Solution to Practice Term Test

Section B

B1 The augmented matrix of this system of linear equations is

$$A = \left(\begin{array}{ccc|c} 1 & 0 & 2 & a \\ 0 & 1 & 1 & 1 \\ 1 & 2 & 4 & 0 \end{array}\right),$$

which is row equivalent to

$$\left(\begin{array}{ccc|c} 1 & 0 & 2 & a \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & -a-2 \end{array}\right).$$

By Rouché-Capelli theorem, the system of linear equations is consistent if and only if there is no pivot in the last column of $\mathrm{rref}(A)$, and this condition is equivalent to $-a-2=0$. So the system is consistent if and only if $a=-2$.

B2

(a) Taking $x=1, y=0$, one has $\begin{pmatrix} 2x+2y \\ x+y \\ x+y \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} \in V$. However, the scalar multiplication $-1 \cdot \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ -1 \\ -1 \end{pmatrix}$ is not in V, as $x+y \neq -1$ for all $x, y \geq 0$, thus the set V is not a vector space.

(b) For any vector $\vec{x} \in W$, $\vec{x}$ and $\begin{pmatrix} 1 \\ 7 \end{pmatrix}$ are linearly dependent. By Definition 2.10, one has $\vec{x} \in \mathrm{Span}(\begin{pmatrix} 1 \\ 7 \end{pmatrix})$ or $\begin{pmatrix} 1 \\ 7 \end{pmatrix} \in \mathrm{Span}(\vec{x})$.

If $\begin{pmatrix} 1 \\ 7 \end{pmatrix} \in \mathrm{Span}(\vec{x})$, then there is some nonzero scalar $c \in \mathbb{R}$ such that $\begin{pmatrix} 1 \\ 7 \end{pmatrix} = c\vec{x}$. This implies $\vec{x} = c^{-1}\begin{pmatrix} 1 \\ 7 \end{pmatrix} \in \mathrm{Span}(\begin{pmatrix} 1 \\ 7 \end{pmatrix})$, so we have $V = \mathrm{Span}(\begin{pmatrix} 1 \\ 7 \end{pmatrix})$, and it is a vector space. The set $\left\{\begin{pmatrix} 1 \\ 7 \end{pmatrix}\right\}$ is linearly independent, thus it is a basis for V, and one has $\dim(W) = 1$.

(c) Any vector $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \in U$ is of the form

$$\begin{pmatrix} -2y-3z \\ y \\ z \end{pmatrix} = y \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} + z \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix} \text{ for some } y, z \in \mathbb{R}.$$

This implies that $U = \mathrm{Span}(\begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix})$, and it is a vector space.

The matrix

$$\mathrm{rref}\left(\begin{pmatrix} -2 & -3 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}\right) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$$

has a pivot in every column. According to Proposition 2.12, the spanning set $\left\{ \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix} \right\}$ of U is linearly independent. So it is a basis for U, and we have $\dim(U) = 2$.

B3

(a) Let $\vec{e}_1, \vec{e}_2, \vec{e}_3, \vec{e}_4$ be the standard basis for $\mathbb{R}^4$. Using the equation defining F, we have

$$F(\vec{e}_1) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \ F(\vec{e}_2) = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \ F(\vec{e}_3) = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \text{ and } F(\vec{e}_4) = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}.$$

By Theorem 4.8, the defining matrix for F is

$$M_F = \begin{pmatrix} F(\vec{e}_1) & F(\vec{e}_2) & F(\vec{e}_3) & F(\vec{e}_4) \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

(b) The reduced row echelon form of M_F is

$$\mathrm{rref}(M_F) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix},$$

which has 3 pivot columns and 1 non-pivot column. By Theorem 5.8, we have $\mathrm{nullity}(F) = \mathrm{nullity}(M_F) = 1$.

(c) Since $\mathrm{rref}(M_F)$ has a column without pivot, F is not injective by Theorem 4.14. Since every row of $\mathrm{rref}(M_F)$ has a pivot, F is surjective by Theorem 4.17.

In conclusion, F is surjective and not injective.

B4

(a) Not enough information.

If we take both invertible matrices A and B to be $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, then $A + B = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ is invertible with inverse matrix $\begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix}$.

If we take invertible matrices A and B to be $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ respectively, then $A + B = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ is not invertible.

