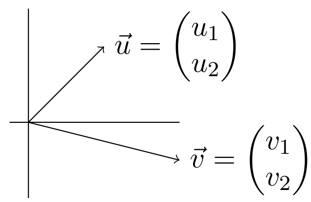
CHAPTER 11 ACTIVITY PACKET *solutions*

Instructions. This packet is due on Quercus no later than **11:59pm on Monday, March 30th**. Please complete your work directly on this packet. We will spend time together during lecture working on most or all of the activities in this packet. You are responsible for completing all portions of this packet, including lecture activities not discussed in class, and completing the definitions included in the packet. Solutions will be posted to the course website after the assignment due date.

Lecture Activity 11.1. Let $\vec{u}$ and $\vec{v}$ be the vectors drawn below.



P1. Find a formula for the length of $\vec{u}$ and $\vec{v}$.

Solution. We can find a formula for the length of a vector using the Pythagorean Theorem. We have that the length of the vector $\vec{u}$ is equal to $\sqrt{u_1^2 + u_2^2}$. Similarly, the length of the vector $\vec{v}$ is equal to $\sqrt{v_1^2 + v_2^2}$.

P2. Find a formula for the distance between $\vec{u}$ and $\vec{v}$. That is, find the distance between the points (u_1, u_2) and (v_1, v_2).

Solution. Observe that the distance between the two vectors is equal to the length of the vector $\vec{u} - \vec{v}$, and so the distance between $\vec{u}$ and $\vec{v}$ is equal to $\sqrt{(u_1 - v_1)^2 + (u_2 - v_2)^2}$

Definition 11.1. Let $\mathcal{B}$ be a basis for $\mathbb{R}^n$ and let $\vec{u}$ and $\vec{v}$ be vectors in $\mathbb{R}^n$ with $\mathcal{B}$-coordinates

$$[\vec{u}]_{\mathcal{B}} = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix} \text{ and } [\vec{v}]_{\mathcal{B}} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}.$$

The DOT PRODUCT of $\vec{u}$ and $\vec{v}$ with respect to the basis $\mathcal{B}$ is the scalar

$$[\vec{u}]_{\mathcal{B}} \cdot [\vec{v}]_{\mathcal{B}} := u_1 v_1 + u_2 v_2 + \cdots + u_n v_n.$$

Lemma 11.4. Let $\vec{u}, \vec{v}$ and $\vec{w}$ be vectors in $\mathbb{R}^n$ and let $c \in \mathbb{R}$ be a scalar. Then, the dot product satisfies the following properties:

1. *Commutativity:* $\boxed{\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}}$

2. *Distributivity with Addition:* $\boxed{(\vec{u} + \vec{v}) \cdot \vec{w} = \vec{u} \cdot \vec{w} + \vec{v} \cdot \vec{w}}$

3. *Distributivity with Scalar Multiplication:* $\boxed{(c\vec{u}) \cdot \vec{v} = c(\vec{u} \cdot \vec{v})}$

Lecture Activity 11.2. Let $\vec{u}$ and $\vec{v}$ be vectors in $\mathbb{R}^2$ and define the following notation

$$\|\vec{u}\| := \sqrt{\vec{u} \cdot \vec{u}}.$$

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P1. Show that the length of $\vec{u}$ is equal to $\sqrt{\vec{u} \cdot \vec{u}}$.

Solution. Using our work from Lecture Activity 11.1, we have the the length of $\vec{u}$ is equal to

$$\sqrt{u_1^2 + u_2^2} = \sqrt{u_1 u_1 + u_2 u_2} = \sqrt{\vec{u} \cdot \vec{u}} = \|\vec{u}\|.$$

P2. Show that the distance between $\vec{u}$ and $\vec{v}$ is equal to $\|\vec{u} - \vec{v}\|$.

