CHAPTER 8 ACTIVITY PACKET *solutions*

Instructions. This packet is due on Quercus no later than **11:59pm on Monday, November 3rd**. Please complete your work directly on this packet. We will spend time together during lecture working on most or all of the activities in this packet. You are responsible for completing all portions of this packet, including lecture activities not discussed in class, and completing the definitions included in the packet. Solutions will be posted to the course website after the assignment due date.

Definition 8.1. Let A be an $n \times n$ matrix. A non-zero vector $\vec{v}$ is an EIGENVECTOR of A if ...

> there is a real number scalar λ such that $A\vec{v} = \lambda\vec{v}$.

The scalar λ is called an EIGENVALUE of A.

Lecture Activity 8.1. For each of the following matrix-vector pairs, determine whether $\vec{v}$ is an eigenvector of the matrix A. If it is, find the corresponding eigenvalue λ.

P1. $A = \begin{pmatrix} 3 & 2 \\ 3 & 8 \end{pmatrix}$ and $\vec{v} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$

Solution. Note that $A\vec{v} = \begin{pmatrix} 3 & 2 \\ 3 & 8 \end{pmatrix} \begin{pmatrix} -2 \\ 1 \end{pmatrix} = \begin{pmatrix} -4 \\ 2 \end{pmatrix} = 2 \begin{pmatrix} -2 \\ 1 \end{pmatrix}$. Therefore $\vec{v}$ is an eigenvector of A with eigenvalue $\lambda = 2$.

P2. $B = \begin{pmatrix} 3 & 2 \\ 3 & 8 \end{pmatrix}$ and $\vec{v} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

Solution. Note that $B\vec{v} = \begin{pmatrix} 3 & 2 \\ 3 & 8 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 5 \\ 11 \end{pmatrix} \neq \lambda \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ for any real number λ. Therefore, the vector $\vec{v}$ is not an eigenvector for B.

P3. $C = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$ and $\vec{v} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$

Note that $C\vec{v} = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} \begin{pmatrix} -2 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = 0 \begin{pmatrix} -2 \\ 1 \end{pmatrix}$. Therefore, $\vec{v}$ is an eigenvector of C with eigenvalue $\lambda = 0$.

Proposition 8.2. For an $n \times n$ matrix A, the set of eigenvectors of A corresponding to an eigenvalue λ is equal to the nonzero vectors in $\mathrm{Nul}(A - \lambda I_n)$.

Prove Proposition 8.2.

Observe $\vec{v} \in \mathrm{Nul}(A - \lambda I_n)$ if and only if

$$(A - \lambda I_n)\vec{v} = \vec{0}$$

$$\Leftrightarrow A\vec{v} - \lambda\vec{v} = \vec{0}$$

$$\Leftrightarrow A\vec{v} = \lambda\vec{v},$$

as needed.

Definition 8.3. Let A be an $n \times n$ matrix with eigenvalue λ.

1. The λ-Eigenspace of A is the vector subspace of $\mathbb{R}^n$ defined by ...

   $E_\lambda := \mathrm{Nul}(A - \lambda I_n)$.

2. The GEOMETRIC MULTIPLICITY of λ is ...

   the dimension of the λ-eigenspace, $\dim(E_\lambda)$.

Lecture Activity 8.2. Let $A = \begin{pmatrix} 3 & 2 \\ 3 & 8 \end{pmatrix}$, and recall from Lecture Activity 8.1 that $\lambda = 2$ is an eigenvalue of A

P1. Find the 2-Eigenspace of A.

Solution. The 2-eigenspace of A is the null space of $A - 2I_2$. Now

$$A - 2I_2 = \begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix}.$$

So,

$$\begin{aligned} \mathrm{Nul}(A - I_2) &= \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 : A \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\} \\ &= \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 : x + 2y = 0 \right\} \\ &= \left\{ \begin{pmatrix} -2y \\ y \end{pmatrix} : y \in \mathbb{R} \right\} \\ &= \mathrm{Span}\left(\begin{pmatrix} -2 \\ 1 \end{pmatrix} \right). \end{aligned}$$

P2. Find the geometric multiplicity of $\lambda = 2$.

