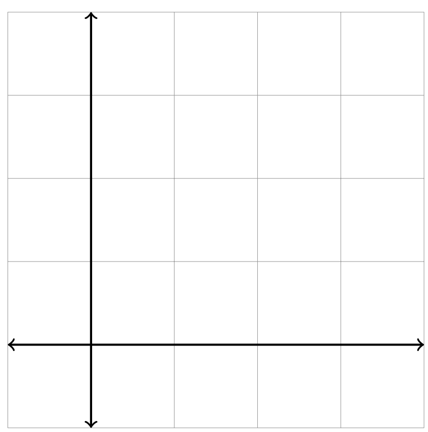
CHAPTER 2 ACTIVITY PACKET NAME:

Instructions. This packet is due on Quercus no later than **11:59pm on Monday, September 15th**. Please complete your work directly on this packet. We will spend time together during lecture working on most or all of the examples and lecture activities in this packet. You are responsible for completing all portions of this packet, including lecture activities not discussed in class, and completing the definitions included in the packet. Solutions will be posted to the course website after the assignment due date.

Lecture Activity 2.1. Let's look at how to calculate total displacement.

P1. Suppose that someone gave you the following directions: (1) from your starting point, walk two blocks east and one block north, then (2) walk one block east and three blocks north. Find the standard coordinate representation of your total displacement.

P2. On the graph below, sketch the path you would take by following the directions from P1. On the same graph, sketch the total displacement vector you found in P1.



P3. Suppose that someone gave you the following directions: (1) from your starting point, walk v_1 blocks east and v_2 blocks north, then (2) walk w_1 blocks east and w_2 blocks north. Find the standard coordinate representation of your total displacement. In this problem, v_1, v_2, w_1, w_2 are unknown real numbers.

Lecture Activity 2.2 (Flight Navigation I)**.** You are piloting an airplane equipped with two fixed-direction thrusters that assist in maneuvering. Each thruster provides thrust in a specific, constant direction and can be fired forward or in reverse for any number of seconds. In this scenario, we assume the airplane is already at cruising altitude, and the two thrusters only affect horizontal position.

- Firing **Thruster A** for one second in the forward direction moves the airplane 17 meters East and 7 meters North; firing Thrusters A for one second in the backward direction moves the airplane 15 meters West and 7 meters South.
- Firing **Thruster B** for one second in the forward direction moves the airplane 5 meters East and 18 meters North; firing Thruster B for one second in the backward direction moves the airplane 5 meters West and 18 meters South.

In this problem, we explore what these thruster directions imply about the airplane's maneuverability at its current fixed altitude.

P1. Suppose you're instructed to reach a waypoint located 235 meters East and 33 meters North of your current position. Can you reach the waypoint using only Thruster A *or* only Thruster B? If yes, determine how many seconds you need to fire the thruster to reach the waypoint. If not, explain why not.

P2. Can you reach the waypoint from P1 using a combination of both thrusters? If yes, determine how many seconds you need to fire each thruster to reach the waypoint. If not, explain why not.

P3. Can you reach every possible waypoint at your current elevation (i.e., in the same horizontal plane) using a combination of Thrusters A and B? Explain your answer.

Lecture Activity 2.3. Find the augmented matrix for the system of linear equations that has the same solution set as the following vector equations. Then, find all solutions to the vector equation.

P1. $x \begin{pmatrix} 1 \\ 2 \end{pmatrix} + y \begin{pmatrix} -1/2 \\ 5 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$

P2. $x \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + y \begin{pmatrix} 3 \\ 4 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 5 \\ 1 \end{pmatrix}$

P3. $x\begin{pmatrix}2\\1\end{pmatrix}+y\begin{pmatrix}5\\3\end{pmatrix}+z\begin{pmatrix}-1\\0\end{pmatrix}=\begin{pmatrix}4\\2\end{pmatrix}$

Definition 2.8. A LINEAR COMBINATION of vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ in $\mathbb{R}^m$ is a vector of the form

where the $c_1, c_2, \ldots, c_n$ are scalars called the COEFFICIENTS of the linear combination.

Definition 2.9. The SPAN of vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ in $\mathbb{R}^m$ is the set

That is $\mathrm{Span}(\vec{v}_1, \ldots, \vec{v}_n)$ is the set of all linear combinations of vectors $\vec{v}_1, \ldots, \vec{v}_n$.

