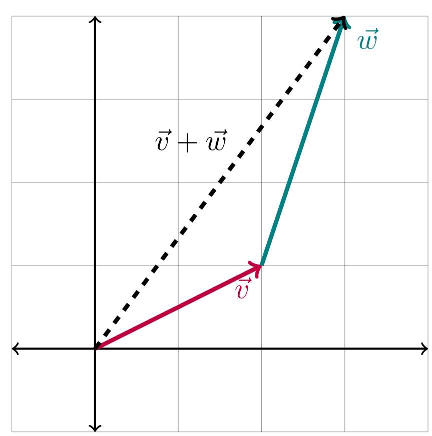
Instructions. This packet is due on Quercus no later than **11:59pm on Monday, September 15th**. Please complete your work directly on this packet. We will spend time together during lecture working on most or all of the examples and lecture activities in this packet. You are responsible for completing all portions of this packet, including lecture activities not discussed in class, and completing the definitions included in the packet. Solutions will be posted to the course website after the assignment due date.

Lecture Activity 2.1. Let's look at how to calculate total displacement.

P1. Suppose that someone gave you the following directions: (1) from your starting point, walk two blocks east and one block north, then (2) walk one block east and three blocks north. Find the standard coordinate representation of your total displacement.

Solution. Our total displacement vector is given by $\begin{pmatrix} 2+1 \\ 1+3 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$. That is, in total we've walked 1 block west and 4 blocks north.

P2. On the graph below, sketch the path you would take by following the directions from P1. On the same graph, sketch the total displacement vector you found in P1.



P3. Suppose that someone gave you the following directions: (1) from your starting point, walk v_1 blocks east and v_2 blocks north, then (2) walk w_1 blocks east and w_2 blocks north. Find the standard coordinate representation of your total displacement. In this problem, v_1, v_2, w_1, w_2 are unknown real numbers.

Solution. Our total displacement vector is given by $\begin{pmatrix} v_1+w_1 \\ v_2+w_2 \end{pmatrix}$. That is, in total we've walked v_1+w_1 block west and v_2+w_2 blocks north.

Lecture Activity 2.2 (Flight Navigation I)**.** You are piloting an airplane equipped with two fixed-direction thrusters that assist in maneuvering. Each thruster provides thrust in a specific, constant direction and can be fired forward or in reverse for any number of seconds. In this scenario, we assume the airplane is already at cruising altitude, and the two thrusters only affect horizontal position.

- Firing **Thruster A** for one second in the forward direction moves the airplane 17 meters East and 7 meters North; firing Thrusters A for one second in the backward direction moves the airplane 17 meters West and 7 meters South.
- Firing **Thruster B** for one second in the forward direction moves the airplane 5 meters East and 18 meters North; firing Thruster B for one second in the backward direction moves the airplane 5 meters West and 18 meters South.

In this problem, we explore what these thruster directions imply about the airplane's maneuverability at its current fixed altitude.

P1. Suppose you're instructed to reach a waypoint located 235 meters East and 33 meters North of your current position. Can you reach the waypoint using only Thruster A *or* only Thruster B? If yes, determine how many seconds you need to fire the thruster to reach the waypoint. If not, explain why not.

Solution. No, it is not possible. Note that the vectors

$$\vec{a} = \begin{pmatrix} 17 \\ 7 \end{pmatrix} \text{ and } \vec{b} = \begin{pmatrix} 5 \\ 18 \end{pmatrix}$$

represent the displacement by firing Thruster A and Thruster B for one second, respectively. If we could reach the desired waypoint using only Thruster A, we would need to be able the solve the vector equation

$$x \begin{pmatrix} 17 \\ 7 \end{pmatrix} = \begin{pmatrix} 235 \\ 33 \end{pmatrix} \Rightarrow 17x = 235 \text{ and } 7x = 33$$

which gives two different values for x, and so is not possible. Similarly, if we could reach the desired waypoint using only Thruster B, we would need to be able to solve the vector equation

$$y \begin{pmatrix} 5 \\ 18 \end{pmatrix} = \begin{pmatrix} 235 \\ 33 \end{pmatrix} \Rightarrow 5y = 235 \text{ and } 18y = 33$$

which again gives two different values for y, and so is not possible.