Solution. We have $\dim \mathrm{Nul}(A - I_2) = 1$ and so the geometric multiplicity is equal to 1.

Proposition 8.5. A real number λ is an eigenvalue of an $n \times n$ matrix A if and only if

$$\det(A - \lambda I_n) = 0.$$

Prove Proposition 8.5.

Proof. Recall that a matrix B is invertible if and only if $\det(B) \neq 0$. Consider the vector equation

$$(A - \lambda I_n)\vec{x} = \vec{0}. \tag{8.1}$$

By definition, $\vec{v}$ is an eigenvector of A with eigenvalue λ precisely when $A\vec{v} = \lambda\vec{v}$, which can be rewritten as

$$(A - \lambda I_n)\vec{v} = \vec{0}.$$

Since eigenvalues are *nonzero*, we know that λ is an eigenvalue of A if and only if there is a *nonzero* solution $\vec{v}$ to Equation (8.1). Since $\vec{0}$ is *also* a solution to Equation (8.1), λ is an eigenvalue of A if and only if Equation (8.1) has infinitely many solutions. By Rouché-Capelli, this takes places precisely when $\text{rref}(A - \lambda I_n)$ has a column without a pivot, which is equivalent to the matrix $A - \lambda I_n$ not being invertible (by the Invertible Matrix Theorem). $\square$

Definition 8.6. For an $n \times n$ matrix A, the CHARACTERISTIC POLYNOMIAL of A is ...

$\chi_A(x) := \det(A - xI_n)$

Lecture Activity 8.3. Find the characteristic polynomial of the following matrices. Then, use Proposition 8.5 to find the eigenvalues of each matrix.

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, B = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{pmatrix}, C = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}.$$

Solution. Solution. For matrix A, we find the characteristic polynomial as follows,

$$\begin{aligned} \chi_A(x) &= \det(A - xI_2) \\ &= \det \begin{pmatrix} 1-x & 1 \\ 1 & 1-x \end{pmatrix} \\ &= (1-x)(1-x) - 1 \cdot 1 \\ &= x^2 - 2x + 1 - 1 \\ &= x^2 - 2x \\ &= x(x-2). \end{aligned}$$

By Proposition 8.5, the eigenvalues of A are the solutions to the equation $\chi_A(x) = 0$. Now $\chi_A(x) = x(x-2) = 0$, when $x = 0$ or $x = 2$. Therefore, A has eigenvalues $\lambda_1 = 0$, and $\lambda_2 = 2$.

Next we find the characteristic polynomial for B,

$$\begin{aligned} \chi_B(x) &= \det(B - xI_3) \\ &= \det \begin{pmatrix} 1-x & 0 & 1 \\ 0 & 1-x & 1 \\ 0 & 0 & 2-x \end{pmatrix} \\ &= (1-x)(1-x)(2-x). \end{aligned}$$

By Proposition 8.5, the eigenvalues of B are the solutions to the equation $\chi_B(x) = 0$. Now $\chi_B(x) = (1-x)^2(2-x) = 0$, when $x = 1$ or $x = 2$. Therefore, B has eigenvalues $\lambda_1 = 1$, and $\lambda_2 = 2$.

Next we find the characteristic polynomial for C,

$$\begin{aligned} \chi_C(x) &= \det(C - xI_2) \\ &= \det \begin{pmatrix} 1-x & -1 \\ 1 & 1-x \end{pmatrix} \\ &= (1-x)(1-x) - (-1) \cdot 1 \\ &= x^2 - 2x + 1 + 1 \\ &= x^2 - 2x + 2 \end{aligned}$$

By Proposition 8.5, the eigenvalues of C are the solutions to the equation $\chi_C(x) = 0$. However, $\chi_C(x) = x^2 - 2x + 2 = 0$ has no real solutions. This means that C does not have any real eigenvalues.