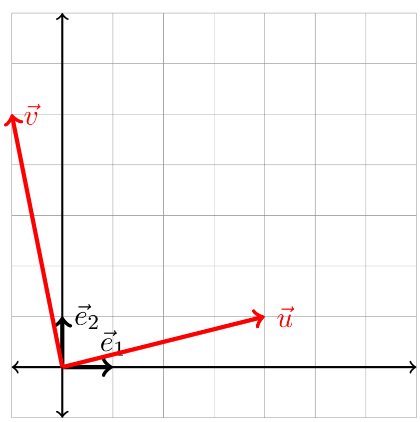
Instructions. This packet is due on Quercus no later than **11:59pm on Monday, November 10th**. Please complete your work directly on this packet. We will spend time together during lecture working on most or all of the activities in this packet. You are responsible for completing all portions of this packet, including lecture activities not discussed in class, and completing the definitions included in the packet. Solutions will be posted to the course website after the assignment due date.

Lecture Activity 9.1. The standard coordinate grid for $\mathbb{R}^2$ is drawn below. Explain how we can use the standard basis $\mathcal{E} = \{\vec{e}_1, \vec{e}_2\}$ to draw the grid lines below. Then, draw the vectors

$$\vec{u} = 4\vec{e}_1 + \vec{e}_2 \text{ and } \vec{v} = -\vec{e}_1 + 5\vec{e}_2$$

on the coordinate grid below.



Solution. Observe that $\text{Span}(\vec{e}_1)$ and $\text{Span}(\vec{e}_2)$ give the horizontal and vertical axes. The lines

$$\vec{e}_1 + \text{Span}(\vec{e}_2),\ 2\vec{e}_1 + \text{Span}(\vec{e}_2),\ 3\vec{e}_1 + \text{Span}(\vec{e}_2), \ldots$$

give our vertical grid lines, and the lines

$$\vec{e}_2 + \text{Span}(\vec{e}_1),\ 2\vec{e}_2 + \text{Span}(\vec{e}_1),\ 3\vec{e}_2 + \text{Span}(\vec{e}_1), \ldots$$

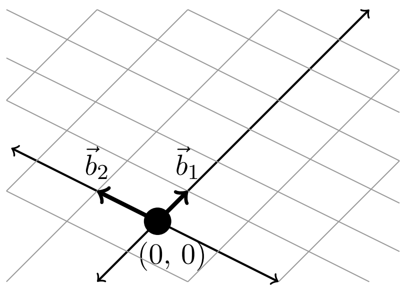
gives the horizontal grid lines.

Lecture Activity 9.2. Let $\mathcal{B} = \{\vec{b}_1, \vec{b}_2\}$ be the basis for $\mathbb{R}^2$ with

$$\vec{b}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \text{ and } \vec{b}_2 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}.$$

The image below shows how we can use $\mathcal{B}$ to create a "nonstandard" coordinate grid for $\mathbb{R}^2$.

P1. Explain how the vectors $\vec{b}_1, \vec{b}_2$ can be used to draw the grid lines below.



Solution. As before, $\text{Span}(\vec{b}_1)$ and $\text{Span}(\vec{b}_2)$ give the axis, and we obtain grid lines via

$$\vec{b}_1 + \text{Span}(\vec{b}_2),\ 2\vec{b}_1 + \text{Span}(\vec{b}_2), \ldots$$