Solution. From Lecture Activity 11.1, we have that the distance between $\vec{u}$ and $\vec{v}$ is

$$\begin{aligned} \sqrt{(u_1 - v_1)^2 + (u_2 - v_2)^2} &= \sqrt{(u_1 - v_1)(u_1 - v_1) + (u_2 - v_2)(u_2 - v_2)} \\ &= \left\| \begin{pmatrix} u_1 - v_1 \\ u_2 - v_2 \end{pmatrix} \right\| \\ &= \|\vec{u} - \vec{v}\| \end{aligned}$$

P3. Show that smaller of the two angles between $\vec{u}$ and $\vec{v}$ is equal to

$$\theta = \arccos \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\| \|\vec{v}\|}.$$

(Hint: use law of cosines, along with Lemma 11.4).

Solution. The law of cosines tell us that

$$\|\vec{u} - \vec{v}\|^2 = \|\vec{u}\|^2 + \|\vec{v}\|^2 - 2\|\vec{u}\|\|\vec{v}\| \cos\theta.$$

Using Proposition 11.4 we can rewrite the left-hand side as

$$\begin{aligned} \|\vec{u} - \vec{v}\|^2 &= (\vec{u} - \vec{v}) \cdot (\vec{u} - \vec{v}) \\ &= \vec{u} \cdot \vec{u} - 2\vec{u} \cdot \vec{v} + \vec{v} \cdot \vec{v} \\ &= \|\vec{u}\|^2 - 2\vec{u} \cdot \vec{v} + \|\vec{v}\|^2 \end{aligned}$$

Plugging this into the equation above gives

$$\begin{aligned} &\|\vec{u}\|^2 - 2\vec{u} \cdot \vec{v} + \|\vec{v}\|^2 = \|\vec{u}\|^2 + \|\vec{v}\|^2 - 2\|\vec{u}\|\|\vec{v}\| \cos\theta \\ &\Rightarrow -2\vec{u} \cdot \vec{v} = -2\|\vec{u}\|\|\vec{v}\| \cos\theta \\ &\Rightarrow \cos\theta = \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\|\|\vec{v}\|}. \end{aligned}$$

And since we're taking the *smaller* of the two angles between $\vec{u}$ and $\vec{v}$ we know that $0 \leq \theta < 180°$, and so we have

$$\theta = \arccos \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\| \|\vec{v}\|}.$$

P4. Use P3 to show that $\vec{u}$ and $\vec{v}$ are perpendicular if and only if $\vec{u} \cdot \vec{v} = 0$.

Solution. Observe that $\vec{u}$ and $\vec{v}$ are perpendicular if and only if the smaller of the two angles between them is $\theta = 90°$. From P1, we have

$$0 = \vec{u} \cdot \vec{v} \Leftrightarrow \cos(\theta) = 0$$

for $0 \leq \theta < 180°$, which occurs if and only if $\theta = 90°$.

Definition 11.5. Let $\vec{u}$ and $\vec{v}$ be vectors in $\mathbb{R}^n$.

1. The NORM of a vector $\vec{u}$ in $\mathbb{R}^n$ is ...

$$\|\vec{u}\| := \sqrt{\vec{u} \cdot \vec{u}}.$$

2. The DISTANCE between vectors $\vec{u}$ and $\vec{v}$ is ...

$$d(\vec{u}, \vec{v}) := \|\vec{u} - \vec{v}\|.$$

3. We say that $\vec{u}$ and $\vec{v}$ are ORTHOGONAL if ...

$$\vec{u} \cdot \vec{v} = 0.$$

Lecture Activity 11.3. Determine which of the following pairs of vectors $\vec{u}$ and $\vec{v}$ are orthogonal.

P1. $\vec{u} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ and $\vec{v} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$

Solution. We have

$$\vec{u} \cdot \vec{v} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 2 \end{pmatrix} = (2)(-1) + (1)(2) = 0,$$

and so the vectors are orthogonal.

P2. $\vec{u} = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$ and $\vec{v} = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$

Solution. We have

$$\vec{u} \cdot \vec{v} = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} = (-1)(2) + (0)(1) + (1)(1) = -1 \neq 0,$$

and so the vectors are **not** orthogonal.

