CHAPTER 3 ACTIVITY PACKET *solutions*

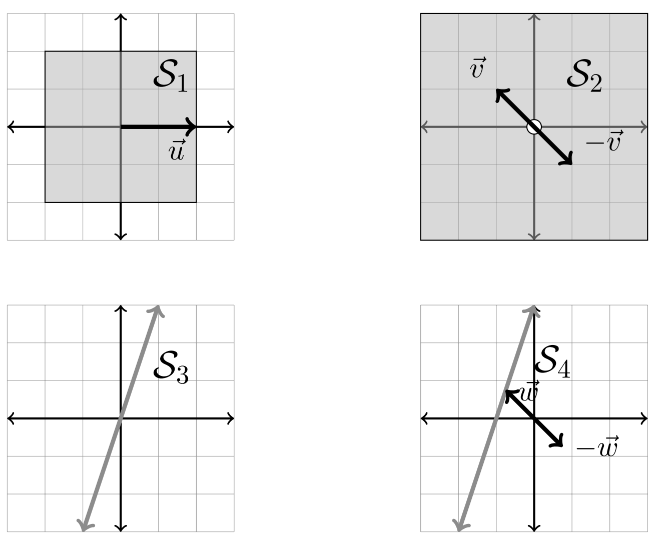
Instructions. This packet is due on Quercus no later than **11:59pm on Monday, September 22nd**. Please complete your work directly on this packet. We will spend time together during lecture working on most or all of the examples and lecture activities in this packet. You are responsible for completing all portions of this packet, including lecture activities not discussed in class, and completing the definitions included in the packet. Solutions will be posted to the course website after the assignment due date.

Definition 3.1. A VECTOR SPACE (over the real numbers) is any set of vectors V in $\mathbb{R}^n$ that satisfies all of the following properties:

1. V is nonempty;
2. V is closed under vector addition: for all $\vec{v}, \vec{w} \in V$ we have $\vec{v} + \vec{w} \in V$;
3. V is closed under scalar multiplication: for all $\vec{v} \in V$ and $c \in \mathbb{R}$ we have $c\vec{v} \in V$.

Lecture Activity 3.1. Determine which of the following sets are vector subspaces of the given ambient space and which are not. Justify your answer.

P1. The subsets of $\mathbb{R}^2$ drawn below (note that the set $\mathcal{S}_2$ in the second image is meant to extend infinitely in all directions)



Solution. We can see geometrically that $\mathcal{S}_3$ is a vector space. To see that sets $\mathcal{S}_1, \mathcal{S}_2$ and $\mathcal{S}_4$ are **not vector spaces**, let $\vec{u}, \vec{v}$ and $\vec{w}$ be the vectors drawn on the images above. We have:

- $\vec{u} \in \mathcal{S}_1$ but $2\vec{u} \notin \mathcal{S}_1$, and so $\mathcal{S}_1$ is not closed under scalar multiplication;
- $\vec{v}, -\vec{v} \in \mathcal{S}_2$ but $\vec{v} + (-\vec{v}) = 0\vec{v} \notin \mathcal{S}_2$ and so $\mathcal{S}_2$ is not closed under vector addition;
- $\vec{w} \in \mathcal{S}_4$ but $-\vec{w} \notin \mathcal{S}_4$ and so $\mathcal{S}_4$ is not closed under scalar multiplication.

P2. The subset $\mathcal{U}$ of $\mathbb{R}^2$ defined by $\mathcal{U} = \left\{ \begin{pmatrix} x \\ 1 \end{pmatrix} : x \in \mathbb{R} \right\}$

Solution. This set is **not a vector space**. For example, observe that

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix} \in \mathcal{U}$$

but we have

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix} \notin \mathcal{U}.$$

P3. The subset $\mathcal{V}$ of $\mathbb{R}^2$ defined by $\mathcal{V} = \left\{ \begin{pmatrix} 2x \\ 0 \end{pmatrix} : x \in \mathbb{R} \right\}$

