CHAPTER 12 ACTIVITY PACKET NAME:

Instructions. This packet is due on Quercus no later than **11:59pm on Tuesday, April 7th**. Please complete your work directly on this packet. We will spend time together during lecture working on most or all of the activities in this packet. You are responsible for completing all portions of this packet, including lecture activities not discussed in class, and completing the definitions included in the packet. Solutions will be posted to the course website after the assignment due date.

Definition 12.1. An $n \times n$ matrix A is called ORTHOGONALLY DIAGONALIZABLE if ...

Lecture Activity 12.1. Consider the matrix $A = \begin{pmatrix} 3 & 2 \\ 2 & 6 \end{pmatrix}$.

P1. Find an orthonormal basis for $\mathbb{R}^2$ of eigenvectors for A, given that

$$\chi_A(x) = (x-2)(x-7).$$

P2. Find an orthogonal matrix Q and diagonal matrix D so that

$$D = Q^\top A Q.$$

P3. Conclude that A is orthogonally diagonalizable.

Proposition 12.2. An $n \times n$ matrix A is orthogonally diagonalizable if and only if there is an orthonormal basis for $\mathbb{R}^n$ consisting of eigenvectors for A.

Proof. Suppose first that A is orthogonally diagonalizable.

Complete the proof: show that there is an orthonormal basis for $\mathbb{R}^n$ consisting of eigenvectors for A.

Conversely, suppose that $\mathcal{B} = \{\vec{v}_1, \ldots, \vec{v}_n\}$ is an orthonormal basis for $\mathbb{R}^n$ consisting of eigenvectors for A.

Complete the proof: show that A is orthogonally diagonalizable.

$\square$

Theorem 12.3 (The Spectral Theorem)**.** An $n \times n$ matrix A is orthogonally diagonalizable if and only if ...

Lemma 12.4. Let A be symmetric. Then,

(1) A has at least one eigenvalue and all eigenvalues of A are real, and

(2) if λ, μ are distinct eigenvalues of A, then for any $\vec{x} \in E_\lambda$ and $\vec{y} \in E_\mu$ we have that $\vec{x}$ and $\vec{y}$ are orthogonal.

Proof. See the course lecture notes for the proof of (1). For (2), suppose that A has eigenvalues $\lambda \neq \mu$ and let $\vec{x} \in E_\lambda$ and $\vec{y} \in E_\mu$. Then we have ...

Complete the proof: show that $\vec{x} \cdot \vec{y} = 0$

□

Definition 12.5. Suppose that A is an $n \times n$ symmetric matrix with eigenvalues $\lambda_1, \ldots, \lambda_n$ and orthonormal basis of eigenvectors $\{\vec{v}_1, \ldots, \vec{v}_n\}$. We call the equality

a SPECTRAL DECOMPOSITION of A, where

Lecture Activity 12.2. Consider the matrix $A = \begin{pmatrix} 3 & 2 \\ 2 & 6 \end{pmatrix}$ from Lecture Activity 12.1.

P1. Use your work from P2 of Lecture Activity 12.2 to find a spectral decomposition for A. That is, find an orthogonal matrix Q and diagonal matrix D so that $A = QDQ^\top$.

P2. Use Chapter Exercise P11.8 to show that Q is a rotation matrix. Recalling that we can write rotation matrices in the form

$$Q = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix},$$

find the angle θ.

P3. Using your work in the previous parts, give a geometric description of how T_A transforms the standard coordinate grid for $\mathbb{R}^2$.

Proposition 12.7. Let A be an $m \times n$ matrix. Then, there exists an orthonormal basis for $\mathbb{R}^n$ of eigenvectors of $A^\top A$ so that $\{A\vec{v}_1, \dots, A\vec{v}_n\}$ is an orthogonal subset of $\mathbb{R}^m$. Furthermore, if we reindex our basis so that $A\vec{v}_1, \dots, A\vec{v}_r$ are nonzero, and $A\vec{v}_{r+1} = \cdots = A\vec{v}_n = \vec{0}$, then $\{A\vec{v}_1, \dots, A\vec{v}_r\}$ forms an orthogonal basis for $\mathrm{Col}(A)$.

Complete the proof: show that $A^\top A$ is symmetric.

So, by the Spectral Theorem, there there exists an orthonormal basis $\{\vec{v}_1, \dots, \vec{v}_n\}$ for $\mathbb{R}^n$ of eigenvectors of $A^\top A$. Suppose that each $\vec{v}_i$ has eigenvalue λ_i.