P3. $\vec{u} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 1 \end{pmatrix}$ and $\vec{v} = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 1 \end{pmatrix}$

Solution. We have

$$\vec{u} \cdot \vec{v} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -1 \\ 0 \\ 1 \end{pmatrix} = (1)(1) + (2)(-1) + (3)(0) + (1)(1) = 0,$$

and so the vectors are orthogonal.

Definition 11.6. A basis $\mathcal{B} = \{\vec{b}_1, \vec{b}_2, \ldots, \vec{b}_n\}$ is ORTHOGONAL if ...

$\vec{b}_i \cdot \vec{b}_j = 0$ for every $i \neq j$.

An basis $\mathcal{B}$ is called orthonormal if it's orthogonal and ...

$\|\vec{b}_i\| = 1$ for every $\vec{b}_i$ in $\mathcal{B}$

Lecture Activity 11.4. Consider the bases $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{D}$ for $\mathbb{R}^2$ given by

$$\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}, \mathcal{C} = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\} \text{ and } \mathcal{D} = \left\{ \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}, \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix} \right\},$$

P1. Verify that $\mathcal{B}$ is not orthogonal, $\mathcal{C}$ is orthogonal but not orthonormal, and that $\mathcal{D}$ is orthonormal.

Solution. For basis $\mathcal{B}$, note that

$$\begin{pmatrix} 1 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} = 3 \neq 0.$$

Since the basis vectors are not orthogonal, the basis $\mathcal{B}$ is not orthogonal either.

For basis $\mathcal{C}$, note that

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -1 \end{pmatrix} = 0.$$

So, the basis $\mathcal{C}$ is also. However, the vectors have norm $\sqrt{2} \neq 1$, and so the basis is not orthonormal. For basis $\mathcal{D}$, note that

$$\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} \cdot \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix} = 0$$

and so $\mathcal{D}$ is orthogonal. We also calculate that the vectors in $\mathcal{D}$ all have norm 1, and so the basis $\mathcal{D}$ is orthonormal.

P2. Calculate $\vec{x} \cdot \vec{y}$ given that $[\vec{x}]_{\mathcal{B}} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ and $[\vec{y}]_{\mathcal{B}} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$.

Solution. We have

$$\begin{aligned} \vec{x} \cdot \vec{y} &= \left(x_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + x_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right) \cdot \left(y_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + y_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right) \\ &= x_1 y_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \end{pmatrix} + x_1 y_2 \begin{pmatrix} 1 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} + x_2 y_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \end{pmatrix} + x_2 y_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ &= 5x_1 y_1 + 3x_1 y_2 + 3x_2 y_1 + 2x_2 y_2. \end{aligned}$$

P3. Calculate $\vec{x} \cdot \vec{y}$ given that $[\vec{x}]_{\mathcal{C}} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ and $[\vec{y}]_{\mathcal{C}} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$.

Solution. We have

$$\begin{aligned}
\vec{x} \cdot \vec{y} &= \left(x_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + x_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right) \cdot \left(y_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + y_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right) \\
&= x_1 y_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} + x_1 y_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -1 \end{pmatrix} + x_2 y_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} + x_2 y_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -1 \end{pmatrix} \\
&= 2x_1 y_1 + 2x_2 y_2 \\
&= 2(x_1 y_1 + x_2 y_2).
\end{aligned}$$

P4. Calculate $\vec{x} \cdot \vec{y}$ given that $[\vec{x}]_{\mathcal{D}} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ and $[\vec{y}]_{\mathcal{D}} = \begin{pmatrix} \vec{y}_1 \\ \vec{y}_2 \end{pmatrix}$.