(b) We write B as $B = \left(\vec{b}_1 \cdots \vec{b}_n\right)$, then the matrix product AB equals $\left(A\vec{b}_1 \cdots A\vec{b}_n\right)$. Since the columns of B are linearly dependent, by Theorem 2.11 there is a non-trivial solution $(c_1, \ldots, c_n)$ to the equation $x_1\vec{b}_1 + \cdots + x_n\vec{b}_n = \vec{0}$.

We know matrix transformations are linear transformations, so

$$c_1 A\vec{b}_1 + \cdots + c_n A\vec{b}_n = A\left(c_1\vec{b}_1 + \cdots + c_n\vec{b}_n\right) = A\vec{0} = \vec{0},$$

which implies that the column vectors of AB are also linearly dependent. According to Proposition 2.12, $\operatorname{rref}(AB)$ has at least one non-pivot column, so this matrix cannot be the identity matrix I_n. By the invertible matrix theorem, AB is not invertible.

(c) Consider the vector equation $x_1T(\vec{v}_1) + \cdots + x_nT(\vec{v}_n) = \vec{0}$, which can be rewritten as $T(x_1\vec{v}_1 + \cdots + x_n\vec{v}_n) = \vec{0}$. As T is injective, we have $x_1\vec{v}_1 + \cdots + x_n\vec{v}_n = \vec{0}$. Because $\{\vec{v}_1, \ldots, \vec{v}_n\}$ is a basis for $\mathbb{R}^n$, one has $x_1 = \cdots = x_n = 0$. This shows that the vector equation has no non-trivial solution, thus the column vectors $T(\vec{v}_1), \ldots, T(\vec{v}_n)$ of A are linearly independent. By Proposition 2.12, the matrix $\operatorname{rref}(A)$ has a pivot in each column, so $\operatorname{rref}(A) = I_n$. By the invertible matrix theorem, A is invertible.

B5 This matrix A has one of the following forms:

$$A = \begin{pmatrix} 0 & 1 & 0 & a & 0 \\ 0 & 0 & 1 & b & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

with at least one of a and b other than 0 or 1, **or**

$$A = \begin{pmatrix} 0 & 1 & c & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \ c \neq 0, 1.$$

Section C

C1 Consider the vector equation

$$x_1(\vec{v}_0 + \vec{v}_1) + x_2(\vec{v}_0 + \vec{v}_2) + \cdots + x_m(\vec{v}_0 + \vec{v}_m) = \vec{0}.$$

The left-hand-side equals

$$(x_1 + \cdots + x_m)\vec{v}_0 + x_1\vec{v}_1 + \cdots + x_m\vec{v}_m.$$

Because $\{\vec{v}_0, \vec{v}_1, \ldots, \vec{v}_m\}$ is linearly independent, one has

$$x_1 + \cdots + x_m = 0,\ x_1 = 0, \ldots,\ x_m = 0.$$

This shows that the vector equation $x_1(\vec{v}_0 + \vec{v}_1) + x_2(\vec{v}_0 + \vec{v}_2) + \cdots + x_m(\vec{v}_0 + \vec{v}_m) = \vec{0}$ has no non-trivial solution, so $\{\vec{v}_0 + \vec{v}_1, \vec{v}_0 + \vec{v}_2, \ldots, \vec{v}_0 + \vec{v}_m\}$ is linearly independent.

C2 True.

Proof. Let r be the number of pivots in $\mathrm{rref}(A)$. Each row has at most 1 pivot, so r is not larger than the number of rows, *i.e.* $r \leq n$. The number of non-pivot columns in $\mathrm{rref}(A)$ is $m - r$. By Theorem 5.8, we have

$$\mathrm{nullity}(A) = m - r \geq m - n > 0. \qquad \square$$

C3 By Theorem 4.17, the surjectivity of T_A implies that every row of $\mathrm{rref}(A)$ has a pivot. As a consequence, $\mathrm{rref}(A)$ has n pivots, thus every column of $\mathrm{rref}(A)$ also has a pivot. Combining this with Theorem 4.14, we obtain that T_A is also injective.

By the invertible matrix theorem, the matrix B is an invertible matrix, so the corresponding matrix transformation T_B is bijective. In particular, T_B is injective.

For any vector $\vec{x} \in \ker(F)$, we have $\vec{0} = F(\vec{x}) = T_A(T_B(\vec{x}))$. As T_A is injective, $T_B(\vec{x}) = \vec{0}$. Because T_B is also injective, we have $\vec{x} = \vec{0}$ thus $\ker(F) = \{\vec{0}\}$.