Complete the proof: show that $(A\vec{v}_i) \cdot (A\vec{v}_j) = 0$ for all $i \neq j$.

Next, reindex our basis as in the theorem statement.

Complete the proof: show that $\mathrm{Col}(A) = \mathrm{Span}(A\vec{v}_1, \dots, A\vec{v}_r)$.

Since $\{A\vec{v}_1, \dots, A\vec{v}_r\}$ is orthogonal, then this set is linearly independent, and hence is a basis. □

Lecture Activity 12.3. Consider the 3×2 matrix $A = \begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 1 & -1 \end{pmatrix}$.

P1. Verify that $A^\top A = \begin{pmatrix} 3 & 2 \\ 2 & 6 \end{pmatrix}$. Then, use your work from Lecture Activity 12.1 to find an orthonormal basis for $\mathbb{R}^2$ of eigenvectors $\{\vec{v}_1, \vec{v}_2\}$ for $A^\top A$.

P2. Use Proposition 12.7 to verify that $\mathcal{B} = \{\vec{u}_1, \vec{u}_2\}$ where

$$\vec{u}_1 = \frac{A\vec{v}_1}{\|A\vec{v}_1\|}, \vec{u}_2 = \frac{A\vec{v}_2}{\|A\vec{v}_2\|}$$

is an orthonormal basis for $\mathrm{im}(T_A)$.

P3. Show that there exists a vector $\vec{u}_3$ so that $\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$ is an orthonormal basis for $\mathbb{R}^3$.

P4. Let's use our work from the previous parts to decompose our transformation $T_A : \mathbb{R}^2 \to \mathbb{R}^3$.

Step 1: Rotate/reflect. Let $Q = \begin{pmatrix} \vec{v}_1 & \vec{v}_2 \end{pmatrix}$ where $\{\vec{v}_1, \vec{v}_2\}$ is the basis from P1. Observe that $T_{Q^\top}$ rotates the plane by $\theta \approx 63^\circ$ degrees clockwise, and that

$$T_{Q^\top}(\vec{v}_1) = \vec{e}_1 \text{ and } T_{Q^\top}(\vec{v}_2) = \vec{e}_2.$$

Step 2: Dilate and embed. Consider the "*block-diagonal*" matrix

$$\Sigma = \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \\ 0 & 0 \end{pmatrix}$$

where $\sigma_1 = \|A\vec{v}_1\|$ and $\sigma_2 = \|A\vec{v}_2\|$. Give a geometric description for the matrix-tranformation $T_\Sigma : \mathbb{R}^2 \to \mathbb{R}^3$.

Step 3: Rotate/reflect. Let

$$U = \begin{pmatrix} \vec{u}_1 & \vec{u}_2 & \vec{u}_3 \end{pmatrix}$$

where $\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$ is the orthonormal basis for $\mathbb{R}^3$ found in P3. Use geometric reasoning to convince yourself that T_U is either a rotation or reflection transformation. *(Bonus: think about what computations you would need to perform to describe this rotation transformation explicitly).*

P5. Use geometric reasoning, along with your work from P4, to show that

$$A = U\Sigma Q^\top,$$

where U and Q are orthogonal matrices, and Σ is a "block-diagonal" matrix. Discuss how this decomposition describes the transformation $T_A : \mathbb{R}^2 \to \mathbb{R}^3$ as a rotation/reflection, followed by dilation/embedding, followed by another rotation/reflection.

Definition 12.9. Let A be an $m \times n$ matrix and $\{\vec{v}_1, \ldots, \vec{v}_n\}$ be an orthonormal basis for $\mathbb{R}^n$ of eigenvectors for $A^\top A$, as above. The SINGULAR VALUES of A are ...

Proposition 12.10. Let A be an $m \times n$ matrix and $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of $A^\top A$. Then, $\lambda_i > 0$ and the singular values of A are given by $\sigma_i = \sqrt{\lambda_i}$.

Prove Proposition 12.10.

Lecture Activity 12.4. Find the singular values of the following matrices. Given your calculations, what can you say about the corresponding transformations?

P1. $A = \begin{pmatrix} 2 & 0 \\ 0 & 2 \\ 1 & 0 \end{pmatrix}$

P2. $B = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & -1 \\ 1 & 1 & 2 \end{pmatrix}$

P3. $C = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$