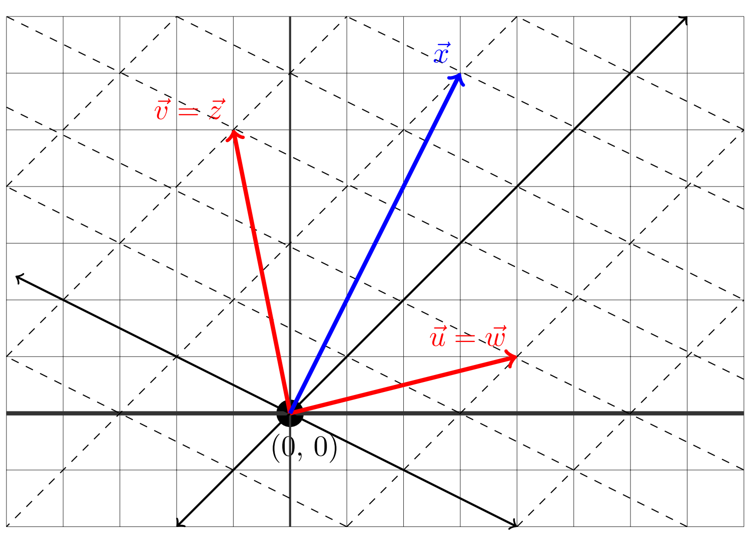
$$\vec{b}_2 + \text{Span}(\vec{b}_1),\ 2\vec{b}_2 + \text{Span}(\vec{b}_1), \ldots$$

P2. The following graph includes the standard coordinate grid defined by the standard basis $\mathcal{E}$ (drawn with solid lines) and the "nonstandard" coordinate grid defined by the basis $\mathcal{B}$ (drawn with dashed lines). Draw the following vectors on the graph below

$$\vec{u} = 4\vec{e}_1 + \vec{e}_2,\ \vec{v} = -\vec{e}_1 + 5\vec{e}_2$$

$$\vec{w} = 2\vec{b}_1 - \vec{b}_2,\ \vec{z} = 3\vec{b}_1 + 2\vec{b}_2$$

What do you notice?



P3. Consider the vector $\vec{x} = 3\vec{e}_1 + 6\vec{e}_2$. Use the graph from P2 to find real numbers x_1, x_2 so that $\vec{x} = x_1\vec{b}_1 + x_2\vec{b}_2$.

Solution. We've drawn $\vec{x}$ in blue on the graph above. Using the grid lines, we see that

$$\vec{x} = 5\vec{b}_1 + \vec{b}_2.$$

Definition 9.1. Let $\mathcal{B} = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$ be an ordered basis for a vector space V. Recall that every vector $\vec{x}$ in V can be written in the form

$$\vec{x} = x_1\vec{v}_1 + \cdots + x_n\vec{v}_n.$$

The $\mathcal{B}$-COORDINATES of $\vec{x}$ is the vector in $\mathbb{R}^n$ given by

$$\boxed{[\vec{x}]_{\mathcal{B}} := \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}.}$$

Lecture Activity 9.3. Let $\mathcal{B} = \{\vec{b}_1, \vec{b}_2, \vec{b}_3\}$ be the ordered basis for $\mathbb{R}^3$ where

$$\vec{b}_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \vec{b}_2 = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}, \text{ and } \vec{b}_3 = \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix}.$$

Find the $\mathcal{B}$-coordinates of the vector $\vec{v} = \begin{pmatrix} 1 \\ 3 \\ 4 \end{pmatrix}$.

Solution. We want to find the scalars $x_1, x_2, x_3 \in \mathbb{R}$ such that $\vec{v} = x_1\vec{b}_1 + x_2\vec{b}_2 + x_3\vec{b}_3$. This is a vector equation which we can express as a system of linear equations with augmented matrix

$$\left(\begin{array}{ccc|c} 1 & 2 & 0 & 1 \\ 1 & 1 & 1 & 3 \\ 0 & -1 & 3 & 4 \end{array}\right).$$

Row reducing, we find that the reduced row echelon form of this matrix is

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 \end{array}\right).$$

This gives us a unique solution, where $c_1 = 3, c_2 = -1$, and $c_3 = 1$. So,

$$[\vec{v}]_{\mathcal{B}} = \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix}.$$

Definition 9.4. Let $\mathcal{C}$ and $\mathcal{B}$ be bases for a vector space V. Then, the CHANGE OF BASIS matrix $M_{\mathcal{C}\leftarrow\mathcal{B}}$ is the matrix satisfying

$$\boxed{M_{\mathcal{C}\leftarrow\mathcal{B}}[\vec{x}]_{\mathcal{B}} = [\vec{x}]_{\mathcal{C}}}$$

for every vector $\vec{x}$ in V.