Solution. This set **is a vector space**. First, observe that $\begin{pmatrix} 0 \\ 0 \end{pmatrix} \in \mathcal{V}$, and so $\mathcal{V}$ is nonempty, as needed. Next, take any vectors $\vec{v}, \vec{w} \in \mathcal{V}$ and scalar $c \in \mathbb{R}$. Then we can write

$$\vec{v} = \begin{pmatrix} 2a \\ 0 \end{pmatrix}, \vec{w} = \begin{pmatrix} 2b \\ 0 \end{pmatrix},$$

for some $a, b \in \mathbb{R}$. This gives

$$\begin{aligned} \vec{v} + \vec{w} &= \begin{pmatrix} 2a \\ 0 \end{pmatrix} + \begin{pmatrix} 2b \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} 2(a+b) \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} 2x \\ 0 \end{pmatrix}, \text{ where } x = a + b \end{aligned}$$

and so $\vec{u} + \vec{w} \in \mathcal{V}$ and $\mathcal{V}$ is closed under vector addition. Now to check closer under scalar multiplication, we have

$$c\vec{v} = \begin{pmatrix} 2ac \\ 0 \end{pmatrix} \in \mathcal{V},$$

as needed. Hence, $\mathcal{V}$ is a vector space.

P4. The subset $\mathcal{W}$ of $\mathbb{R}^3$ defined by $\mathcal{W} = \left\{ \begin{pmatrix} x-y \\ x+y+2z \\ y+z \end{pmatrix} : x, y, z \in \mathbb{R} \right\}$

First note that $\mathcal{W}$ is non-empty, since $\vec{0} \in \mathcal{W}$ (where we take $x = y = z = 0$). Now let $\vec{u}, \vec{v} \in \mathcal{W}$ and let $c \in \mathbb{R}$ be a scalar. Then for some $u_1, u_2, u_3, v_1, v_2, v_3 \in \mathbb{R}$, we have

$$\vec{u} = \begin{pmatrix} u_1 - u_2 \\ u_1 + u_2 + 2u_3 \\ u_2 + u_3 \end{pmatrix}, \vec{v} = \begin{pmatrix} v_1 - v_2 \\ v_1 + v_2 + v_3 \\ v_2 + v_3 \end{pmatrix}.$$

Now

$$\begin{aligned}
\vec{u} + \vec{v} &= \begin{pmatrix} u_1 - u_2 \\ u_1 + u_2 + 2u_3 \\ u_2 + u_3 \end{pmatrix} + \begin{pmatrix} v_1 - v_2 \\ v_1 + v_2 + 2v_3 \\ v_2 + v_3 \end{pmatrix} \\
&= \begin{pmatrix} u_1 - u_2 + v_1 - v_2 \\ u_1 + u_2 + 2u_3 + v_1 + v_2 + 2v_3 \\ u_2 + u_3 + v_2 + v_3 \end{pmatrix} \\
&= \begin{pmatrix} (u_1 + v_1) - (u_2 + v_2) \\ (u_1 + v_1) + (u_2 + v_2) + 2(u_3 + v_3) \\ (u_2 + v_2) + (u_3 + v_3) \end{pmatrix} \\
&= \begin{pmatrix} x - y \\ x + y + 2z \\ y + z \end{pmatrix},
\end{aligned}$$

where $x = u_1 + v_1, y = u_2 + v_2, z = u_3 + v_3$. So $\vec{u} + \vec{v} \in \mathcal{W}$ and $\mathcal{W}$ is closed under vector addition. Now to check closure under scalar multiplication,

$$\begin{aligned}
c\vec{u} &= c \begin{pmatrix} u_1 - u_2 \\ u_1 + u_2 + 2u_3 \\ u_2 + u_3 \end{pmatrix} \\
&= \begin{pmatrix} c(u_1 - u_2) \\ c(u_1 + u_2 + 2u_3) \\ c(u_2 + u_3) \end{pmatrix} \\
&= \begin{pmatrix} cu_1 - cu_2 \\ cu_1 + cu_2 + 2cu_3 \\ cu_2 + cu_3 \end{pmatrix} \\
&= \begin{pmatrix} x - y \\ x + y + 2z \\ y + z \end{pmatrix},
\end{aligned}$$

where $x = cu_1, y = cu_2, z = cu_3$ and so $c\vec{u} \in \mathcal{W}$, as needed. Hence, $\mathcal{W}$ is a vector space.