Solution. We have

$$\begin{aligned}
\vec{x} \cdot \vec{y} &= \left(x_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + x_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right) \cdot \left(y_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + y_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right) \\
&= x_1 y_1 \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} \cdot \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} + x_1 y_2 \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} \cdot \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix} + x_2 y_1 \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix} \cdot \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} + x_2 y_2 \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix} \cdot \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix} \\
&= x_1 y_1 + x_2 y_2.
\end{aligned}$$

P5. What did you notice in your calculations?

Solution. Observe that $\vec{x} \cdot \vec{y} = [\vec{x}]_{\mathcal{D}} \cdot [\vec{x}]_{\mathcal{D}}$. That is, orthonormal bases preserve the dot product.

Proposition 11.7. Let $\mathcal{B}$ be an orthonormal basis for $\mathbb{R}^n$ and take any vectors $\vec{x}, \vec{y}$ in $\mathbb{R}^n$. Then

$$[\vec{x}]_{\mathcal{B}} \cdot [\vec{y}]_{\mathcal{B}} = \vec{x} \cdot \vec{y}.$$

Proof. Suppose that $\mathcal{B} = \{\vec{b}_1, \vec{b}_2, \ldots, \vec{b}_n\}$ is an orthonormal basis for $\mathbb{R}^n$ and write

$$[\vec{x}]_{\mathcal{B}} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \text{ and } [\vec{y}]_{\mathcal{B}} = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}.$$

That is, $\vec{x} = x_1\vec{b}_1 + \cdots + x_n\vec{b}_n$ and $\vec{y} = y_1\vec{b}_1 + \cdots + y_n\vec{b}_n$, and so we have

$$\vec{x} \cdot \vec{y} = (x_1\vec{b}_1 + \cdots + x_n\vec{b}_n) \cdot (y_1\vec{b}_1 + \cdots + y_n\vec{b}_n).$$

Complete the proof: use Lemma 11.4 to show that $\vec{x} \cdot \vec{y} = x_1y_1 + \cdots + x_ny_n$.

Using the distributive properties of the dot product (Lemma 11.4), gives

$$\begin{aligned} \vec{x} \cdot \vec{y} = & x_1y_1(\vec{b}_1 \cdot \vec{b}_1) + x_1y_2(\vec{b}_1 \cdot \vec{b}_2) + \cdots + x_1y_n(\vec{b}_1 \cdot \vec{b}_n) \\ & + x_2y_1(\vec{b}_2 \cdot \vec{b}_1) + x_2y_2(\vec{b}_2 \cdot \vec{b}_2) + \cdots + x_2y_n(\vec{b}_2 \cdot \vec{b}_n) \\ & \vdots \\ & + x_ny_1(\vec{b}_n \cdot \vec{b}_1) + x_ny_2(\vec{b}_n \cdot \vec{b}_2) + \cdots + x_ny_n(\vec{b}_n \cdot \vec{b}_n) \end{aligned}$$

But, since we know that $\vec{b}_i \cdot \vec{b}_j = 0$ whenever $i \neq j$ then our calculation above gives

$$\vec{x} \cdot \vec{y} = x_1y_1\vec{b}_1 \cdot \vec{b}_1 + \cdots x_ny_n\vec{b}_n \cdot \vec{b}_n.$$

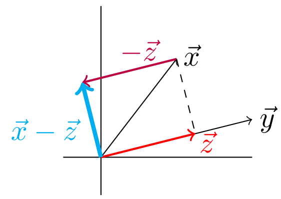
Finally, we have $\vec{b}_i \cdot \vec{b}_i = \|\vec{b}_i\|^2 = 1$, since our basis is orthonormal. So, we have

$$\vec{x} \cdot \vec{y} = x_1y_1 + \cdots + x_ny_n.$$

Thus, $\vec{x} \cdot \vec{y} = x_1y_1 + x_2y_2 + \cdots + x_ny_n = [\vec{x}]_{\mathcal{B}} \cdot [\vec{y}]_{\mathcal{B}}$, as needed. □

Lecture Activity 11.5. Let $\vec{x}$ and $\vec{y}$ be vectors in $\mathbb{R}^2$, and let $\vec{z}$ be the closest point in Span($\vec{y}$) to $\vec{x}$. That is, $\vec{z}$ is the point in Span($\vec{y}$) so that $d(\vec{x}, \vec{z})$ is as small as possible.