P2. Can you reach the waypoint from P1 using a combination of both thrusters? If yes, determine how many seconds you need to fire each thruster to reach the waypoint. If not, explain why not.

Solution. If we fire Thruster A for x seconds and Thruster B for y seconds (letting negative seconds denote firing the thruster in the backward direction), our total displacement is given by

$$x\begin{pmatrix}17\\7\end{pmatrix}+y\begin{pmatrix}5\\18\end{pmatrix}.$$

To reach the desired waypoint, we need to solve the vector equation

$$x\begin{pmatrix}17\\7\end{pmatrix}+y\begin{pmatrix}5\\18\end{pmatrix}=\begin{pmatrix}235\\33\end{pmatrix}$$

which can be rewritten in the form

$$\begin{pmatrix}17x+5y\\7x+18y\end{pmatrix}=\begin{pmatrix}235\\33\end{pmatrix}.$$

Since a vector is uniquely determined by its standard coordinates, solutions to our vector equation are precisely solutions to the following system of linear equations

$$\begin{cases}17x+5y=235\\7x+18y=33.\end{cases}$$

We row reduce the augmented matrix of this system

$$\left(\begin{array}{cc|c}17&5&235\\7&18&33\end{array}\right)\sim\left(\begin{array}{cc|c}1&0&15\\0&1&4\end{array}\right).$$

So, yes we can reach the waypoint using both of our Thrusters. To do so, we would need to use Thruster A in the forward direction for 15 seconds, and Thruster B in the backward direction for 4 seconds.

P3. Can you reach every possible waypoint at your current elevation (i.e., in the same horizontal plane) using a combination of Thrusters A and B? Explain your answer.

Solution. Yes, we can reach all possible waypoints at our current elevation. To see this, consider an arbitrary waypoint that's c meters East and d meters North of our current location (letting negative meters denote distance in the opposite direction). As in the previous problem, we need to consider the vector equation

$$x \begin{pmatrix} 17 \\ 7 \end{pmatrix} + y \begin{pmatrix} 5 \\ 18 \end{pmatrix} = \begin{pmatrix} c \\ d \end{pmatrix}.$$

This has the same solution set as the system of linear equations with augmented matrix

$$\left(\begin{array}{cc|c} 17 & 5 & c \\ 7 & 18 & d \end{array}\right),$$

which has reduced row echelon form

$$\left(\begin{array}{cc|c} 1 & 0 & * \\ 0 & 1 & * \end{array}\right),$$

where $*$ denote some real numbers that depend on c and d (note that these could be calculated explicitly, but we can answer our question without this information). Observe that, no matter the values of c and d, this matrix does not have a pivot in the last column, and so by Rouché-Capelli, this system *always* has a solution.

Lecture Activity 2.3. Find the augmented matrix for the system of linear equations that has the same solution set as the following vector equations. Then, find all solutions to the vector equation.

P1. $x\begin{pmatrix}1\\2\end{pmatrix}+y\begin{pmatrix}-1/2\\5\end{pmatrix}=\begin{pmatrix}3\\4\end{pmatrix}$

Solution. This vector equation has the same solution set as the system of linear equations with augmented matrix

$$\left(\begin{array}{cc|c}1 & -1/2 & 3\\ 2 & 5 & 4\end{array}\right)$$

which has reduced row echelon form

$$\left(\begin{array}{cc|c}1 & 0 & 17/6\\ 0 & 1 & -1/3\end{array}\right)$$

and so this vector equation has solution $\boxed{x=17/6, y=-1/3}$.

