vec{u}_2$. We omit the details of this calculation, and note that such a vector could be found explicitly.

P4. Let's use our work from the previous parts to decompose our transformation $T_A : \mathbb{R}^2 \to \mathbb{R}^3$.

Step 1: Rotate/reflect. Let $Q = \begin{pmatrix} \vec{v}_1 & \vec{v}_2 \end{pmatrix}$ where $\{\vec{v}_1, \vec{v}_2\}$ is the basis from P1. Observe that $T_{Q^\top}$ rotates the plane by $\theta \approx 63°$ degrees clockwise, and that

$$T_{Q^\top}(\vec{v}_1) = \vec{e}_1 \text{ and } T_{Q^\top}(\vec{v}_2) = \vec{e}_2.$$

Solution. By P2 of Lecture Activity 12.2 we know that T_Q rotates vectors counterclockwise by $\approx 63°$. Since Q is orthogonal, we have that $Q^\top = Q^{-1}$ will rotate vectors *clockwise* by the same amount. Furthermore, since $T_Q : \vec{e}_i \mapsto \vec{v}_i$ we have the the inverse function will do the opposite. That is, $T_{Q^\top} : \vec{v}_i \mapsto \vec{e}_i$.

Step 2: Dilate and embed. Consider the "*block-diagonal*" matrix

$$\Sigma = \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \\ 0 & 0 \end{pmatrix}$$

where $\sigma_1 = \|A\vec{v}_1\|$ and $\sigma_2 = \|A\vec{v}_2\|$. Give a geometric description for the matrix-tranformation $T_\Sigma : \mathbb{R}^2 \to \mathbb{R}^3$.

Solution. Observe that

$$T_\Sigma(\vec{e}_1) = \begin{pmatrix} \sigma_1 \\ 0 \\ 0 \end{pmatrix}, T_\Sigma(\vec{e}_2) = \begin{pmatrix} 0 \\ \sigma_2 \\ 0 \end{pmatrix}.$$

So, this transformation stretches the x-axis by σ_1 and the y-axis by σ_2, and then "embeds" $\mathbb{R}^2$ into $\mathbb{R}^3$ by placing it on the xy-plane.

Step 3: Rotate/reflect. Let

$$U = \begin{pmatrix} \vec{u}_1 & \vec{u}_2 & \vec{u}_3 \end{pmatrix}$$

where $\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$ is the orthonormal basis for $\mathbb{R}^3$ found in P3. Use geometric reasoning to convince yourself that T_U is either a rotation or reflection transformation. *(Bonus: think about what computations you would need to perform to describe this rotation transformation explicitly).*

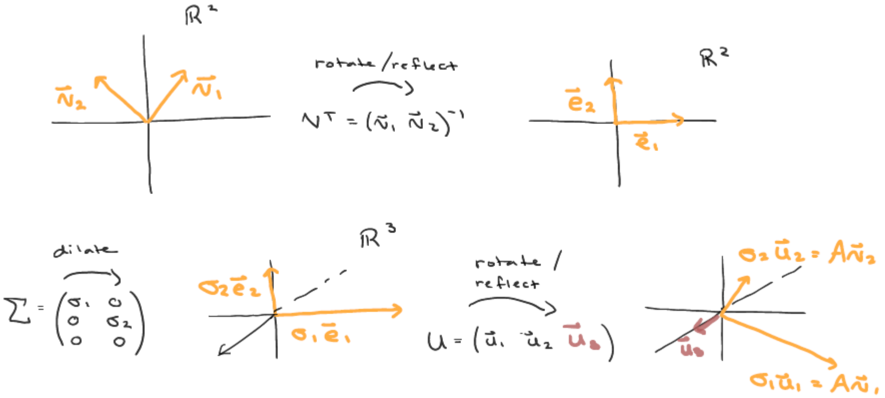
This transformation sends our standard basis $\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$ to an orthonormal basis $\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$. So, the image of the unit cube remains a unit cube, which can only be obtained by rotation or reflection.

P5. Use geometric reasoning, along with your work from P4, to show that

$$A = U\Sigma Q^{\top},$$

where U and Q are orthogonal matrices, and Σ is a "block-diagonal" matrix. Discuss how this decomposition describes the transformation $T_A : \mathbb{R}^2 \to \mathbb{R}^3$ as a rotation/reflection, followed by dilation/embedding, followed by another rotation/reflection.