P1. Use the picture below to argue that $\vec{y}$ is orthogonal to $\vec{x} - \vec{z}$.



Solution. Observe that the vector $\vec{x} - \vec{z}$, when drawn at the origin, is parallel to the dashed line. By the Pythagorean Theorem, the dashed line must make a right angle with the line spanned by $\vec{y}$. Indeed, if this were not a right angle, the distance from $\vec{x}$ to Span($\vec{y}$) would be equal to the hypotenuse of a right triangle. But the hypotenuse of a right triangle is always larger than the length of its legs, and so $d(\vec{x}, \vec{z})$ would not be as small as possible.

P2. Since $\vec{z}$ is in Span($\vec{y}$), we can write $\vec{z} = c\vec{y}$ for some real number c. Use the previous part to show that

$$c = \frac{\vec{x} \cdot \vec{y}}{\vec{y} \cdot \vec{y}}.$$

Solution. From the previous part, we know that $\vec{y} \cdot (\vec{x} - \vec{z}) = 0$. But this means that $\vec{y} \cdot (\vec{x} - c\vec{y}) = 0$, and we have

$$\begin{aligned} \vec{y} \cdot (\vec{x} - c\vec{y}) &= 0 \\ \Rightarrow \vec{y} \cdot \vec{x} - c\vec{y} \cdot \vec{y} &= 0 \\ \Rightarrow \vec{x} \cdot \vec{y} &= c\vec{y} \cdot \vec{y} \\ c &= \frac{\vec{x} \cdot \vec{y}}{\vec{y} \cdot \vec{y}}. \end{aligned}$$

P3. Conclude that the closest point on Span($\vec{y}$) to $\vec{x}$ is given by

$$\vec{z} = \frac{\vec{x} \cdot \vec{y}}{\vec{y} \cdot \vec{y}}\vec{y}.$$

Recalling that $\vec{z} = c\vec{y}$, then using our calculation from P2 we have

$$\vec{z} = \frac{\vec{x} \cdot \vec{y}}{\vec{y} \cdot \vec{y}}\vec{y}.$$

Definition 11.8. For vectors $\vec{x}, \vec{y}$ in $\mathbb{R}^n$, the ORTHOGONAL PROJECTION of $\vec{x}$ onto $\vec{y}$ is ...

$$\operatorname{proj}_{\vec{y}} \vec{x} := \frac{\vec{x} \cdot \vec{y}}{\vec{y} \cdot \vec{y}} \vec{y}.$$

Theorem 11.9 (The Gram-Schmidt Process)**.** Every vector space has an orthogonal basis. Furthermore, if V is a vector subspace of $\mathbb{R}^n$ with basis $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m\}$, and we let

$$\begin{aligned}
\vec{u}_1 &= \vec{v}_1 \\
\vec{u}_2 &= \vec{v}_2 - \operatorname{proj}_{\vec{u}_1} \vec{v}_2 \\
\vec{u}_3 &= \vec{v}_3 - \operatorname{proj}_{\vec{u}_1} \vec{v}_3 - \operatorname{proj}_{\vec{u}_2} \vec{v}_3 \\
&\vdots \\
\vec{u}_m &= \vec{v}_m - \operatorname{proj}_{\vec{u}_1} \vec{v}_m - \operatorname{proj}_{\vec{u}_2} \vec{v}_m - \cdots - \operatorname{proj}_{\vec{u}_{m-1}} \vec{v}_m,
\end{aligned}$$

then, $\{\vec{u}_1, \ldots, \vec{u}_m\}$ is an orthogonal basis for V. Furthermore,

$$\left\{ \frac{\vec{u}_1}{\|\vec{u}_2\|}, \ldots, \frac{\vec{u}_m}{\|\vec{u}_m\|} \right\}$$

is an orthonormal basis for V.