P2. $x\begin{pmatrix}1\\1\\1\end{pmatrix}+y\begin{pmatrix}3\\4\\1\end{pmatrix}=\begin{pmatrix}2\\5\\1\end{pmatrix}$

Solution. This vector equation has the same solution set as the system of linear equations with augmented matrix

$$\left(\begin{array}{cc|c}1 & 3 & 2\\ 1 & 4 & 5\\ 1 & 1 & 1\end{array}\right)$$

which has reduced row echelon form

$$\left(\begin{array}{cc|c}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{array}\right).$$

Since there's a pivot in the last column, this system has no solutions. Hence, our vector equation has $\boxed{\text{no solutions}}$.

P3. $x\begin{pmatrix}2\\1\end{pmatrix} + y\begin{pmatrix}5\\3\end{pmatrix} + z\begin{pmatrix}-1\\0\end{pmatrix} = \begin{pmatrix}4\\2\end{pmatrix}$

Solution. This system of linear equations has the same solution set as the system of linear equations with augmented matrix

$$\left(\begin{array}{ccc|c} 2 & 5 & -1 & 4 \\ 1 & 3 & 0 & 2 \end{array}\right)$$

which has reduced row echelon form

$$\left(\begin{array}{ccc|c} 1 & 0 & -3 & 2 \\ 0 & 1 & 1 & 0 \end{array}\right).$$

The system is consistent with variable z being free. We can parameterize our basic variables as following

$$x - 3z = 2 \Rightarrow x = 2 + 3z, \text{ and } y + z = 0 \Rightarrow y = -z.$$

Hence, the solution set to our vector equation is given by

$$\boxed{\{(2+3z, -z, z) : z \in \mathbb{R}\}}.$$

Definition 2.8. A LINEAR COMBINATION of vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ in $\mathbb{R}^m$ is a vector of the form

$$\vec{w} = c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n$$

where the $c_1, c_2, \ldots, c_n$ are scalars called the COEFFICIENTS of the linear combination.

Definition 2.9. The SPAN of vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ in $\mathbb{R}^m$ is the set

$$\text{Span}(\vec{v}_1, \ldots, \vec{v}_n) = \{c_1\vec{v}_1 + \cdots + c_n\vec{v}_n \mid c_1, c_2, \ldots, c_n \in \mathbb{R}\}.$$

That is $\text{Span}(\vec{v}_1, \ldots, \vec{v}_n)$ is the set of all linear combinations of vectors $\vec{v}_1, \ldots, \vec{v}_n$.

Lecture Activity 2.4 (Flight Navigation II)**.** You are piloting an airplane equipped with four fixed-direction thrusters that assist in maneuvering. Each thruster provides thrust in a specific, constant direction and can be fired forward or in reverse for any number of seconds.

- Firing **Thruster A** for one second in the forward direction moves the airplane 10 meters East, 9 meters North, and 2 meters Up. Firing Thruster A for one second in the backward direction has the opposite effect; that is, moves the airplane 10 meters West, 9 meters South, and 2 meters Down.
- Firing **Thruster B** for one second in the forward direction moves the airplane 4 meters East, 7 meters North, and 1 meters Up. Firing B in reverse has the opposite effect.
- Firing **Thruster C** for one second in the forward direction moves the airplane 0 meters East, 2 meter North, and 3 meter Up. Firing C in reverse has the opposite effect.
- Firing **Thruster D** for one second in the forward direction moves the airplane 2 meter East, 5 meters South, and 0 meters Up. Firing D in reverse has the opposite effect.

In this problem, we will explore what these thruster directions imply about the airplane's maneuverability in 3 dimensional space.

P1. Show that you can reach any waypoint using all four thrusters.

