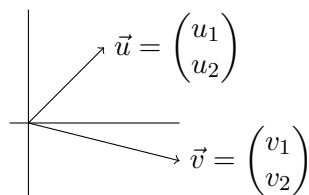


**Instructions.** This packet is due on Quercus no later than **11:59pm on Monday, March 30th**. Please complete your work directly on this packet. We will spend time together during lecture working on most or all of the activities in this packet. You are responsible for completing all portions of this packet, including lecture activities not discussed in class, and completing the definitions included in the packet. Solutions will be posted to the course website after the assignment due date.

**Lecture Activity 11.1.** Let  $\vec{u}$  and  $\vec{v}$  be the vectors drawn below.



P1. Find a formula for the length of  $\vec{u}$  and  $\vec{v}$ .

P2. Find a formula for the distance between  $\vec{u}$  and  $\vec{v}$ . That is, find the distance between the points  $(u_1, u_2)$  and  $(v_1, v_2)$ .

**Definition 11.1.** Let  $\mathcal{B}$  be a basis for  $\mathbb{R}^n$  and let  $\vec{u}$  and  $\vec{v}$  be vectors in  $\mathbb{R}^n$  with  $\mathcal{B}$ -coordinates

$$[\vec{u}]_{\mathcal{B}} = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix} \text{ and } [\vec{v}]_{\mathcal{B}} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}.$$

The DOT PRODUCT of  $\vec{u}$  and  $\vec{v}$  with respect to the basis  $\mathcal{B}$  is the scalar

**Lemma 11.4.** Let  $\vec{u}, \vec{v}$  and  $\vec{w}$  be vectors in  $\mathbb{R}^n$  and let  $c \in \mathbb{R}$  be a scalar. Then, the dot product satisfies the following properties:

1. *Commutativity:*  $\vec{u} \cdot \vec{v} =$   $\vec{v} \cdot \vec{u}$

2. *Distributivity with Addition:*  $(\vec{u} + \vec{v}) \cdot \vec{w} =$   $\vec{u} \cdot \vec{w} + \vec{v} \cdot \vec{w}$

3. *Distributivity with Scalar Multiplication:*  $(c\vec{u}) \cdot \vec{v} =$   $c(\vec{u} \cdot \vec{v})$

**Lecture Activity 11.2.** Let  $\vec{u}$  and  $\vec{v}$  be vectors in  $\mathbb{R}^2$  and define the following notation

$$\|\vec{u}\| := \sqrt{\vec{u} \cdot \vec{u}}.$$

P1. Show that the length of  $\vec{u}$  is equal to  $\|\vec{u}\|$ .

P2. Show that the distance between  $\vec{u}$  and  $\vec{v}$  is equal to  $\|\vec{u} - \vec{v}\|$ .

P3. Show that smaller of the two angles between  $\vec{u}$  and  $\vec{v}$  is equal to

$$\theta = \arccos \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\| \|\vec{v}\|}.$$

(Hint: use law of cosines, along with Lemma 11.4).

P4. Use P3 to show that  $\vec{u}$  and  $\vec{v}$  are perpendicular if and only if  $\vec{u} \cdot \vec{v} = 0$ .

**Definition 11.5.** Let  $\vec{u}$  and  $\vec{v}$  be vectors in  $\mathbb{R}^n$ .

1. The NORM of a vector  $\vec{u}$  in  $\mathbb{R}^n$  is ...

2. The DISTANCE between vectors  $\vec{u}$  and  $\vec{v}$  is ...

3. We say that  $\vec{u}$  and  $\vec{v}$  are ORTHOGONAL if ...

**Lecture Activity 11.3.** Determine which of the following pairs of vectors  $\vec{u}$  and  $\vec{v}$  are orthogonal.

P1.  $\vec{u} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$  and  $\vec{v} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$

P2.  $\vec{u} = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$  and  $\vec{v} = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$

P3.  $\vec{u} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 1 \end{pmatrix}$  and  $\vec{v} = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 1 \end{pmatrix}$

**Definition 11.6.** A basis  $\mathcal{B} = \{\vec{b}_1, \vec{b}_2, \dots, \vec{b}_n\}$  is ORTHOGONAL if ...

An basis  $\mathcal{B}$  is called orthonormal if it's orthogonal and ...

