Instructions. This packet is due on Quercus no later than **11:59pm on Monday, October 6th**. Please complete your work directly on this packet. We will spend time together during lecture working on most or all of the activities in this packet. You are responsible for completing all portions of this packet, including lecture activities not discussed in class, and completing the definitions included in the packet. Solutions will be posted to the course website after the assignment due date.

Definition 4.18. A function $f : X \to Y$ is called BIJECTIVE if ...

Lecture Activity 4.9. Show that a linear tranformation $F : \mathbb{R}^n \to \mathbb{R}^m$ can be bijective if and only if $n = m$.

Definition 4.19. Let V be a subspace of $\mathbb{R}^n$ and W a subspace of $\mathbb{R}^m$. An ISOMORPHISM between V and W is ...

If an isomorphism exists between two vector spaces, we say these spaces are ISOMORPHIC, and we write $V \cong W$.

Definition 5.2. Let $F : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation.

1. The KERNEL of F is the subset $\ker(F) \subseteq \mathbb{R}^n$ defined by

2. The IMAGE of F is the subset $\operatorname{im}(F) \subseteq \mathbb{R}^m$ defined by

Lecture Activity 5.1. Let $F = T_C$ where C is our matrix from Lecture Activity 4.3

$$C = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

P1. Find a vector $\vec{x} \in \ker(F)$.

P2. Find a vector $\vec{y} \in \operatorname{im}(F)$.

P3. Find a vector $\vec{v}$ so that $\ker(F) = \mathrm{Span}(\vec{v})$. Conclude that $\ker(F)$ is a vector space.

P4. Find a vector $\vec{w}$ so that $\mathrm{im}(F) = \mathrm{Span}(\vec{w})$. Conclude that $\mathrm{im}(F)$ is a vector space.

Definition 5.4. Let $F : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation.

1. The RANK of F is ...

   and is denoted by $\operatorname{rank}(F)$.

2. The NULLITY of F is ...

   and is denoted by $\operatorname{nullity}(F)$.

Lecture Activity 5.2. Let $F : \mathbb{R}^3 \to \mathbb{R}^2$ be given by

$$F\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = \begin{pmatrix} x+y \\ x+z \end{pmatrix}.$$

P1. Calculate $\operatorname{rank}(F)$.

P2. Calculate nullity(F).

Definition 5.5. Let A be an $m \times n$ matrix with column vectors $A = \begin{pmatrix} \vec{v}_1 & \cdots & \vec{v}_n \end{pmatrix}$.

1. The COLUMN SPACE of A is the subspace of $\mathbb{R}^m$ given by ...

2. The NULL SPACE of A is the subspace of $\mathbb{R}^n$ given by

Proposition 5.6. Let $F : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation with defining matrix M_F. Then, $\ker(F) = \mathrm{Nul}(M_F)$ and $\mathrm{im}(F) = \mathrm{Col}(M_F)$.

Proof. Let F have defining matrix $M = M_F$.

Complete the proof: show that $\ker(F) = \mathrm{Nul}(M)$.

Next, suppose that M has column vectors $M = \begin{pmatrix} \vec{v}_1 & \cdots & \vec{v}_n \end{pmatrix}$.

Complete the proof: show that $\mathrm{im}(F) = \mathrm{Col}(M)$.

$\square$

Definition 5.7. Let A be a matrix.

1. The NULLITY of A is ...

   and is denoted by $\mathrm{nullity}(A)$.

2. The RANK of A is ...

   and is denoted by $\mathrm{rank}(A)$.

Lecture Activity 5.3. Calculate the rank and nullity of

$$A = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 2 & 1 & 1 \end{pmatrix}$$

Lecture Activity 5.4. Let A and B be $m \times 3$ matrices.

P1. Suppose that $\operatorname{rref}(A)$ has exactly two pivots, which are located in columns 1 and 3. Show that $\operatorname{rank}(A) = 2$ and $\operatorname{nullity}(A) = 1$.

P2. Suppose that $\operatorname{rref}(B)$ has exactly one pivot, which is located in column 1. Show that $\operatorname{rank}(B) = 1$ and $\operatorname{nullity}(B) = 2$.

Definition 5.10. A system of linear equations is called HOMOGENEOUS if . . .

Lecture Activity 5.5. Consider the homogeneous system of linear equations

$$\begin{cases} x + 2y + 4z = 0 \\ x + y - z = 0 \\ y + 5z = 0 \end{cases}$$

P1. Find a matrix C so that the solution set to this system is equal to $\mathrm{Nul}(C)$.

P2. Calculate $\mathrm{nullity}(C)$.

P3. Recall from Section 1.7 that the solution set to this system is equal to the set of intersection points of planes in $\mathbb{R}^3$. Given your work in the previous parts, do these planes intersect at a point, a line, or a plane in $\mathbb{R}^3$?

Theorem 5.12. The solution set to a consistent system of linear equations in with coefficient matrix C is equal to

$$\vec{p} + \mathrm{Nul}(C) := \{\vec{p} + \vec{v} \mid \vec{v} \in \mathrm{Nul}(C)\}$$

where $\vec{p}$ is any particular solution to the system of linear equations.

Proof. Suppose that our system of linear equations has coefficient matrix C and particular solution $\vec{p}$. Then, the system has the same solution set as the matrix-vector equation $C\vec{x} = \vec{b}$ for a vector $\vec{b} \in \mathbb{R}^n$. Let $\vec{s}$ be any solution to this matrix-vector equation.

Complete the proof: show that $\vec{s} - \vec{p}$ is a solution to $C\vec{x} = 0$.

Since the solution set of $C\vec{x} = \vec{0}$ is equal to $\mathrm{Nul}(C)$, then by above we can write $\vec{s} - \vec{p} \in \mathrm{Nul}(C)$.

Complete the proof: conclude that the solution $\vec{s}$ is in $\vec{p} + \mathrm{Nul}(C)$.

Conversely, take any $\vec{p} + \vec{v} \in \vec{p} + \mathrm{Nul}(C)$.

Complete the proof: show that $\vec{p} + \vec{v}$ is a solution to $C\vec{x} = \vec{b}$.

$\square$

Lecture Activity 5.6. Use Theorem 5.12 to determine whether the solution set for each of the following systems is empty, a point, a line, or a plane in $\mathbb{R}^3$.

P1. $$\begin{cases} x + 2y + 4z = 1 \\ x + y - z = 2 \\ y + 5z = -1 \end{cases}$$

P2. $$\begin{cases} x + 2y + 2z = 5 \\ x + y + z = 0 \\ 3x + 3z = 1 \end{cases}$$

P3. $$\begin{cases} x + 2y + 4z = 1 \\ x + y - z = 2 \\ y + 5z = 1 \end{cases}$$