Solution. Note that the following vectors represent the airplane's displacement after firing Thurster A, B, C, and D for 1 second, respectively

$$\vec{a} = \begin{pmatrix} 10 \\ 9 \\ 2 \end{pmatrix}, \vec{b} = \begin{pmatrix} 4 \\ 7 \\ 1 \end{pmatrix}, \vec{c} = \begin{pmatrix} 0 \\ 2 \\ 3 \end{pmatrix}, \text{ and } \vec{d} = \begin{pmatrix} 2 \\ -5 \\ 0 \end{pmatrix}.$$

We want show that $\text{Span}(\vec{a}, \vec{b}, \vec{c}, \vec{d})$ contains every vector in $\mathbb{R}^3$. Consider a waypoint that's located at displacement $\vec{f}$ from our current location, where $\vec{f}$ is an arbitrary vector in $\mathbb{R}^3$. The vector equation

$$x\vec{a} + y\vec{b} + z\vec{c} + w\vec{d} = \vec{f}$$

has the same solution set as the system of linear equations

$$\left(\begin{array}{cccc|c} 10 & 4 & 0 & 2 & * \\ 9 & 7 & 2 & -5 & * \\ 2 & 1 & 3 & 0 & * \end{array}\right)$$

where $*$ denotes the unknown real number coordinates of the vector $\vec{f}$. This matrix has reduced row echelon form

$$\left(\begin{array}{cccc|c} 1 & 0 & 0 & 1 & ** \\ 0 & 1 & 0 & -2 & ** \\ 0 & 0 & 1 & 0 & ** \end{array}\right)$$

where $**$ denotes some unknown real numbers. Since this matrix does not have a pivot in the last column, no matter what vector $\vec{f}$ we chose, the system always has a solution. Hence, every vector in $\mathbb{R}^3$ can be reached as a linear combination of vectors $\vec{a}, \vec{b}, \vec{c}, \vec{d}$.

P2. Show that you can reach any waypoint using only Thrusters A, B and C.

Solution. We can solve this problem similarly to P1, but we present a different solution to make an important observation. Using our work from P1, we can see that

$$\vec{d} = \vec{a} - 2\vec{b}. \tag{2.1}$$

Consider a waypoint that's located at displacement $\vec{f}$ from our current location. Using our work in P1 we know that $\vec{f}$ is in $\mathrm{Span}(\vec{a}, \vec{b}, \vec{c}, \vec{d})$ and so there exist real numbers x, y, z, w so that

$$\vec{f} = x\vec{a} + y\vec{b} + z\vec{c} + w\vec{d}.$$

Substituting Equation (2.1) in for $\vec{d}$ in the equation above gives

$$\vec{f} = x\vec{a} + y\vec{b} + z\vec{c} + w(a - 2\vec{b})$$

and rearranging gives

$$\vec{f} = (x + w)\vec{a} + (y - 2w)\vec{b} + z\vec{c}.$$

Hence, $\vec{f}$ is in $\mathrm{Span}(\vec{a}, \vec{b}, \vec{c})$, and so we can reach any vector in $\mathbb{R}^3$ as a linear combination of vectors $\vec{a}, \vec{b}, \vec{c}$.

Note: The key observation in our work above was that since $\vec{d}$ was already in $\mathrm{Span}(\vec{a}, \vec{b}, \vec{c})$, it didn't "contribute" anything new to our maneuverability in space.

P3. Do you think it's possible to reach any waypoint using only two Thrusters?

Solution. **No**, it is not possible. Let's provide reasoning for Thrusters A and B and note that the argument for the other pairs of Thrusters follow identically. Consider a waypoint that's located at displacement $\vec{f}$ from our current location, where $\vec{f}$ is an arbitrary vector in $\mathbb{R}^3$. The vector equation

$$x\vec{a} + y\vec{b} = \vec{f}$$

has augmented matrix $\left(\vec{a} \quad \vec{b} \mid \vec{f}\right)$. Observe that we can find a vector $\vec{f}$ so that this matrix is row equivalent to a matrix of the form

$$\left(\begin{array}{cc|c} 1 & 0 & * \\ 0 & 1 & * \\ 0 & 0 & \text{nonzero} \end{array}\right).$$

Hence, the reduced row echelon form of this matrix will have a pivot in the last column, and hence $\vec{f}$ will not be a linear combination of $\vec{a}$ and $\vec{b}$.

Definition 2.10. A set of vectors $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ in $\mathbb{R}^m$ is called LINEARLY DEPENDENT if ...

