General Information.

- The Final Exam will take place from **9am to 12pm on Monday, December 15th**.
- Details about the location of your sitting can be found on this page.
- The Final Exam will be cumulative. That is, all material from Chapters 1 through 11 of the course lecture notes may appear on the exam.
- This document contains a practice test (starting on page 2 of the pdf). While the problems are different than what will appear on the actual final exam, the instructions and format are identical.
- Please keep in mind that reviewing this practice exam **is not enough** to prepare you for the actual exam. The practice exam is only meant to give you an example of the style and format of the assessment.
- Solutions to the practice exam will be posted by the end of the day on **Tuesday, December 9th**.

Tips for Success

Here is a (non-exhaustive) list of tips that might help you prepare for the term test.

1. Start early, and spread your studying out over several days.
2. Study all definitions and proofs from the Chapter Activity Packets (CAP1-6).
3. Make flashcards for definitions. Consider using a Frayer model.
4. Use active recall techniques when studying proofs from the CAPs.
5. Complete the attached practice test on your own without help. Only ask for help after you've given yourself time to get stuck and have attempted to solve all problems on your own.
6. Redo material from the course (including Lecture Activities from the CAPs, Webwork, and Chapter Exercises). Don't just "review" or "look over" what we've done. **Rework problems without the solutions**. Only use the solutions to check your work.
7. Carefully review the feedback you were given on all Problem Set Quizzes.
8. Solve the remaining chapter exercises problems that were not assigned on the problem sets.
9. Use the additional resources list in the preface of the course lecture notes (located on page 6 of the pdf) to find new problems to solve.
10. Form a study group. Write new problems and exchange them with your classmates for extra practice.
11. Don't write anything down that you do not completely understand. Be stubborn. If you can't understand something after giving it a good effort, ask for help.
12. Attend office hours. Challenge yourself to ask questions and to let the instructor know when something doesn't make sense.

University of Toronto
Faculty of Arts and Sciences
PRACTICE MAT223H1S Final Exam

Fall 2025
Duration: 180 minutes
Aids Allowed: None

Name (First then Last): ______________________________

University Email Address: ______________________________@mail.utoronto.ca

Student Number: ______________________________

GENERAL INSTRUCTIONS:

- Fill out your name, student number, and email address at the top of this page.
- This test contains three sections:
  - **Section A** (9 points available) includes definition statements and theorem proofs.
  - **Section B** (24 points available) includes computational and multiple choice problems.
  - **Section C** (20 points available) includes conceptual and proof-based problems.

  Please read the instructions at the beginning of each section carefully.
- No calculators, notes, or electronics are permitted. Turn off and place all cell phones, smart watches, electronic devices, and unauthorized study materials in your bag under your desk. These devices may not be left in your pockets.
- Place your TCard on your desk so that it can be seen by the invigilators.
- All work must be completed in the space provided. There is additional space at the back of this packet if needed. Do not detach these pages.
- Please ask questions if anything is unclear.
- Once you've finished working, close your exam and then raise your hand. We will verify your name against your TCard and collect your exam.
- If you are still working when time is called, promptly close the test packet and wait for an invigilator to come collect your test.

SPECIAL INSTRUCTIONS:

- Write legibly and darkly. If we cannot read your work, we will not grade the problem.
- Erase or cross out any work you do not wish to have scored, and clearly indicate if there is work on another page you want scored.
- Fill in your bubbles completely.

Good: ● A ○ B   Bad: ◐ A ⊙ B ⊠ C

Section A.

INSTRUCTIONS:

1. The problems in this section will ask you to **complete a definition** or to **prove a theorem** from the course lecture notes.

2. Definitions must be stated precisely as they are in the course lecture notes (up to rewording). Each definition statement is worth **one point** and no partial credit will be given.

3. Theorem proofs will each be worth **five points**, which will be awarded using our standard rubric (which is available in the Section C instructions).

On exam day, in this section there will be **four definitions** (each worth 1 point) and **one scaffolded proof** (worth 5 points) chosen from the Chapter Activity Packets (CAP1-CAP11, including CAP11, part 2). Note that these questions will be scaffolded precisely as they appear in the Chapter Activity Packets.

Section B.

INSTRUCTIONS:

1. Each problem in this section is worth **three points**.
2. Problems with multiple parts will be worth one point each. Otherwise, no partial credit will be given.
3. You do not need to show your work or provide justification on any problem in Section B.
4. **Your answer must be placed in the answer box provided.**
5. We have provided extra space for your scratch work on each problem, but nothing outside of the answer box will be considered toward your score on the Section B problems.