Solution. The steps from P4 are demonstrated in the image below



By Proposition 12.7 we see that this composition of transformations is precisely the transformation T_A, as demonstrated in the image below

Definition 12.9. Let A be an $m \times n$ matrix and $\{\vec{v}_1, \dots, \vec{v}_n\}$ be an orthonormal basis for $\mathbb{R}^n$ of eigenvectors for $A^\top A$, as above. The SINGULAR VALUES of A are …

the scalars $\sigma_i := \|A\vec{v}_i\|$, for $i = 1, \dots, n$.

Proposition 12.10. Let A be an $m \times n$ matrix and $\lambda_1, \dots, \lambda_n$ be the eigenvalues of $A^\top A$. Then, $\lambda_i > 0$ and the singular values of A are given by $\sigma_i = \sqrt{\lambda_i}$.

Prove Proposition 12.10.

Proof. We have

$$\begin{aligned}
\sigma_i^2 &= \|A\vec{v}_i\|^2 \\
&= (A\vec{v}_i) \cdot (A\vec{v}_i) \\
&= (A\vec{v}_i)^\top (A\vec{v}_i) \\
&= \vec{v}_i^\top A^\top A\vec{v}_i \\
&= \vec{v}_i^\top \lambda_i \vec{v}_i \\
&= \lambda_i \vec{v}_i \cdot \vec{v}_i \\
&= \lambda_i \|\vec{v}_i\|^2 \\
&= \lambda_i,
\end{aligned}$$

where the final equality follows because the $\vec{v}_i$ form an ortho**normal** set. Hence, $\sigma_i^2 = \lambda_i$ and so $\lambda_i > 0$ and $\sigma_i = \sqrt{\lambda_i}$. □

Lecture Activity 12.4. Find the singular values of the following matrices. Given your calculations, what can you say about the corresponding transformations?

P1. $A = \begin{pmatrix} 2 & 0 \\ 0 & 2 \\ 1 & 0 \end{pmatrix}$

Solution. The singular values of A are the square roots of the eigenvalues of

$$A^\top A = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 2 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 5 & 0 \\ 0 & 4 \end{pmatrix}.$$

Since the eigenvalues of $A^\top A$ are 4 and 5, the singular values of A are 2 and $\sqrt{5}$.

By Proposition 12.7, we know that there's an orthonormal basis $\{\vec{v}_1, \vec{v}_2\}$ so that $\{A\vec{v}_1, A\vec{v}_2\}$ is an orthogonal basis for $\mathrm{im}(T_A)$. So, T_A maps a unit square (with sides given by $\vec{v}_1, \vec{v}_2$) to a rectangle of side lengths $2, \sqrt{5}$. We can see that T_A is injective, but not surjective, and that this transformation "stretches out" space (that is, our grid lines get further apart).

P2. $B = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & -1 \\ 1 & 1 & 2 \end{pmatrix}$

Solution. The singular values of B are the square roots of the eigenvalues of

$$B^\top B = \begin{pmatrix} 1 & 0 & 1 \\ 2 & 1 & 1 \\ 1 & -1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & -1 \\ 1 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 2 & 3 & 3 \\ 3 & 6 & 3 \\ 3 & 3 & 6 \end{pmatrix}.$$

Now $\chi_{B^\top B}(x) = -x^3 + 14x^2 - 33x = x(x-3)(x-11)$. Since the eigenvalues of $B^\top B$ are 0, 3 and 11, the singular values of B are 0, $\sqrt{3}$ and $\sqrt{11}$.

By Proposition 12.7, we know there's an orthonormal basis $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ for $\mathbb{R}^3$ so that $\{B\vec{v}_2, B\vec{v}_3\}$ is a basis for $\mathrm{im}(T_B)$. We see then that T_B sends the unit cube (with sides given by $\vec{v}_1, \vec{v}_2, \vec{v}_3$) to a 2-dimensional rectangle with sides $\sqrt{3}$ and $\sqrt{11}$. So, T_B is not injective nor surjective (since there must be some collapsing). We can see that T_B collapses $\mathbb{R}^3$ onto a 2-dimensional subspaces, which gets "stretched out".

P3. $C = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$

Solution. The singular values of C are the square roots of the eigenvalues of

$$C^\top C = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}.$$

Since $C^\top C$ has only one eigenvalue equal to 2, and so the singular values of C are $\sqrt{2}$.

Solution. By Proposition 12.7, we know that T_C sends a unit square to another square with side length $\sqrt{2}$. So, T_C rotates/reflects $\mathbb{R}^2$ and stretches the grid lines out equally in both directions by a factor of $\sqrt{2}$.