> at least one of the vectors is a linear combination of the others. That is, for at least one $i \in \{1, \ldots, n\}$ we have
>
> $$\vec{v}_i \in \operatorname{Span}(\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_{i-1}, \vec{v}_{i+1}, \ldots, \vec{v}_n).$$

Otherwise, the vectors are called LINEARLY INDEPENDENT.

Lecture Activity 2.5. Use the definition of linear dependence to determine which of the sets are linearly dependent and which are linearly independent. For the sets that are linearly dependent, demonstrate how to write one of the vectors as a linear combination of the others.

P1. $S = \left\{ \begin{pmatrix} -1 \\ 4 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix}, \right\}$

Solution. Observe that

$$\begin{pmatrix} -1 \\ 4 \\ 0 \end{pmatrix} = 2 \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix} + 3 \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix}$$

and so the set S is linearly dependent.

P2. $T = \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix} \right\}$.

Solution. First, we ask whether $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \overset{?}{\in} \operatorname{Span}\left(\begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix} \right)$. The vector equation

$$x \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix} + y \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

has the same solution set as the system of linear equations with augmented matrix

$$\left(\begin{array}{cc|c} -1 & 3 & 1 \\ 2 & 1 & 1 \\ 0 & 2 & 0 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right)$$

and so this system is inconsistent. Hence, $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \notin \operatorname{Span}\left(\begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix} \right)$.

Arguing similarly, we see that

$$\begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix} \notin \operatorname{Span}\left(\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix} \right)$$

and

$$\begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix} \notin \operatorname{Span}\left(\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix} \right).$$

So, no vector is in the span of the other two, and so this system is linearly independent.

Theorem 2.11. A set of vectors $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ in $\mathbb{R}^m$ is linearly dependent if and only if the vector equation

$$x_1\vec{v}_1 + x_2\vec{v}_2 + \cdots + x_n\vec{v}_n = \vec{0}$$

has a "nontrivial" solution; that is, a solution other than $(x_1, x_2, \ldots, x_n) = (0, 0, \ldots, 0)$.

Proof. Suppose that the set $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ is linearly dependent. By relabeling, we may assume that $\vec{v}_1 \in \text{Span}(\vec{v}_2, \ldots, \vec{v}_n)$. So, there are real numbers $c_2, \ldots, c_n$ so that

$$\vec{v}_1 = c_2\vec{v}_2 + \cdots + c_n\vec{v}_n.$$

Complete the proof: show that the system has a nontrivial solution.

Subtracting both sides by $\vec{v}_1$ gives

$$-\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n = \vec{0}.$$

Hence, the vector equation

$$x_1\vec{v}_1 + x_2\vec{v}_2 + \cdots + x_n\vec{v}_n = \vec{0}$$

has $(x_1, \ldots, x_n) = (-1, c_2, \ldots, c_n)$ as a solution. Since $x_1 = -1 \neq 0$, this solution is nontrivial, as needed.

Conversely, suppose that the vector equation

$$x_1\vec{v}_1 + x_2\vec{v}_2 + \cdots + x_n\vec{v}_n = \vec{0}$$

has a nontrivial solution $(c_1, c_2, \ldots, c_n)$. Then, one of the c_i is nonzero. By relabeling, we may assume that $c_1 \neq 0$.

Complete the proof: show that the set $\{\vec{v}_1, \ldots, \vec{v}_n\}$ is linearly dependent.

This gives

$$c_1\vec{v}_1 = -c_2\vec{v}_2 - \cdots - c_n\vec{v}_n$$

and so dividing on both sides by c_1 (which we can do since we know c_1 is nonzero) gives

$$\vec{v}_1 = -\frac{c_2}{c_1}\vec{v}_2 - \cdots - \frac{c_n}{c_1}\vec{v}_n \in \text{Span}(\vec{v}_2, \ldots, \vec{v}_n).$$

Hence, $\vec{v}_1 \in \text{Span}(\vec{v}_2, \ldots, \vec{v}_n)$, and so the set $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ is linearly dependent.