B1. (3 points) For the systems of linear equations described below, determine whether the system has no solution, exactly one solution, or infinitely many solutions.

a) A system whose coefficient matrix is invertible.

> **Answer:** The system of linear equations has
>
> ○ No solutions ○ Exactly one solution ○ Infinitely many solutions

b) A system whose augmented matrix is invertible.

> **Answer:** The system of linear equations has
>
> ○ No solutions ○ Exactly one solution ○ Infinitely many solutions

c) The system with augmented matrix A^T, where A is the augmented matrix representing the system in part (a).

> **Answer:** The system of linear equations must have
>
> ○ No solutions ○ Exactly one solution ○ Infinitely many solutions

B2. (3 points) For each linear transformation defined below, determine whether the reduced row echelon form of its standard defining matrix has a pivot in every row, every column, both, or neither.

a) $F : \mathbb{R}^4 \to \mathbb{R}^3$ satisfying that $\{F(\vec{e}_1), F(\vec{e}_3), F(\vec{e}_4)\}$ is linearly independent.

Answer: rref(M_F) has ...

- ○ Pivots in every row and every column
- ○ Pivots in every row but not every column
- ○ Pivots in every column but not every row
- ○ None of the above

b) $G = T_Q$, where Q is an $n \times n$ orthogonal matrix.

Answer: rref(Q) has ...

- ○ Pivots in every row and every column
- ○ Pivots in every row but not every column
- ○ Pivots in every column but not every row
- ○ None of the above

c) $H : \mathbb{R}^2 \to \mathbb{R}^4$ with $H(\vec{e}_1) \neq \vec{0}$ and $H(\vec{e}_2) \neq \vec{0}$.

Answer: rref(M_H) has ...

- ○ Pivots in every row and every column
- ○ Pivots in every row but not every column
- ○ Pivots in every column but not every row
- ○ None of the above

B3. (3 points) Calculate the following determinants.

a) $\det(AB)$ where $A = \begin{pmatrix} 3 & 40 & -1 \\ 0 & 7 & 3 \\ 0 & 0 & -3 \end{pmatrix}$ and $B = A^T$, the transpose of A.

$\det(A) =$

b) $\det(G)$, where $G\colon \mathbb{R}^2 \to \mathbb{R}^2$ is the linear transformation which stretches vectors in the $\vec{e}_1 + \vec{e}_2$ direction by -3 and leaves the $\vec{e}_1 - \vec{e}_2$ direction unchanged.

$\det(F) =$

c) Let C be standard defining matrix of G from part (b). Is it possible that C similar to the matrix AB from part (a)?

○ Yes, it is possible ○ No, it is not possible

B4. (3 points) Determine which of the following matrices are invertible. If there is not enough information to determine whether the matrix is invertible or not invertible, select "could be either".

a) A 3×3 matrix N satisfying that N^3 is the zero matrix.

○ Is invertible ○ Is not invertible ○ Could be either

b) A symmetric matrix.

○ Is invertible ○ Is not invertible ○ Could be either

c) The defining matrix of the linear transformation in $\mathbb{R}^3$ that rotates vectors about the z-axis by an angle of θ and leaves vectors in the xy-plane fixed.

○ Is invertible ○ Is not invertible ○ Could be either

B5. (3 points) Let A be a 3×3 matrix with eigenvalues $0, 1, 2$. Find the eigenvalues of the following matrices:

a) The matrix A^2.

b) The matrix $A - I_3$.

c) The matrix $3(A^T)^2$.

B6. (3 points) Let $\mathcal{E}$ be the standard basis of $\mathbb{R}^3$. Consider the basis $\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}$.

a) Find $[\vec{v}]_{\mathcal{E}}$ given that $[\vec{v}]_{\mathcal{B}} = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$.

$$[\vec{v}]_{\mathcal{E}} = \begin{pmatrix} \square \\ \square \\ \square \end{pmatrix}$$

b) Find the change of basis matrix from $\mathcal{B}$ to $\mathcal{E}$.

$$M_{\mathcal{E} \leftarrow \mathcal{B}} = \begin{pmatrix} \square & \square & \square \\ \square & \square & \square \\ \square & \square & \square \end{pmatrix}$$

c) Find the change of basis matrix from $\mathcal{E}$ to $\mathcal{B}$.

$$M_{\mathcal{B} \leftarrow \mathcal{E}} = \begin{pmatrix} \square & \square & \square \\ \square & \square & \square \\ \square & \square & \square \end{pmatrix}$$

B7. (3 points) Let

$$A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{pmatrix}$$

.

a) Find the characteristic polynomial of A.