Lecture Activity 9.4. Consider the basis $\mathcal{B} = \{\vec{b}_1, \vec{b}_2\}$ for $\mathbb{R}^2$ where

$$\vec{b}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \text{ and } \vec{b}_2 = \begin{pmatrix} 5 \\ 3 \end{pmatrix}.$$

P1. Find $M_{\mathcal{E}\leftarrow\mathcal{B}}$.

Solution. Suppose that

$$[\vec{x}]_{\mathcal{B}} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

Then, $\vec{x} = x_1\vec{b}_1 + x_2\vec{b}_2$. We can write this in standard coordinates as

$$\begin{aligned} [\vec{x}]_{\mathcal{E}} &= x_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} + x_2 \begin{pmatrix} 5 \\ 3 \end{pmatrix} \\ &= \begin{pmatrix} 2 & 5 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\ &= \begin{pmatrix} 2 & 5 \\ 1 & 3 \end{pmatrix} [\vec{x}]_{\mathcal{B}}. \end{aligned}$$

Hence, $M_{\mathcal{E}\leftarrow\mathcal{B}} = \begin{pmatrix} 2 & 5 \\ 1 & 3 \end{pmatrix}$.

P2. Show that $M_{\mathcal{E}\leftarrow\mathcal{B}}$ is invertible with $M_{\mathcal{B}\leftarrow\mathcal{E}} = M_{\mathcal{E}\leftarrow\mathcal{B}}^{-1}$.

Solution. Note that the columns of $M_{\mathcal{E}\leftarrow\mathcal{B}}$ consist of basis elements for $\mathbb{R}^2$, and so by Proposition 2.12 we know that $\operatorname{rref}(M_{\mathcal{E}\leftarrow\mathcal{B}})$ has a pivot in every column. Hence, by the Invertible Matrix Theorem this matrix is invertible. Alternatively, we could calculate

$$\det(M_{\mathcal{E}\leftarrow\mathcal{B}}) = 1 \neq 0,$$

and so $M_{\mathcal{E}\leftarrow\mathcal{B}}$ is invertible by Theorem 7.16.

P3. Use your work in P2 to find $M_{\mathcal{B}\leftarrow\mathcal{E}}$.

Solution. We need to calculate the inverse of the matrix we found in P1. We have

$$\left(\begin{array}{cc|cc} 2 & 5 & 1 & 0 \\ 1 & 3 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{cc|cc} 1 & 0 & 3 & -5 \\ 0 & 1 & -1 & 2 \end{array}\right)$$

Lemma 9.6. Let $\mathcal{C}$ be a basis for a vector space V. Then, for any $\vec{x}, \vec{y} \in V$ and scalar $k \in \mathcal{R}$ ...

$$[\vec{x} + \vec{y}]_{\mathcal{C}} = [\vec{x}]_{\mathcal{C}} + [\vec{y}]_{\mathcal{C}}$$

$$[k\vec{x}]_{\mathcal{C}} = k[\vec{x}]_{\mathcal{C}}$$

Lemma 9.7. Let $\{\vec{b}_1, \ldots, \vec{b}_n\}$ be a linearly independent subset of a vector space V. Then, for any basis $\mathcal{C}$ of V, the set $\{[\vec{b}_1]_{\mathcal{C}}, \ldots, [\vec{b}_n]_{\mathcal{C}}\}$ is ...

linearly independent.