Lecture Activity 11.6. In this activity, we'll see how to represent the matrix product using dot products for 2×2 matrices.

P1. Let

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \text{ and } B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$$

be 2×2 matrices. Using the definition of the matrix product, show that AB has column vectors $AB = \begin{pmatrix} \vec{u}_1 & \vec{u}_2 \end{pmatrix}$ where

$$\vec{u}_1 = \begin{pmatrix} a_{11}b_{11} + a_{12}b_{21} \\ a_{21}b_{11} + a_{22}b_{21} \end{pmatrix} \text{ and } \vec{u}_2 = \begin{pmatrix} a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{12} + a_{22}b_{22} \end{pmatrix}.$$

Solution. By Definition 6.3, we have $AB = \begin{pmatrix} \vec{u}_1 & \vec{u}_2 \end{pmatrix}$ where

$$\vec{u}_1 = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} b_{11} \\ b_{21} \end{pmatrix} = b_{11} \begin{pmatrix} a_{11} \\ a_{21} \end{pmatrix} + b_{21} \begin{pmatrix} a_{12} \\ a_{22} \end{pmatrix} = \begin{pmatrix} a_{11}b_{11} + a_{12}b_{21} \\ a_{21}b_{11} + a_{22}b_{21} \end{pmatrix}$$

and

$$\vec{u}_2 = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} b_{12} \\ b_{22} \end{pmatrix} = b_{12} \begin{pmatrix} a_{11} \\ a_{21} \end{pmatrix} + b_{22} \begin{pmatrix} a_{12} \\ a_{22} \end{pmatrix} = \begin{pmatrix} a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{12} + a_{22}b_{22} \end{pmatrix}.$$

P2. Let A and B be 2×2 matrices, and suppose that A has **row vectors** $A = \begin{pmatrix} \vec{a}_1^\top \\ \vec{a}_2^\top \end{pmatrix}$ and that B has column vectors $B = \begin{pmatrix} \vec{b}_1 & \vec{b}_2 \end{pmatrix}$. Show that

$$AB = \begin{pmatrix} \vec{a}_1 \cdot \vec{b}_1 & \vec{a}_1 \cdot \vec{b}_2 \\ \vec{a}_2 \cdot \vec{b}_1 & \vec{a}_2 \cdot \vec{b}_2 \end{pmatrix}.$$

Solution. We have

$$\vec{a}_1 \cdot \vec{b}_1 = \begin{pmatrix} a_{11} \\ a_{12} \end{pmatrix} \cdot \begin{pmatrix} b_{11} \\ b_{21} \end{pmatrix} = a_{11}b_{11} + a_{12}b_{21} \qquad \vec{a}_1 \cdot \vec{b}_2 = \begin{pmatrix} a_{11} \\ a_{12} \end{pmatrix} \cdot \begin{pmatrix} b_{12} \\ b_{22} \end{pmatrix} = a_{11}b_{12} + a_{12}b_{22}$$

$$\vec{a}_2 \cdot \vec{b}_1 = \begin{pmatrix} a_{21} \\ a_{22} \end{pmatrix} \cdot \begin{pmatrix} b_{11} \\ b_{21} \end{pmatrix} = a_{21}b_{11} + a_{22}b_{21} \qquad \vec{a}_2 \cdot \vec{b}_2 = \begin{pmatrix} a_{21} \\ a_{22} \end{pmatrix} \cdot \begin{pmatrix} b_{12} \\ b_{22} \end{pmatrix} = a_{21}b_{12} + a_{22}b_{22}$$

and since

$$AB = \begin{pmatrix} \vec{u}_1 & \vec{u}_2 \end{pmatrix}$$

then the result follows by P1.

Lecture Activity 11.7. Suppose that $\mathcal{B} = \{\vec{b}_1, \vec{b}_2\}$ is an orthonormal basis for $\mathbb{R}^2$, and consider the matrix $Q = \begin{pmatrix} \vec{b}_1 & \vec{b}_2 \end{pmatrix}$.