Lecture Activity 2.4 (Flight Navigation II)**.** You are piloting an airplane equipped with four fixed-direction thrusters that assist in maneuvering. Each thruster provides thrust in a specific, constant direction and can be fired forward or in reverse for any number of seconds.

- Firing **Thruster A** for one second in the forward direction moves the airplane 10 meters East, 9 meters North, and 2 meters Up. Firing Thruster A for one second in the backward direction has the opposite effect; that is, moves the airplane 10 meters West, 9 meters South, and 2 meters Down.
- Firing **Thruster B** for one second in the forward direction moves the airplane 4 meters East, 7 meters North, and 1 meters Up. Firing B in reverse has the opposite effect.
- Firing **Thruster C** for one second in the forward direction moves the airplane 0 meters East, 2 meter North, and 3 meter Up. Firing C in reverse has the opposite effect.
- Firing **Thruster D** for one second in the forward direction moves the airplane 2 meter East, 5 meters South, and 0 meters Up. Firing D in reverse has the opposite effect.

In this problem, we will explore what these thruster directions imply about the airplane's maneuverability in 3 dimensional space.

P1. Show that you can reach any waypoint using all four thrusters.

P2. Show that you can reach any waypoint using only Thrusters A, B and C.

P3. Do you think it's possible to reach any waypoint using only two Thrusters?

Definition 2.10. A set of vectors $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ in $\mathbb{R}^m$ is called LINEARLY DEPENDENT if ...

Otherwise, the vectors are called LINEARLY INDEPENDENT.

Lecture Activity 2.5. Use the definition of linear dependence to determine which of the sets are linearly dependent and which are linearly independent. For the sets that are linearly dependent, demonstrate how to write one of the vectors as a linear combination of the others.

P1. $S = \left\{ \begin{pmatrix} -1 \\ 4 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix}, \right\}$

P2. $T = \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix} \right\}.$

Theorem 2.11. A set of vectors $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ in $\mathbb{R}^m$ is linearly dependent if and only if the vector equation $x_1\vec{v}_1 + x_2\vec{v}_2 + \cdots + x_n\vec{v}_n = \vec{0}$ has a "nontrivial" solution; that is, a solution other than $(x_1, x_2, \ldots, x_n) = (0, 0, \ldots, 0)$.

Proof. Suppose that the set $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ is linearly dependent. By relabeling, we may assume that $\vec{v}_1 \in \text{Span}(\vec{v}_2, \ldots, \vec{v}_n)$. So, there are real numbers $c_2, \ldots, c_n$ so that $\vec{v}_1 = c_2\vec{v}_2 + \cdots + c_n\vec{v}_n$.

Complete the proof: show that the system has a nontrivial solution.

Conversely, suppose that the vector equation $x_1\vec{v}_1 + x_2\vec{v}_2 + \cdots + x_n\vec{v}_n = \vec{0}$ has a nontrivial solution $(c_1, c_2, \ldots, c_n)$. Then, one of the c_i is nonzero. By relabeling, we may assume that $c_1 \neq 0$.

Complete the proof: show that the set $\{\vec{v}_1, \ldots, \vec{v}_n\}$ is linearly independent.

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Lecture Activity 2.6. Use Theorem 2.11 to determine which of the following sets are linearly dependent and which are linearly independent.

P1. $S = \left\{ \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix}, \begin{pmatrix} -1 \\ 4 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix} \right\}$

P2. $T = \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix} \right\}$

P3. $U = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix} \right\}$

Proposition 2.12. A set of vectors $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$ in $\mathbb{R}^m$ is linearly independent if and only if the reduced row echelon form of the matrix $\begin{pmatrix} \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \end{pmatrix}$ has a pivot in every column.

Proof. Suppose that the set $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$ is linearly independent. Then, by Theorem 2.11 the vector equation $x_1\vec{v}_1 + x_2\vec{v}_2 + \cdots + x_n\vec{v}_n = \vec{0}$ has a nontrivial solution.

Complete the proof: use Rouché-Capelli to show that the reduced row echelon form of the matrix $\begin{pmatrix} \vec{v}_1 & \cdots & \vec{v}_n \end{pmatrix}$ has a pivot in every column.

Conversely, suppose that the reduced row echelon form of the matrix $\begin{pmatrix} \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \end{pmatrix}$ has a pivot in every column. Then, by Rouché-Capelli, the system of linear equations with augmented matrix $\left(\begin{array}{cccc|c} \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n & \vec{0} \end{array}\right)$ only has one solution.

Complete the proof: use Theorem 2.11 to conclude that the set $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$ is linearly independent.

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