Proposition 3.2. The span of any set of vectors in $\mathbb{R}^n$ is a vector subspace of $\mathbb{R}^n$.

Proof. Suppose that $V = \mathrm{Span}\,(\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m)$ for vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m$ in $\mathbb{R}^n$.

Complete the proof: show that V is a vector space.

Observe that $\vec{0} \in V$ since we can write

$$\vec{0} = 0\vec{v}_1 + \cdots + 0\vec{v}_m,$$

and so V is not empty. Next, take any $\vec{v}, \vec{w}$ in V. Then we can write

$$\vec{v} = a_1\vec{v}_1 + \cdots + a_m\vec{v}_m$$

$$\vec{w} = b_1\vec{v}_1 + \cdots + b_m\vec{v}_m$$

for scalars $a_i, b_i \in \mathbb{R}$ and so

$$\vec{v} + \vec{w} = (a_1 + b_1)\vec{v}_1 + \cdots + (a_m + b_m)\vec{v}_m \in \mathrm{Span}(\vec{v}_1, \ldots, \vec{v}_m).$$

Hence, $\vec{v} + \vec{w} \in V$ and so V is closed under vector addition. Finally, for any scalar $c \in \mathbb{R}$ we have

$$c\vec{v} = (ca_1)\vec{v}_1 + \cdots + (ca_m)\vec{v}_m \in \mathrm{Span}(\vec{v}_1, \ldots, \vec{v}_m).$$

Hence, $c\vec{v} \in V$ and so V is closed under scalar multiplication. Therefore, V is a vector space.

To see that V is a subset of $\mathbb{R}^n$, note $\vec{v}_1, \ldots, \vec{v}_m \in \mathbb{R}^n$. Furthermore, by definition we know that $\mathbb{R}^n$ is closed under scalar multiplication and so $c_1\vec{v}_1, \ldots, c_n\vec{v}_n \in \mathbb{R}^n$. Again by definition we know that $\mathbb{R}^n$ is closed under vector addition and so

$$\vec{v} = c_1\vec{v}_1 + \cdots + c_m\vec{v}_m \in \mathbb{R}^n$$

for all $\vec{v} \in V$ and so $V \subseteq \mathbb{R}^n$. $\square$

Definition 3.4. Let V be a vector subspace of $\mathbb{R}^n$. A SPANNING SET (also known as a GENERATING SET) for V is ...

any subset B of V so that $V = \mathrm{Span}(B)$.

Lecture Activity 3.2. Show that the following sets are vector spaces by finding a generating set. Compare with your work in Lecture Activity 3.1

P1. $V = \left\{ \begin{pmatrix} 2x \\ 0 \end{pmatrix} : x \in \mathbb{R} \right\}$

Solution. Observe that we can write

$$V = \left\{ x \begin{pmatrix} 2 \\ 0 \end{pmatrix} : x \in \mathbb{R} \right\} = \mathrm{Span}\left(\begin{pmatrix} 2 \\ 0 \end{pmatrix} \right).$$

So, by Proposition 3.2, V is a vector space.

P2. $W = \left\{ \begin{pmatrix} x - y \\ x + y + 2z \\ y + z \end{pmatrix} : x, y, z \in \mathbb{R} \right\}$

Solution. Observe that we can write

$$\begin{aligned} W &= \left\{ \begin{pmatrix} x - y \\ x + y + 2z \\ y + z \end{pmatrix} : x, y, z \in \mathbb{R} \right\} \\ &= \left\{ \begin{pmatrix} x \\ x \\ 0 \end{pmatrix} + \begin{pmatrix} -y \\ y \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ 2z \\ z \end{pmatrix} : x, y, z \in \mathbb{R} \right\} \\ &= \left\{ x \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + y \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} + z \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} : x, y, z \in \mathbb{R} \right\} \\ &= \mathrm{Span}\left(\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} \right) \end{aligned}$$

So, by Proposition 3.2, W is a vector space.

Definition 3.5. A subset $\mathcal{B}$ of a vector space V is called a BASIS if ...