Answer: $\chi_A(x) =$

b) Find the dimension of the 1-eigenspace E_1.

Answer: $\dim E_1 =$

c) Find an invertible matrix C so that $C^{-1}AC$ is a diagonal matrix.

$$C = \begin{pmatrix} \square & \square & \square \\ \square & \square & \square \\ \square & \square & \square \end{pmatrix}$$

B8. (3 points) Determine which of the following statements are always true and which are always false. If there's not enough information to determine whether a statement is always true or always false, select "could be true or false".

a) If $\vec{v}$ and $\vec{w}$ are two vectors in $\mathbb{R}^n$ such that $\vec{v} \cdot \vec{w} = 0$, then $\{\vec{v}, \vec{w}\}$ is a linearly independent set.

○ Always true ○ Always false ○ Could be true or false

b) If a linear transformation $F : \mathbb{R}^3 \to \mathbb{R}^3$ preserves the angle between every pair of vectors, then its standard defining matrix M_F is orthogonal.

○ Always true ○ Always false ○ Could be true or false

c) Let V be a vector subspace of $\mathbb{R}^n$. Every basis of V has n elements.

○ Always true ○ Always false ○ Could be true or false

Section C.

INSTRUCTIONS:

1. Each problem in this section is worth **5 points**.
2. You must provide justification for all of your answers in Section C.
3. Points will be awarded based on the rubric below. Note that half points may be awarded, and further rubric items may be added to cover potential cases not outlined below.

Points	**Rubric**
5	Solution is presented with clear justification that is logically complete and correct. May include minor typos and computational errors if they do not majorly impact the argument. No important steps are missing or assumed. All assumptions and special cases have been covered. All suggestions for improvement come under the category of "improvements for clarity" rather than "correcting logical errors". Omission of details will be judged depending on context of the material, with simpler steps being acceptable for omission when covering more advanced topics.
4	Solution is close to full and complete, but contains either a computational error or an error in reasoning that majorly impacts the argument. This score is also appropriate for solutions that are mathematically sound but confusingly written.
3	Solution is incorrect, but understanding of the problem was demonstrated and student provided a clear outline of a potential approach with information about where they got stuck **-or-** solution is correct but justification is insufficient or so confusingly written that it cannot be followed with a reasonable amount of effort.
2	Solution is incorrect, but student demonstrated understanding of the problem **-or-** solution is correct and student did not provide justification for their answer.
1	Solution is incorrect and student did not demonstrate understanding of the problem, but did demonstrate some knowledge of relevant material.
0	Solution is incorrect or incomplete, and there was no demonstration of knowledge of relevant material.

C1. (5 points) Let A be an $m \times (n+1)$ matrix, and suppose that the system of linear equations in n variables with augmented matrix A has at least one solution. Show that the homogeneous system of linear equations in $n+1$ variables with coefficient matrix A has infinitely many solutions

Proof.

C2. (5 points) Let $B = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_6\}$ be a set of vectors in $\mathbb{R}^4$. Then B cannot be a basis of $\mathbb{R}^4$. **If true, provide a proof. If false, provide a counterexample, and justify why this is one.**

○ True ○ False

Proof or Counterexample:

C3. (5 points) Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation and $\mathcal{B}$ a basis for $\mathbb{R}^n$. Show that if the defining matrix A_F is invertible, then $A_{F,\mathcal{B}}$ is also invertible.

Proof.

C4. (5 points) Let A be an 6×7 matrix. Is it possible that the nullity of A equals the nullity of its transpose A^T? **If yes, find an example and prove that it is an example. If no, prove it.**

Proof or Example.

YOU MUST SUBMIT THIS PAGE.

If you would like work on this page scored, then clearly indicate to which question the work belongs and indicate on the page containing the original question that there is work on this page to score.

YOU MUST SUBMIT THIS PAGE.

If you would like work on this page scored, then clearly indicate to which question the work belongs and indicate on the page containing the original question that there is work on this page to score.

YOU MUST SUBMIT THIS PAGE.

If you would like work on this page scored, then clearly indicate to which question the work belongs and indicate on the page containing the original question that there is work on this page to score.

YOU MUST SUBMIT THIS PAGE.

If you would like work on this page scored, then clearly indicate to which question the work belongs and indicate on the page containing the original question that there is work on this page to score.