Theorem 9.8. Let V be a vector space with basis $\mathcal{C} = \{\vec{c}_1, \ldots, \vec{c}_n\}$. Then, a subset $\mathcal{B} = \{\vec{b}_1, \ldots, \vec{b}_n\}$ of V is a basis for V if and only if $\begin{pmatrix} [\vec{b}_1]_{\mathcal{C}} & \cdots & [\vec{b}_n]_{\mathcal{C}} \end{pmatrix}$ is invertible. In this case, we have

$$M_{\mathcal{C}\leftarrow\mathcal{B}} = \begin{pmatrix} [\vec{b}_1]_{\mathcal{C}} & \cdots & [\vec{b}_n]_{\mathcal{C}} \end{pmatrix}.$$

and furthermore $= M_{\mathcal{B}\leftarrow\mathcal{C}} = M_{\mathcal{C}\leftarrow\mathcal{B}}^{-1}$.

Proof. Suppose first that $\mathcal{B}$ is a basis. Then, for any $\vec{x}$ in V, let $[\vec{x}]_{\mathcal{B}} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$. That is, we have

$$\vec{x} = x_1\vec{b}_1 + \cdots + x_n\vec{b}_n.$$

Complete the proof: Use Lemma 9.6 to show that $M_{\mathcal{C}\leftarrow\mathcal{B}} = \begin{pmatrix} [\vec{b}_1]_{\mathcal{C}} & \cdots & [\vec{b}_n]_{\mathcal{C}} \end{pmatrix}$

Writing the equation above in $\mathcal{C}$-coordinates gives

$$[\vec{x}]_{\mathcal{C}} = [x_1\vec{b}_1 + \cdots + x_n\vec{b}_n]_{\mathcal{C}}.$$

By Lemma 9.6 this gives

$$\begin{aligned} [\vec{x}]_{\mathcal{C}} &= x_1[\vec{b}_1]_{\mathcal{C}} + \cdots + x_n[\vec{b}_n]_{\mathcal{C}} \\ &= \begin{pmatrix} [\vec{b}_1]_{\mathcal{C}} & \cdots & [\vec{b}_n]_{\mathcal{C}} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \\ &= \begin{pmatrix} [\vec{b}_1]_{\mathcal{C}} & \cdots & [\vec{b}_n]_{\mathcal{C}} \end{pmatrix} [\vec{x}]_{\mathcal{B}}, \end{aligned}$$

and so $M_{\mathcal{C}\leftarrow\mathcal{B}} = \begin{pmatrix} [\vec{b}_1]_{\mathcal{C}} & \cdots & [\vec{b}_n]_{\mathcal{C}} \end{pmatrix}$.

Now, by Lemma 9.7, we know that the columns of $M_{\mathcal{C}\leftarrow\mathcal{B}}$ are linearly independent.

Complete the proof: show that $M_{\mathcal{C}\leftarrow\mathcal{B}}$ is invertible, and that $M_{\mathcal{C}\leftarrow\mathcal{B}}^{-1} = M_{\mathcal{B}\leftarrow\mathcal{C}}$.

Since the columns of $M_{\mathcal{C}\leftarrow\mathcal{B}}$ are linearly independent, then by Proposition 2.12 we know that $\mathrm{rref}(M_{\mathcal{E}\leftarrow\mathcal{B}})$ has a pivot in every column. Since this matrix is square, it must also have a pivot in every row. This gives $\mathrm{rref}(M_{\mathcal{E}\leftarrow\mathcal{B}}) = I_n$ and so by the Invertible Matrix Theorem this matrix is invertible. So,

$$[\vec{x}]_{\mathcal{C}} = M_{\mathcal{C}\leftarrow\mathcal{B}}[\vec{x}]_{\mathcal{B}} \Rightarrow [\vec{x}]_{\mathcal{B}} = M_{\mathcal{C}\leftarrow\mathcal{B}}^{-1}[\vec{x}]_{\mathcal{C}}.$$

Hence, we have $M_{\mathcal{C}\leftarrow\mathcal{B}}^{-1} = M_{\mathcal{B}\leftarrow\mathcal{C}}$.

Conversely, suppose that the matrix $\begin{pmatrix}[\vec{b}_1]_{\mathcal{C}} & \cdots & [\vec{b}_n]_{\mathcal{C}}\end{pmatrix}$ is invertible. To see that $\mathcal{B}$ is linearly independent, consider the vector equation

$$x_1\vec{b}_1 + \cdots + x_n\vec{b}_n = \vec{0}.$$

Complete the proof: show that $\{\vec{b}_1, \ldots, \vec{b}_n\}$ is linearly independent.