1. B is a spanning set for V, and
2. B is linearly independent.

Lecture Activity 3.3. Determine which of the following sets are bases for $\mathbb{R}^3$.

$$\mathcal{B}_1 = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} \right\}$$

Solution. Note that

$$\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix},$$

and so the vectors in $\mathcal{B}_1$ are linearly dependent. So, $\mathcal{B}_1$ is **not a basis** for $\mathbb{R}^3$.

$$\mathcal{B}_2 = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}$$

Solution. Since the vector

$$\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \notin \mathrm{Span}\left(\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right),$$

it follows that $\mathcal{B}_2$ is not a generating set for $\mathbb{R}^3$, and therefore **is not a basis** for $\mathbb{R}^3$.

$$\mathcal{B}_3 = \left\{ \begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix} \right\}$$

Solution. By Proposition 2.4, we see that $\mathcal{B}_3$ is linearly independent. Furthermore, every vector in $\mathbb{R}^3$ we be written in the form

$$\begin{pmatrix}x\\y\\z\end{pmatrix} = x\begin{pmatrix}1\\0\\0\end{pmatrix} + y\begin{pmatrix}0\\1\\0\end{pmatrix} + z\begin{pmatrix}0\\0\\1\end{pmatrix}$$

and so $\mathcal{B}_3$ also generates. Hence, $\mathcal{B}_3$ **is a basis** for $\mathbb{R}^3$.

$$\mathcal{B}_4 = \left\{ \begin{pmatrix}1\\0\\1\end{pmatrix}, \begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix} \right\}$$

Solution. The set $\mathcal{B}_4$ **is a basis** for $\mathbb{R}^3$. To see this, observe that

$$\operatorname{rref}\begin{pmatrix}1 & 1 & 0 = \\ 0 & 1 & 0 \\ 1 & 0 & 1\end{pmatrix} = \begin{pmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{pmatrix}$$

and so $\mathcal{B}_4$ is linearly independent by Proposition 2.12. Now to show that the set generates, take any vector $\vec{v} = \begin{pmatrix}a\\b\\c\end{pmatrix}$ in $\mathbb{R}^3$ and consider the vector equation

$$x\begin{pmatrix}1\\0\\1\end{pmatrix} + y\begin{pmatrix}1\\1\\0\end{pmatrix} + z\begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}a\\b\\c\end{pmatrix},$$

We have

$$\operatorname{rref}\left(\begin{array}{ccc|c}1 & 1 & 0 & x\\0 & 1 & 0 & y\\1 & 0 & 1 & z\end{array}\right) = \left(\begin{array}{ccc|c}1 & 0 & 0 & a-b\\0 & 1 & 0 & b\\0 & 0 & 1 & c-a+b\end{array}\right),$$

and so this vector equation has solution $x = a - b, y = b, c = c - a + b$. Hence, $\vec{v} \in \operatorname{Span}(\mathcal{B}_4)$ for every vector $\vec{v}$ in $\mathbb{R}^3$, and so $\mathcal{B}_4$ is a generating set for $\mathbb{R}^3$.

Definition 3.8. Let V be a nonzero vector subspace of $\mathbb{R}^n$. Then, the DIMENSION of V, denoted $\dim V$, is ...

the size of any basis for V.

Definition 3.9. The STANDARD BASIS for $\mathbb{R}^n$ is the set $\mathcal{E} := \{\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n\}$ where $\vec{e}_i$ is ...

the vector with 1 in the ith coordinate and 0 in all other coordinates. That is,

$$\vec{e}_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \vec{e}_2 = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \ldots, \vec{e}_n = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}.$$

Lecture Activity 3.4. Show that the standard basis is a basis for $\mathbb{R}^n$. Conclude that $\dim(\mathbb{R}^n) = n$.