**Lecture Activity 11.4.** Consider the bases  $\mathcal{B}$ ,  $\mathcal{C}$  and  $\mathcal{D}$  for  $\mathbb{R}^2$  given by

$$\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}, \mathcal{C} = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\} \text{ and } \mathcal{D} = \left\{ \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}, \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix} \right\},$$

P1. Verify that  $\mathcal{B}$  is not orthogonal,  $\mathcal{C}$  is orthogonal but not orthonormal, and that  $\mathcal{D}$  is orthonormal.

P2. Calculate  $\vec{x} \cdot \vec{y}$  given that  $[\vec{x}]_{\mathcal{B}} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$  and  $[\vec{y}]_{\mathcal{B}} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$ .

P3. Calculate  $\vec{x} \cdot \vec{y}$  given that  $[\vec{x}]_{\mathcal{C}} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$  and  $[\vec{y}]_{\mathcal{C}} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$ .

P4. Calculate  $\vec{x} \cdot \vec{y}$  given that  $[\vec{x}]_{\mathcal{D}} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$  and  $[\vec{y}]_{\mathcal{D}} = \begin{pmatrix} \vec{y}_1 \\ \vec{y}_2 \end{pmatrix}$ .

P5. What did you notice in your calculations?

**Proposition 11.7.** Let  $\mathcal{B}$  be an orthonormal basis for  $\mathbb{R}^n$  and take any vectors  $\vec{x}, \vec{y}$  in  $\mathbb{R}^n$ . Then

$$[\vec{x}]_{\mathcal{B}} \cdot [\vec{y}]_{\mathcal{B}} = \vec{x} \cdot \vec{y}.$$

*Proof.* Suppose that  $\mathcal{B} = \{\vec{b}_1, \vec{b}_2, \dots, \vec{b}_n\}$  is an orthonormal basis for  $\mathbb{R}^n$  and write

$$[\vec{x}]_{\mathcal{B}} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \text{ and } [\vec{y}]_{\mathcal{B}} = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}.$$

That is,  $\vec{x} = x_1\vec{b}_1 + \dots + x_n\vec{b}_n$  and  $\vec{y} = y_1\vec{b}_1 + \dots + y_n\vec{b}_n$ , and so we have

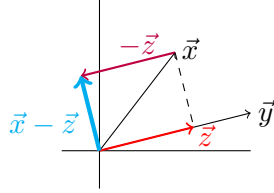
$$\vec{x} \cdot \vec{y} = (x_1\vec{b}_1 + \dots + x_n\vec{b}_n) \cdot (y_1\vec{b}_1 + \dots + y_n\vec{b}_n).$$

**Complete the proof: use Lemma 11.4 to show that  $\vec{x} \cdot \vec{y} = x_1y_1 + \dots + x_ny_n$ .**

Thus,  $\vec{x} \cdot \vec{y} = x_1y_1 + x_2y_2 + \dots + x_ny_n = [\vec{x}]_{\mathcal{B}} \cdot [\vec{y}]_{\mathcal{B}}$ , as needed. □

**Lecture Activity 11.5.** Let  $\vec{x}$  and  $\vec{y}$  be vectors in  $\mathbb{R}^2$ , and let  $\vec{z}$  be the closest point in  $\text{Span}(\vec{y})$  to  $\vec{x}$ . That is,  $\vec{z}$  is the point in  $\text{Span}(\vec{y})$  so that  $d(\vec{x}, \vec{z})$  is as small as possible.

P1. Use the picture below to argue that  $\vec{y}$  is orthogonal to  $\vec{x} - \vec{z}$ .



