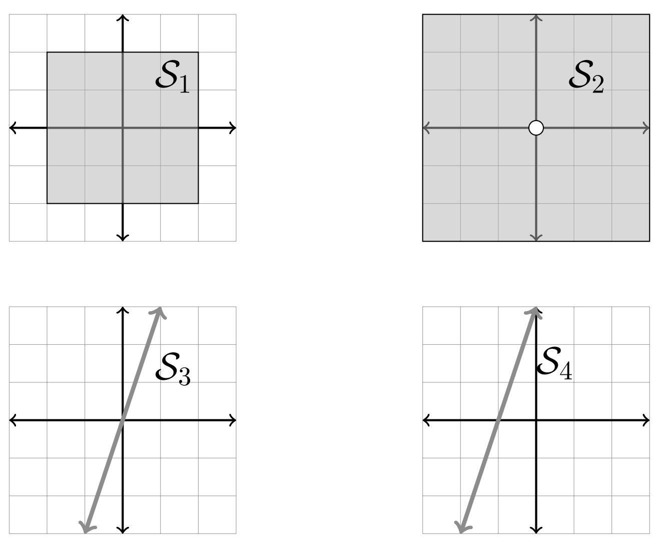
CHAPTER 3 ACTIVITY PACKET   NAME:

Instructions. This packet is due on Quercus no later than **11:59pm on Monday, September 22nd**. Please complete your work directly on this packet. We will spend time together during lecture working on most or all of the examples and lecture activities in this packet. You are responsible for completing all portions of this packet, including lecture activities not discussed in class, and completing the definitions included in the packet. Solutions will be posted to the course website after the assignment due date.

Definition 3.1. A VECTOR SPACE (over the real numbers) is any set of vectors V in $\mathbb{R}^n$ that satisfies all of the following properties:

Lecture Activity 3.1. Determine which of the following sets are vector subspaces of the given ambient space and which are not. Justify your answer.

P1. The subsets of $\mathbb{R}^2$ drawn below (note that the set $\mathcal{S}_2$ in the second image is meant to extend infinitely in all directions)



P2. The subset $\mathcal{U}$ of $\mathbb{R}^2$ defined by $\mathcal{U} = \left\{ \begin{pmatrix} x \\ 1 \end{pmatrix} : x \in \mathbb{R} \right\}$

P3. The subset $\mathcal{V}$ of $\mathbb{R}^2$ defined by $\mathcal{V} = \left\{ \begin{pmatrix} 2x \\ 0 \end{pmatrix} : x \in \mathbb{R} \right\}$

P4. The subset $\mathcal{W}$ of $\mathbb{R}^3$ defined by $\mathcal{W} = \left\{ \begin{pmatrix} x - y \\ x + y + 2z \\ y + z \end{pmatrix} : x, y, z \in \mathbb{R} \right\}$

Proposition 3.2. The span of any set of vectors in $\mathbb{R}^n$ is a vector subspace of $\mathbb{R}^n$.

Proof. Suppose that $V = \text{Span}\,(\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m)$ for vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m$ in $\mathbb{R}^n$.

Complete the proof: show that V is a vector space.

To see that V is a subset of $\mathbb{R}^n$, note $\vec{v}_1, \ldots, \vec{v}_m \in \mathbb{R}^n$. Furthermore, by definition we know that $\mathbb{R}^n$ is closed under scalar multiplication and so $c_1\vec{v}_1, \ldots, c_n\vec{v}_n \in \mathbb{R}^n$. Again by definition we know that $\mathbb{R}^n$ is closed under vector addition and so

$$\vec{v} = c_1\vec{v}_1 + \cdots + c_m\vec{v}_m \in \mathbb{R}^n$$

for all $\vec{v} \in V$ and so $V \subseteq \mathbb{R}^n$. $\square$

Definition 3.4. Let V be a vector subspace of $\mathbb{R}^n$. A SPANNING SET (also known as a GENERATING SET) for V is ...

Lecture Activity 3.2. Show that the following sets are vector spaces by finding a generating set. Compare with your work in Lecture Activity 3.1

P1. $V = \left\{ \begin{pmatrix} 2x \\ 0 \end{pmatrix} : x \in \mathbb{R} \right\}$

P2. $W = \left\{ \begin{pmatrix} x - y \\ x + y + 2z \\ y + z \end{pmatrix} : x, y, z \in \mathbb{R} \right\}$

Definition 3.5. A subset $\mathcal{B}$ of a vector space V is called a BASIS if ...

Lecture Activity 3.3. Determine which of the following sets are bases for $\mathbb{R}^3$.

$$\mathcal{B}_1 = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} \right\}$$

$$\mathcal{B}_2 = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}$$

$$\mathcal{B}_3 = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

$$\mathcal{B}_4 = \left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

Definition 3.8. Let V be a nonzero vector subspace of $\mathbb{R}^n$. Then, the DIMENSION of V, denoted $\dim V$, is ...

Definition 3.9. The STANDARD BASIS for $\mathbb{R}^n$ is the set $\mathcal{E} := \{\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n\}$ where $\vec{e}_i$ is ...

Lecture Activity 3.4. Show that the standard basis is a basis for $\mathbb{R}^n$. Conclude that $\dim(\mathbb{R}^n) = n$.

Lecture Activity 3.5. In this problem we'll find a basis for the vector space W from Lecture Activities 3.1 and 3.2, defined by

$$W = \left\{ \begin{pmatrix} x - y \\ x + y + 2z \\ y + z \end{pmatrix} : x, y, z \in \mathbb{R} \right\}.$$

P1. Use your work from Lecture Activity 3.2 to observe that we can write $W = \text{Span}(\vec{u}, \vec{v}, \vec{w})$.

P2. Let $A = \begin{pmatrix} \vec{u} & \vec{v} & \vec{w} \end{pmatrix}$ and observe that

$$\text{rref}(A) = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Use this calculation to show that $\vec{w} \in \text{Span}(\vec{u}, \vec{v})$.

P3. Use P2 to show that $\text{Span}(\vec{u}, \vec{v}, \vec{w}) = \text{Span}(\vec{u}, \vec{v})$.

P4. Use your work in the previous parts to find a basis for W. Then, find the dimension of W.

Lemma 3.10. Let A be an $m \times n$ matrix of the form

$$A = \begin{pmatrix} \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \end{pmatrix}$$

where the $\vec{v}_i$ are vectors in $\mathbb{R}^m$. If the nth column of $\mathrm{rref}(A)$ does not have a pivot, then the vector $\vec{v}_n$ is in $\mathrm{Span}(\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_{n-1})$.

Prove Lemma 3.10.

Lecture Activity 3.6. Find a basis for the following vector spaces, and state their dimension.

P1. $V = \text{Span}\left(\begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right)$

P2. $W = \text{Span}\left(\begin{pmatrix} 1 \\ 0 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 0 \\ 6 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 7 \\ 2 \\ 12 \\ 7 \end{pmatrix} \right)$