Solution. The matrix

$$\begin{pmatrix} \vec{e}_1 & \vec{e}_2 & \cdots & \vec{e}_n \end{pmatrix} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}$$

is already in reduced row echelon form and has a pivot in every column, and so by Proposition 2.4, $\mathcal{E} = \{\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n\}$ is a linearly independent set. Furthermore, we defined vectors by their standard coordinates (see Definition 2.3). So, for any $\vec{v} \in \mathbb{R}^n$ we can write

$$\vec{v} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = x_1\vec{e}_1 + x_2\vec{e}_2 + \cdots + x_n\vec{e}_n,$$

for $x_1, x_2, \ldots, x_n \in \mathbb{R}$. Hence, $\vec{v} \in \operatorname{Span}(\mathcal{E})$ and so $\mathcal{E}$ generates $\mathbb{R}^n$. Thus, $\mathcal{E}$ is a basis for $\mathbb{R}^n$ containing n elements, and so $\dim(\mathbb{R}^n) = n$.

Lecture Activity 3.5. In this problem we'll find a basis for the vector space W from Lecture Activities 3.1 and 3.2, defined by

$$W = \left\{ \begin{pmatrix} x - y \\ x + y + 2z \\ y + z \end{pmatrix} : x, y, z \in \mathbb{R} \right\}.$$

P1. Use your work from Lecture Activity 3.2 to observe that we can write $W = \operatorname{Span}(\vec{u}, \vec{v}, \vec{w})$.

Solution. From Lecture Activity 3.2 we can set

$$\vec{u} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \vec{v} = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}, \vec{w} = \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}.$$

P2. Let $A = \begin{pmatrix} \vec{u} & \vec{v} & \vec{w} \end{pmatrix}$ and observe that

$$\operatorname{rref}(A) = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Use this calculation to show that $\vec{w} \in \operatorname{Span}(\vec{u}, \vec{v})$.

Solution. Consider the vector equation

$$x\vec{u} + y\vec{v} = \vec{w}. \tag{3.1}$$

Since $\operatorname{rref}\begin{pmatrix} \vec{u} & \vec{v} & \vec{w} \end{pmatrix}$ does not have a pivot in the last column, then by Rouché-Capelli, we know that this vector equation has a solution. In particular, we use $\operatorname{rref}(A)$ to solve $x = y = 1$ which gives

$$\vec{w} = \vec{u} + \vec{v}.$$

P3. Use P2 to show that $\text{Span}(\vec{u}, \vec{v}, \vec{w}) = \text{Span}(\vec{u}, \vec{v})$.

Solution. Note that this problem is asking us to prove set equality, so we need to show that two statements are true: $\text{Span}(\vec{u}, \vec{v}, \vec{w}) \subseteq \text{Span}(\vec{u}, \vec{v})$ **and** $\text{Span}(\vec{u}, \vec{v}) \subseteq \text{Span}(\vec{u}, \vec{v}, \vec{w})$.

For the first set inclusion, take any $\vec{x} \in \text{Span}(\vec{u}, \vec{v}, \vec{w})$. Then there are real numbers a, b, c so that $\vec{x} = a\vec{u} + b\vec{v} + c\vec{w}$. But, recalling that $\vec{w} = \vec{u} + \vec{v}$ gives

$$\vec{x} = a\vec{u} + b\vec{v} + c(\vec{u} + \vec{v}) = (a + c)\vec{u} + (b + c)\vec{v} \in \text{Span}(\vec{u}, \vec{v}).$$

Hence, $\text{Span}(\vec{u}, \vec{v}, \vec{w}) \subseteq \text{Span}(\vec{u}, \vec{v})$, as needed.

For the opposite set inclusion, take any $\vec{y} \in \text{Span}(\vec{u}, \vec{v})$. Then, there are real numbers d, e so that

$$\vec{y} = d\vec{u} + e\vec{v} = d\vec{u} + e\vec{v} + 0\vec{w} \in \text{Span}(\vec{u}, \vec{v}, \vec{w}).$$

Hence, $\text{Span}(\vec{u}, \vec{v}) \subseteq \text{Span}(\vec{u}, \vec{v}, \vec{w})$ and so the desired set equality follows.

P4. Use your work in the previous parts to find a basis for W. Then, find the dimension of W.