□

Lecture Activity 2.6. Use Theorem 2.11 to determine which of the following sets are linearly dependent and which are linearly independent.

P1. $S = \left\{ \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix}, \begin{pmatrix} -1 \\ 4 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix} \right\}$

Solution. The vector equation

$$x \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix} + y \begin{pmatrix} -1 \\ 4 \\ 0 \end{pmatrix} + z \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

has the same solution set as the system of linear equations with augmented matrix

$$\left(\begin{array}{ccc|c} 1 & -1 & -1 & 0 \\ 2 & 4 & 0 & 0 \\ -3 & 0 & 2 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & -2/3 & 0 \\ 0 & 1 & 1/3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

Observe that this system is consistent and the rref of the coefficient matrix has a column without a pivot. So, by Rouché-Capelli this system has infinitely many solutions. Hence, one of those solutions must be nontrivial, and so this set is **linearly dependent**.

P2. $T = \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix} \right\}$

Solution. The vector equation

$$x \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + y \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix} + z \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

has the same solution set as the augmented matrix

$$\left(\begin{array}{ccc|c} 1 & 3 & -1 & 0 \\ 1 & 1 & 2 & 0 \\ 0 & 2 & 0 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right),$$

and so the only solution to the vector equation above is the trivial solution $(x, y, z) = (0, 0, 0)$. Hence, the set S is **linearly independent**.

P3. $U = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix} \right\}$

Solution. The vector equation

$$x \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + y \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + z \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} + w \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

has the same solution set as the system of linear equations with augmented matrix

$$\left(\begin{array}{cccc|c} 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right).$$

Observe that this system is consistent and the rref of the coefficient matrix has a column without a pivot. So, by Rouché-Capelli this system has infinitely many solutions. Hence, one of those solutions must be nontrivial, and so this set is **linearly dependent**.

Proposition 2.12. A set of vectors $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ in $\mathbb{R}^m$ is linearly independent if and only if the reduced row echelon form of the matrix $\begin{pmatrix} \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \end{pmatrix}$ has a pivot in every column.

Proof. Suppose that the set $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ is linearly independent. Then, by Theorem 2.11 the vector equation $x_1\vec{v}_1 + x_2\vec{v}_2 + \cdots + x_n\vec{v}_n = \vec{0}$ only has the trivial solution.

Complete the proof: use Rouché-Capelli to show that the reduced row echelon form of the matrix $\begin{pmatrix} \vec{v}_1 & \cdots & \vec{v}_n \end{pmatrix}$ has a pivot in every column.

This equation has the same solution set as the system of linear equations with augmented matrix

$$\left(\begin{array}{cccc|c} \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n & \vec{0} \end{array}\right).$$

Note that this equation always has the trivial solution $(x_1, x_2, \ldots, x_n) = (0, 0, \ldots, 0)$, and so the system is consistent. Hence, by Rouché-Capelli, the reduced row echelon form of the matrix

$$\begin{pmatrix} \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \end{pmatrix}$$

must have a pivot in every column, as needed.

Conversely, suppose that the reduced row echelon form of the matrix $\begin{pmatrix} \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \end{pmatrix}$ has a pivot in every column. Then, by Rouché-Capelli, the system of linear equations with augmented matrix

$$\left(\begin{array}{cccc|c} \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n & \vec{0} \end{array}\right)$$

only has one solution.

Complete the proof: use Theorem 2.11 to conclude that the set $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ is linearly independent.

Since $(0, 0, \ldots, 0)$ is always a solution to this equation, it must be the only solution. So, the vector equation

$$x_1\vec{v}_1 + x_2\vec{v}_2 + \cdots + x_n\vec{v}_n = \vec{0}$$

only has the trivial solution. Therefore, by Theorem 2.11, the set $\{\vec{v}_1, \ldots, \vec{v}_n\}$ is linearly independent. □