P2. Since  $\vec{z}$  is in  $\text{Span}(\vec{y})$ , we can write  $\vec{z} = c\vec{y}$  for some real number  $c$ . Use the previous part to show that

$$c = \frac{\vec{x} \cdot \vec{y}}{\vec{y} \cdot \vec{y}}.$$

P3. Conclude that the closest point on  $\text{Span}(\vec{y})$  to  $\vec{x}$  is given by

$$\vec{z} = \frac{\vec{x} \cdot \vec{y}}{\vec{y} \cdot \vec{y}} \vec{y}.$$

**Definition 11.8.** For vectors  $\vec{x}, \vec{y}$  in  $\mathbb{R}^n$ , the ORTHOGONAL PROJECTION of  $\vec{x}$  onto  $\vec{y}$  is ...

**Theorem 11.9** (The Gram-Schmidt Process). Every vector space has an orthogonal basis. Furthermore, if  $V$  is a vector subspace of  $\mathbb{R}^n$  with basis  $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_m\}$ , and we let

then,  $\{\vec{u}_1, \dots, \vec{u}_m\}$  is an orthogonal basis for  $V$ . Furthermore,

is an orthonormal basis for  $V$ .

**Lecture Activity 11.6.** In this activity, we'll see how to represent the matrix product using dot products for  $2 \times 2$  matrices.

P1. Let

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \text{ and } B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$$

be  $2 \times 2$  matrices. Using the definition of the matrix product, show that  $AB$  has column vectors  $AB = (\vec{u}_1 \ \vec{u}_2)$  where

$$\vec{u}_1 = \begin{pmatrix} a_{11}b_{11} + a_{12}b_{21} \\ a_{21}b_{11} + a_{22}b_{21} \end{pmatrix} \text{ and } \vec{u}_2 = \begin{pmatrix} a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{12} + a_{22}b_{22} \end{pmatrix}.$$

P2. Let  $A$  and  $B$  be  $2 \times 2$  matrices, and suppose that  $A$  has **row vectors**  $A = \begin{pmatrix} \vec{a}_1^\top \\ \vec{a}_2^\top \end{pmatrix}$  and that  $B$  has column vectors  $B = \begin{pmatrix} \vec{b}_1 & \vec{b}_2 \end{pmatrix}$ . Show that

$$AB = \begin{pmatrix} \vec{a}_1 \cdot \vec{b}_1 & \vec{a}_1 \cdot \vec{b}_2 \\ \vec{a}_2 \cdot \vec{b}_1 & \vec{a}_2 \cdot \vec{b}_2 \end{pmatrix}.$$

**Lecture Activity 11.7.** Suppose that  $\mathcal{B} = \{\vec{b}_1, \vec{b}_2\}$  is an orthonormal basis for  $\mathbb{R}^2$ , and consider the matrix  $Q = \begin{pmatrix} \vec{b}_1 & \vec{b}_2 \end{pmatrix}$ .

P1. Use Lemma 11.11 to observe that

$$Q^\top Q = \begin{pmatrix} \vec{b}_1 \cdot \vec{b}_1 & \vec{b}_1 \cdot \vec{b}_2 \\ \vec{b}_2 \cdot \vec{b}_1 & \vec{b}_2 \cdot \vec{b}_2 \end{pmatrix}.$$

P2. Use P1 to show that  $Q^\top Q = I_2$ .

P3. Conclude that  $Q$  is invertible with  $Q^{-1} = Q^\top$ .

**Definition 11.13.** We call an  $n \times n$  matrix  $Q$  ORTHOGONAL if ...

Equivalently,  $Q$  is called orthogonal if  $Q^\top = Q^{-1}$ .

**Theorem 11.15.** Let  $Q$  be an  $n \times n$  orthogonal matrix. Then, for any  $\vec{u}, \vec{v}$  in  $\mathbb{R}^n$  we have

$$Q\vec{u} \cdot Q\vec{v} = \vec{u} \cdot \vec{v}.$$

In particular,  $\|Q\vec{u}\| = \|\vec{u}\|$  and  $\vec{u}$  is orthogonal to  $\vec{v}$  if and only if  $Q\vec{u}$  is orthogonal to  $Q\vec{v}$ .

**Prove Theorem 11.15.**