Solution. By P1 and P3, we have that $W = \text{Span}(\vec{u}, \vec{v}, \vec{w}) = \text{Span}(\vec{u}, \vec{v})$, and so $\{\vec{u}, \vec{v}\}$ is a generating set for W. Furthermore, since $\text{rref}\begin{pmatrix}\vec{u} & \vec{v} & \vec{w}\end{pmatrix}$ has a pivot in the first two columns, then $\text{rref}\begin{pmatrix}\vec{u} & \vec{v}\end{pmatrix}$ has a pivot in every column. Hence, by Proposition 2.12, the set $\{\vec{u}, \vec{v}\}$ is linearly independent. Therefore, $\{\vec{u}, \vec{v}\}$ is a basis for W and so $\dim(W) = 2$.

Lemma 3.10. Let A be an $m \times n$ matrix of the form

$$A = \begin{pmatrix} \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \end{pmatrix}$$

where the $\vec{v}_i$ are vectors in $\mathbb{R}^m$. If the nth column of $\operatorname{rref}(A)$ does not have a pivot, then the vector $\vec{v}_n$ is in $\operatorname{Span}(\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_{n-1})$.

Prove Lemma 3.10.

Proof. Our argument will follow similarly to our work from P2 of Lecture Activity 3.5

Consider the vector equation

$$x_1\vec{v}_1 + +x_2\vec{v}_2 + \cdots + x_{n-1}\vec{v}_{n-1} = \vec{v}_n. \tag{3.2}$$

Note that $A = \begin{pmatrix} \vec{v}_1 & \cdots & \vec{v}_n \end{pmatrix}$ is the augmented matrix of the corresponding system of linear equations. Since there is no pivot in the nth column of $\operatorname{rref}(A)$, then by Rouché-Capelli we know this system is consistent. That is, there exists (at least one) real number solution $(c_1, \ldots, c_{n-1})$ to Equation 3.2, which gives

$$\vec{v}_n = c_1\vec{v}_1 + \cdots + c_{n-1}\vec{v}_{n-1} \in \operatorname{Span}(\vec{v}_1, \ldots, \vec{v}_{n-1}),$$

as needed. $\square$

Lecture Activity 3.6. Find a basis for the following vector spaces, and state their dimension.

P1. $V = \operatorname{Span}\left(\begin{pmatrix}2\\1\\-1\end{pmatrix}, \begin{pmatrix}1\\0\\-1\end{pmatrix}, \begin{pmatrix}0\\1\\1\end{pmatrix}\right)$

Solution. We have

$$\operatorname{rref}\begin{pmatrix}2 & 1 & 0\\1 & 0 & 1\\-1 & -1 & 1\end{pmatrix} = \begin{pmatrix}1 & 0 & 1\\0 & 1 & -2\\0 & 0 & 0\end{pmatrix},$$

which has pivots in columns 1 and 2, but no pivot in column 3. Hence, by Theorem 3.11, V has basis

$$\left\{\begin{pmatrix}2\\1\\-1\end{pmatrix}, \begin{pmatrix}1\\0\\-1\end{pmatrix}\right\}$$

and hence $\dim(V) = 2$.

P2. $W = \operatorname{Span}\left(\begin{pmatrix}1\\0\\2\\1\end{pmatrix}, \begin{pmatrix}3\\0\\6\\3\end{pmatrix}, \begin{pmatrix}1\\1\\1\\1\end{pmatrix}, \begin{pmatrix}7\\2\\12\\7\end{pmatrix}\right)$

Solution. We have

$$\operatorname{rref}\begin{pmatrix}1 & 3 & 1 & 7\\0 & 0 & 1 & 2\\2 & 6 & 1 & 12\\1 & 3 & 1 & 7\end{pmatrix} = \begin{pmatrix}1 & 3 & 0 & 5\\0 & 0 & 1 & 2\\0 & 0 & 0 & 0\\0 & 0 & 0 & 0\end{pmatrix},$$

which has pivots in columns 1 and 3, but no pivot in columns 2 or 4. Hence, by Theorem 3.11, W has basis

$$\left\{\begin{pmatrix}1\\0\\2\\1\end{pmatrix}, \begin{pmatrix}7\\1\\1\\1\end{pmatrix}\right\}$$

and hence $\dim(W) = 2$.