Writing this equation in $\mathcal{C}$-coordinates and applying Lemma 9.6 gives

$$\alpha_1[\vec{b}_1]_{\mathcal{C}} + \cdots + \alpha_n[\vec{b}_n]_{\mathcal{C}} = \begin{pmatrix}0\\ \vdots\\ 0\end{pmatrix}.$$

Since $\begin{pmatrix}[\vec{b}_1]_{\mathcal{C}} & \cdots & [\vec{b}_n]_{\mathcal{C}}\end{pmatrix}$ is invertible, its columns must be linearly independent. So,

$$x_1 = \cdots = x_n = 0,$$

as needed.

Since V is n-dimensional, we have that $\mathcal{B}$ is a basis for V by Theorem 3.12. $\square$

Lecture Activity 9.5. Let V be the plane in $\mathbb{R}^3$ spanned by

$$\mathcal{C} = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

P1. Show that $\mathcal{B}$ is also a basis for V, where

$$\mathcal{B} = \left\{ \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 4 \\ 2 \\ 3 \end{pmatrix} \right\}.$$

Solution. First let's find $[\vec{b}_1]_{\mathcal{C}}$ and $[\vec{b}_2]_{\mathcal{C}}$. We need to find solutions to the following vector equations

$$\vec{b}_1 = x_1\vec{c}_1 + x_2\vec{c}_2,$$
$$\vec{b}_2 = y_2\vec{c}_1 + y_2\vec{c}_2.$$

Expressing these as systems of linear equations and solving them, we get $\vec{b}_1 = \vec{c}_1 + \vec{c}_2$, and $\vec{b}_2 = 2\vec{c}_1 + \vec{c}_2$. So that $[\vec{b}_1]_{\mathcal{C}} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, and $[\vec{b}_2]_{\mathcal{C}} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$. Now,

$$\begin{pmatrix} [\vec{b}_1]_{\mathcal{C}} & [\vec{b}_2]_{\mathcal{C}} \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}$$

is invertible because its determinant is equal to $1 \cdot 1 - 2 \cdot 1 = -1 \neq 0$. By Theorem 9.8, we can conclude that $\mathcal{B}$ is a basis for V.

P2. Find the change of basis matrices $M_{\mathcal{C}\leftarrow\mathcal{B}}$ and $M_{\mathcal{C}\leftarrow\mathcal{B}}$.

Solution. By Theorem 9.8,

$$M_{\mathcal{C}\leftarrow\mathcal{B}} = \begin{pmatrix} [\vec{b}_1]_{\mathcal{C}} & [\vec{b}_2]_{\mathcal{C}} \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}.$$

Also,

$$M_{\mathcal{B}\leftarrow\mathcal{C}} = M_{\mathcal{C}\leftarrow\mathcal{B}}^{-1} = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} -1 & 2 \\ 1 & -1 \end{pmatrix}.$$

P3. Find a new basis $\mathcal{D}$ for V that's not equal to $\mathcal{B}$ or $\mathcal{C}$.

Solution. Let $\mathcal{D} = \{\vec{d}_1, \vec{d}_2\}$. Then $\mathcal{D}$ is a basis for V if $\begin{pmatrix} [\vec{d}_1]_{\mathcal{C}} & [\vec{d}_2]_{\mathcal{C}} \end{pmatrix}$ is invertible. Let $\begin{pmatrix} [\vec{d}_1]_{\mathcal{C}} & [\vec{d}_2]_{\mathcal{C}} \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 3 & 0 \end{pmatrix}$, the determinant of this matrix is equal to $1 \cdot 0 - 2 \cdot 3 = -6 \neq 0$, and so the matrix is invertible. This matrix gives us the vectors $\vec{d}_1$ and $\vec{d}_2$,

$$\vec{d}_1 = \vec{c}_1 + 3\vec{c}_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + 3 \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 7 \\ 1 \\ 4 \end{pmatrix},$$

$$\vec{d}_2 = 2\vec{c}_1 + 0\vec{c}_2 = 2 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + 0 \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}.$$