P1. Use Lemma 11.11 to observe that

$$Q^\top Q = \begin{pmatrix} \vec{b}_1 \cdot \vec{b}_1 & \vec{b}_1 \cdot \vec{b}_2 \\ \vec{b}_2 \cdot \vec{b}_1 & \vec{b}_2 \cdot \vec{b}_2 \end{pmatrix}.$$

Solution. Observe that $Q^\top$ has row vectors

$$Q^\top = \begin{pmatrix} \vec{b}_1^\top \\ \vec{b}_2^\top \end{pmatrix}$$

and so the result follows directly by Lemma 11.11 (or by our work in Lecture Activity 11.6).

P2. Use P1 to show that $Q^\top Q = I_2$.

Since $\mathcal{B}$ is an orthonormal basis, we have $\vec{b}_1 \cdot \vec{b}_2 = \vec{b}_2 \cdot \vec{b}_1 = 0$ and

$$\vec{b}_1 \cdot \vec{b}_1 = \|\vec{b}_1\|^2 = 1, \vec{b}_2 \cdot \vec{b}_2 = \|\vec{b}_2\|^2 = 1.$$

Thus, by P1 we have

$$Q^\top Q = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I_2.$$

P3. Conclude that Q is invertible with $Q^{-1} = Q^\top$.

Solution. Since $Q^\top Q = I_2$ then by Lemma 6.9 we have that $Q^{-1} = Q^\top$ and so Q is invertible.

Definition 11.13. We call an $n \times n$ matrix Q ORTHOGONAL if ...

its column vectors form an orthonormal basis for $\mathbb{R}^n$.

Equivalently, Q is called orthogonal if $Q^\top = Q^{-1}$.

Theorem 11.15. Let Q be an $n \times n$ orthogonal matrix. Then, for any $\vec{u}, \vec{v}$ in $\mathbb{R}^n$ we have

$$Q\vec{u} \cdot Q\vec{v} = \vec{u} \cdot \vec{v}.$$

In particular, $\|Q\vec{u}\| = \|\vec{u}\|$ and $\vec{u}$ is orthogonal to $\vec{v}$ if and only if $Q\vec{u}$ is orthogonal to $Q\vec{v}$.

Prove Theorem 11.15.

Proof. Suppose that $Q = \begin{pmatrix} \vec{b}_1 & \cdots & \vec{b}_n \end{pmatrix}$ is orthogonal, and let $\vec{u}$ and $\vec{v}$ be vectors in $\mathbb{R}^n$ with standard basis coordinates

$$\vec{u} = \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix} \text{ and } \vec{v} = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}.$$

Since $\mathcal{B}$ is an orthonormal basis, we have

$$\begin{aligned} Q\vec{v} \cdot Q\vec{w} &= (u_1\vec{b}_1 + \cdots + u_n\vec{b}_n) \cdot (v_1\vec{b}_1 + \cdots + v_n\vec{b}_n) \\ &= u_1v_1(\vec{b}_1 \cdots \vec{b}_1) + \cdots + u_nv_n(\vec{b}_n \cdot \vec{b}_n), \text{ since } \vec{b}_i \cdot \vec{b}_j = 0 \text{ when } i \neq j \\ &= u_1v_1 + \cdots + u_nv_n, \text{ since } \vec{b}_i \cdot \vec{b}_i = \|\vec{b}_i\|^2 = 1 \\ &= \vec{u} \cdot \vec{v}. \end{aligned}$$

Thus, we have

$$\|Q\vec{u}\| = \vec{u} \cdot \vec{u} = \|\vec{u}\|,$$

and since $Q\vec{u} \cdot Q\vec{v} = \vec{u} \cdot \vec{v}$ then we have $\vec{u} \cdot \vec{v} = 0 \Leftrightarrow Q\vec{u} \cdot Q\vec{v} = 0$.

□