Theorem 10.1. Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation and $\mathcal{B}$ be any basis for $\mathbb{R}^n$. Then, there exists a unique $n \times n$ matrix M so that $[F(\vec{x})]_{\mathcal{B}} = M[\vec{x}]_{\mathcal{B}}$. Furthermore, we have

$$M = \begin{pmatrix} [F(\vec{b}_1)]_{\mathcal{B}} & \cdots & [F(\vec{b}_n)]_{\mathcal{B}} \end{pmatrix}.$$

Prove Theorem 10.1

Proof. Take any $\vec{x} \in \mathbb{R}^n$ and write $\vec{x} = x_1\vec{b}_1 + \cdots + x_n\vec{b}_n$. Then,

$$\begin{aligned} [F(\vec{x})]_{\mathcal{B}} &= [F(x_1\vec{b}_1 + \cdots + x_n\vec{b}_n)]_{\mathcal{B}} \\ &= [x_1F(\vec{b}_1) + \cdots + x_nF(\vec{b}_n)]_{\mathcal{B}} && \text{since } F \text{ is linear} \\ &= x_1[F(\vec{b}_1)]_{\mathcal{B}} + \cdots + x_n[F(\vec{b}_n)]_{\mathcal{B}} && \text{by Lemma 9.6} \\ &= \begin{pmatrix} [F(\vec{b}_1)]_{\mathcal{B}} & \cdots & [F(\vec{b}_1)]_{\mathcal{B}} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \\ &= \begin{pmatrix} [F(\vec{b}_1)]_{\mathcal{B}} & \cdots & [F(\vec{b}_1)]_{\mathcal{B}} \end{pmatrix} [\vec{x}]_{\mathcal{B}}, \end{aligned}$$

as needed. □

Definition 10.5. Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation, and $\mathcal{B}$ be any basis for $\mathbb{R}^n$. Then, the DEFINING MATRIX OF F WITH RESPECT TO THE BASIS $\mathcal{B}$ is the matrix M so that ...

$$[F(\vec{x})]_{\mathcal{B}} = M[\vec{x}]_{\mathcal{B}}.$$

We use the notation $M = M_{F,\mathcal{B}}$.

Lecture Activity 10.1. Let $\mathcal{B} = \{\vec{b}_1, \vec{b}_2\}$ be the basis with

$$\vec{b}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \vec{b}_2 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}.$$

P2. Let $F : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation which stretches vectors in the $\vec{b}_1$ direction by 2 and leaves vectors in the $\vec{b}_2$ direction fixed. That is,

$$F(x_1\vec{b}_1 + x_2\vec{b}_2) = 2x_1\vec{b}_1 + x_2\vec{b}_2.$$

Find the defining matrix $M_{F,\mathcal{B}}$.

Solution. By Theorem 10.1, we have

$$M_{G,\mathcal{B}} = \begin{pmatrix} [F(\vec{b}_1)]_{\mathcal{B}} & [F(\vec{b}_2)]_{\mathcal{B}} \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}.$$

P2. Let $G : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation with linearly independent eigenvectors $\vec{v}$ and $\vec{w}$ with corresponding eigenvalues λ_1, λ_2. Letting $\mathcal{B} = \{\vec{v}, \vec{w}\}$, find the defining matrix $M_{H,\mathcal{B}}$.

Solution. Note that $G(\vec{v}) = \lambda_1\vec{v}$ and $G(\vec{w}) = \lambda_2\vec{w}$. So, by Theorem 10.1, we have

$$M_{G,\mathcal{B}} = \begin{pmatrix} [G(\vec{b}_1)]_{\mathcal{B}} & [G(\vec{b}_2)]_{\mathcal{B}} \end{pmatrix} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}.$$